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LOCAL VOLATILITIES FOR QUANTO OPTION PRICES WITH VARIOUS TYPES OF PAYOFFS

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Abstract. This paper is about the derivations of local volatilities for European quanto call option prices according to various types of payoffs. We derive the explicit formulas of local volatilities with constant foreign and domestic interest rates by adapting the method of Derman-Kani.

1. Introduction

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable. A quanto option is a cash-settled, cross-currency derivative whose underlying asset has a payoff in one currency, but the payoff is converted to another currency when the option is settled. For that reason, the correlation between underlying asset and currency exchange rate plays an important role in pricing quanto option. At exercise, the value of the option is calculated as the options intrinsic value in foreign currency, which is then converted to the domestic currency. This allows investors to obtain exposure to foreign assets without the corresponding exchange rate risk.

Pricing options based on the classical Black-Scholes [1] model, on which most of the research on quanto options has focused, has a problem of assuming a constant volatility which leads to smiles and skews in the implied volatility for the underlying asset price. One way to overcome such handicaps of constant volatility is using a local volatility which treats the volatility as a deterministic function of the underlying asset price, current time, maturity and the strike price.

Indeed, the concept of a local volatility was introduced and developed by Dupire [3] and Derman-Kani [2] as they found that there is a unique diffusion process consistent with the risk-neutral densities derived from the market prices of European options. The main advantage to use local volatilities is that

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the only source of randomness is the price of underlying asset, making local volatility easy to calibrate.

Most recently, Lee and Lee [5] provided the explicit formula of the local volatility with constant foreign and domestic interest rates by adapting the methods of [3] and [2]. Furthermore, they obtained the analogue of the Dupire equation for the local volatility with stochastic interest rates.

In this paper, extending to [5], we modify and adjust the various types of payoffs and the method of [2] to obtain the explicit formulas of local volatilities for quanto option prices with constant foreign and domestic riskless rates.

In Section 2, we derive the risk-neutral dynamics of processes for the underlying asset with respect to different currencies. Then, in Section 3, under the models specified in Section 2, we derive the explicit formulas of local volatilities for the European quanto call option prices according to various types of payoffs.

2. Risk-neutral dynamics on currencies

Let S_t be the asset price on a non-dividend paying asset in foreign currency with the local volatility $\sigma_S(t, S_t)$, and let V_t be the foreign exchange rate in foreign currency per unit of the domestic currency with the constant volatility σ_V, which have the following risk-neutral dynamics:

$$\begin{cases} dS_t = r_f S_t dt + \sigma_S(t, S_t) S_t dB_t^f, \\ dV_t = (r_f - r_d) V_t dt + \sigma_V V_t dW_t^f, \end{cases} \tag{1}$$

where r_f and r_d is the constant foreign and domestic riskless rates, respectively. Also, B_t^f and W_t^f are two standard Brownian motions in foreign currency with the correlation ρ.

Then the risk-neutral dynamics of (1) in domestic currency are given by

$$\begin{cases} dS_t = \{r_f - \rho\sigma_S(t, S_t)\sigma_V\} S_t dt + \sigma_S(t, S_t) S_t dB_t^d, \\ dV_t = (r_d - r_f) V_t dt + \sigma_V V_t dW_t^d, \end{cases} \tag{2}$$

where B_t^d and W_t^d are two correlated standard Brownian motions in domestic currency. It was minutely described in [5].

3. Local volatilities for various quanto payoffs

Derman-Kani [2] characterized the local volatility as a risk-neutral expectation of the instantaneous volatility, conditional on the final asset price being equal to the strike price. The following theorems in next subsections adapt the method of [2] to obtain the quanto option framework with constant foreign and domestic riskless rates according to various types of payoffs listed in Chapter 3.4.7 of [4].

From now on, we will denote $\mathbb{E}[\cdot]$ by the risk-neutral conditional expectation with respect to the filtration $\mathcal{F}_t$.

3.1. Type 1

The foreign equity call struck in foreign currency has a payoff given by

$$V_T \max\left(S_T - K_f, 0\right), \tag{3}$$

where K_f is the foreign strike price.

Theorem 3.1. *Suppose that $C_q^{(1)}$ is the price of a European quanto call option at time t in domestic currency with the payoff* (3)*. Then the local volatility is expressed by*

$$\sigma_S\left(S_t; K_f, T\right) = \frac{1}{K_f}\sqrt{\frac{\frac{\partial C_q^{(1)}}{\partial T} + r_f K_f \frac{\partial C_q^{(1)}}{\partial K_f}}{\frac{1}{2}\frac{\partial^2 C_q^{(1)}}{\partial K_f^2}}}.$$

Proof. We may write $C_q^{(1)}$ as

$$C_q^{(1)}\left(S_t; K_f, T\right) = \mathbb{E}\left[e^{-r_d(T-t)} V_T \max\left(S_T - K_f, 0\right)\right]. \tag{4}$$

Differentiating (4) with respect to K_f, it gives

$$\frac{\partial C_q^{(1)}}{\partial K_f} = -\mathbb{E}\left[e^{-r_d(T-t)} V_T H\left(S_T - K_f\right)\right], \tag{5}$$

where $H(\cdot)$ denotes the Heaviside function. Differentiating again (5) with respect to K_f, it gives

$$\frac{\partial^2 C_q^{(1)}}{\partial K_f^2} = \mathbb{E}\left[e^{-r_d(T-t)} V_T \delta\left(S_T - K_f\right)\right],$$

where $\delta(\cdot)$ denotes the Dirac-delta function. Also, differentiating (4) with respect to T, it gives

$$\frac{\partial C_q^{(1)}}{\partial T} = -r_d C_q^{(1)} + e^{-r_d(T-t)}\frac{\partial}{\partial T}\mathbb{E}\left[V_T \max\left(S_T - K_f, 0\right)\right].$$

Applying the Itô formula to this payoff, we have

$$\begin{aligned}
&d\left\{V_T \max\left(S_T - K_f, 0\right)\right\} \\
&= \frac{\partial}{\partial S_T}\left\{V_T \max\left(S_T - K_f, 0\right)\right\} dS_T + \frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left\{V_T \max\left(S_T - K_f, 0\right)\right\}\left(dS_T\right)^2 \\
&\quad + \frac{\partial}{\partial V_T}\left\{V_T \max\left(S_T - K_f, 0\right)\right\} dV_T + \frac{1}{2}\frac{\partial^2}{\partial V_T^2}\left\{V_T \max\left(S_T - K_f, 0\right)\right\}\left(dV_T\right)^2 \\
&\quad + \frac{\partial^2}{\partial S_T \partial V_T}\left\{V_T \max\left(S_T - K_f, 0\right)\right\} dS_T dV_T \\
&= V_T H\left(S_T - K_f\right)\left\{\left(r_f - \rho\sigma_S\sigma_V\right) S_T dT + \sigma_S S_T dB_T^d\right\} \\
&\quad + \frac{1}{2} V_T \delta\left(S_T - K_f\right)\sigma_S^2 S_T^2 dT + \max\left(S_T - K_f, 0\right)\left\{\left(r_d - r_f\right) V_T dT + \sigma_V V_T dW_T^d\right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + H\left(S_T - K_f\right)\rho\sigma_S\sigma_V V_T S_T dT \\
&= V_T H\left(S_T - K_f\right)\left(r_f S_T dT + \sigma_S S_T dB_T^d\right) + \frac{1}{2}V_T\delta\left(S_T - K_f\right)\sigma_S^2 S_T^2 dT \\
&\quad + \max\left(S_T - K_f, 0\right)\left\{\left(r_d - r_f\right)V_T dT + \sigma_V V_T dW_T^d\right\}
\end{aligned}
$$

from (2).

Now, taking the expectation on both sides, it follows that

$$
\begin{aligned}
&\quad d\mathbb{E}\left[V_T \max\left(S_T - K_f, 0\right)\right] \\
&= r_f\mathbb{E}\left[V_T S_T H\left(S_T - K_f\right)\right]dT + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 V_T S_T^2\delta\left(S_T - K_f\right)\right]dT \\
&\quad + \left(r_d - r_f\right)\mathbb{E}\left[V_T \max\left(S_T - K_f, 0\right)\right]dT \\
&= r_f\mathbb{E}\left[V_T\left(S_T - K_f\right)H\left(S_T - K_f\right)\right]dT + r_f K_f\mathbb{E}\left[V_T H\left(S_T - K_f\right)\right]dT \\
&\quad + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 V_T S_T^2\delta\left(S_T - K_f\right)\right]dT + \left(r_d - r_f\right)\mathbb{E}\left[V_T\max\left(S_T - K_f, 0\right)\right]dT,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\quad \frac{\partial}{\partial T}\mathbb{E}\left[V_T \max\left(S_T - K_f, 0\right)\right] \\
&= r_f\mathbb{E}\left[V_T\max\left(S_T - K_f, 0\right)\right] + r_f K_f\mathbb{E}\left[V_T H\left(S_T - K_f\right)\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 V_T S_T^2\delta\left(S_T - K_f\right)\right] + \left(r_d - r_f\right)\mathbb{E}\left[V_T\max\left(S_T - K_f, 0\right)\right] \\
&= r_f K_f\mathbb{E}\left[V_T H\left(S_T - K_f\right)\right] + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 V_T S_T^2\delta\left(S_T - K_f\right)\right] \\
&\quad + r_d\mathbb{E}\left[V_T\max\left(S_T - K_f, 0\right)\right].
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\frac{\partial C_q^{(1)}}{\partial T} &= -r_d C_q^{(1)} - r_f K_f\frac{\partial C_q^{(1)}}{\partial K_f} \\
&\quad + \frac{1}{2}e^{-r_f(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_S^2 V_T S_T^2\delta\left(S_T - K_f\right)\middle| S_T = K_f\right]\right] + r_d C_q^{(1)} \\
&= -r_f K_f\frac{\partial C_q^{(1)}}{\partial K_f} + \frac{1}{2}K_f^2 e^{-r_f(T-t)}\mathbb{E}\left[\sigma_S^2\middle| S_T = K_f\right]\mathbb{E}\left[V_T\delta\left(S_T - K_f\right)\right] \\
&= -r_f K_f\frac{\partial C_q^{(1)}}{\partial K_f} + \frac{1}{2}K_f^2\frac{\partial^2 C_q}{\partial K_f^2}\mathbb{E}\left[\sigma_S^2\middle| S_T = K_f\right],
\end{aligned}
$$

which follows that

$$
\frac{\partial C_q^{(1)}}{\partial T} + r_f K_f\frac{\partial C_q^{(1)}}{\partial K_f} - \frac{1}{2}K_f^2\frac{\partial^2 C_q^{(1)}}{\partial K_f^2}\mathbb{E}\left[\sigma_S^2\middle| S_T = K_f\right] = 0.
$$

Regarding $\sigma_S\left(S_t; K_f, T\right) = \sqrt{\mathbb{E}\left[\sigma_S^2\middle| S_T = K_f\right]}$, we get the desired result. □

As shown in [4], the option pricing formula in this type does not include the exchange rate risk. At any time, the option price is converted from foreign currency into domestic currency using the prevailing exchange rate. Accordingly, we could get the local volatility for the price of a European vanilla call option in foreign currency.

3.2. Type II

The foreign equity call struck in domestic currency has a payoff given by

$$\max\left(V_T S_T - K_d, 0\right), \tag{6}$$

where K_d is the domestic strike price.

Theorem 3.2. *Suppose that $C_q^{(2)}$ is the price of a European quanto call option at time t in domestic currency with the payoff* (6)*. Then the local volatility is expressed by*

$$\sigma_S\left(S_t; K_d, T\right) = -\rho\sigma_V \pm \sqrt{\left(\rho^2-1\right)\sigma_V^2 + \frac{2\left(\frac{\partial C_q^{(2)}}{\partial T} + r_d K_d \frac{\partial C_q^{(2)}}{\partial K_d}\right)}{K_d^2 \frac{\partial^2 C_q^{(2)}}{\partial K_d^2}}}.$$

Proof. We may write $C_q^{(2)}$ as

$$C_q^{(2)}\left(S_t; K_d, T\right) = \mathbb{E}\left[e^{-r_d(T-t)}\max\left(\hat{S}_T - K_d, 0\right)\right], \tag{7}$$

where $\hat{S}_T = V_T S_T$. We note that the risk-neutral dynamic of $\hat{S}_t$ in domestic currency is given by

$$d\hat{S}_t = r_d \hat{S}_t dt + \sigma_S \hat{S}_t dB_t^d + \sigma_V \hat{S}_t dW_t^d \tag{8}$$

from (2).

Differentiating (7) with respect to K_d, it gives

$$\frac{\partial C_q^{(2)}}{\partial K_d} = -\mathbb{E}\left[e^{-r_d(T-t)} H\left(\hat{S}_T - K_d\right)\right]. \tag{9}$$

Differentiating again (9) with respect to K_d, it gives

$$\frac{\partial^2 C_q^{(2)}}{\partial K_d^2} = \mathbb{E}\left[e^{-r_d(T-t)} \delta\left(\hat{S}_T - K_d\right)\right].$$

Also, differentiating (7) with respect to T, it gives

$$\frac{\partial C_q^{(2)}}{\partial T} = -r_d C_q^{(2)} + e^{-r_d(T-t)} \frac{\partial}{\partial T}\mathbb{E}\left[\max\left(\hat{S}_T - K_d, 0\right)\right].$$

Applying the Itô formula to this payoff, we have

$$d\max\left(\hat{S}_T - K_d, 0\right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \hat{S}_T} \max\left(\hat{S}_T - K_d, 0\right) d\hat{S}_T + \frac{1}{2} \frac{\partial^2}{\partial \hat{S}_T^2} \max\left(\hat{S}_T - K_d, 0\right) \left(d\hat{S}_T\right)^2 \\
&= H\left(\hat{S}_T - K_d\right) \left(r_d \hat{S}_T dT + \sigma_S \hat{S}_T dB_T^d + \sigma_V \hat{S}_T dW_T^d\right) \\
&\quad + \frac{1}{2} \delta\left(\hat{S}_T - K_d\right) \left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V\right) \hat{S}_T^2 dT
\end{aligned}
$$

from (8).

Now, taking the expectation on both sides, it follows that

$$
\begin{aligned}
&d\mathbb{E}\left[\max\left(\hat{S}_T - K_d, 0\right)\right] \\
&= r_d \mathbb{E}\left[\hat{S}_T H\left(\hat{S}_T - K_d\right)\right] dT + \frac{1}{2}\mathbb{E}\left[\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V\right) \hat{S}_T^2 \delta\left(S_T - K_d\right)\right] dT \\
&= r_d \mathbb{E}\left[\left(\hat{S}_T - K_d\right) H\left(\hat{S}_T - K_d\right)\right] dT + r_d K_d \mathbb{E}\left[H\left(\hat{S}_T - K_d\right)\right] dT \\
&\quad + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right] dT + \frac{1}{2}\sigma_V^2 \mathbb{E}\left[\hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right] dT \\
&\quad + \rho\sigma_V \mathbb{E}\left[\sigma_S \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right] dT,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\frac{\partial}{\partial T}\mathbb{E}\left[\max\left(\hat{S}_T - K_d, 0\right)\right] \\
&= r_d \mathbb{E}\left[\max\left(\hat{S}_T - K_d, 0\right)\right] + r_d K \mathbb{E}\left[H\left(\hat{S}_T - K\right)\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\sigma_S^2 \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right] + \frac{1}{2}\sigma_V^2 \mathbb{E}\left[\hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right] \\
&\quad + \rho\sigma_V \mathbb{E}\left[\sigma_S \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right)\right].
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\frac{\partial C_q^{(2)}}{\partial T} &= -r_d C_q^{(2)} + r_d C_q^{(2)} - r_d K_d \frac{\partial C_q^{(2)}}{\partial K_d} \\
&\quad + \frac{1}{2} e^{-r_d(T-t)} \mathbb{E}\left[\mathbb{E}\left[\sigma_S^2 \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right) \middle| \hat{S}_T = K_d\right]\right] \\
&\quad + \frac{1}{2}\sigma_V^2 e^{-r_d(T-t)} \mathbb{E}\left[\mathbb{E}\left[\hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right) \middle| \hat{S}_T = K_d\right]\right] \\
&\quad + \rho\sigma_V^2 e^{-r_d(T-t)} \mathbb{E}\left[\mathbb{E}\left[\sigma_S \hat{S}_T^2 \delta\left(\hat{S}_T - K_d\right) \middle| \hat{S}_T = K_d\right]\right] \\
&= -r_d K \frac{\partial C_q}{\partial K_d} + \frac{1}{2} K_d^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S^2 \middle| \hat{S}_T = K_d\right] \mathbb{E}\left[\delta\left(\hat{S}_T - K_d\right)\right] \\
&\quad + \frac{1}{2}\sigma_V^2 K_d^2 e^{-r_d(T-t)} \mathbb{E}\left[\delta\left(\hat{S}_T - K_d\right)\right] \\
&\quad + \rho\sigma_V K_d^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S \middle| \hat{S}_T = K_d\right] \mathbb{E}\left[\delta\left(\hat{S}_T - K_d\right)\right]
\end{aligned}
$$

$$= -r_d K_d \frac{\partial C_q^{(2)}}{\partial K_d} + \frac{1}{2} K_d^2 \frac{\partial^2 C_q^{(2)}}{\partial K_d^2} \mathbb{E}\left[\sigma_S^2 \middle| \hat{S}_T = K_d \right] + \frac{1}{2} \sigma_V^2 K_d^2 \frac{\partial^2 C_q^{(2)}}{\partial K_d^2}$$
$$+ \rho \sigma_V K_d^2 \frac{\partial^2 C_q^{(2)}}{\partial K_d^2} \mathbb{E}\left[\sigma_S \middle| \hat{S}_T = K_d \right],$$

which follows that

$$\frac{\partial C_q^{(2)}}{\partial T} + r_d K_d \frac{\partial C_q^{(2)}}{\partial K_d}$$
$$- \frac{1}{2} K_d^2 \frac{\partial^2 C_q^{(2)}}{\partial K_d^2} \left(\mathbb{E}\left[\sigma_S^2 \middle| \hat{S}_T = K_d \right] + \sigma_V^2 + 2\rho\sigma_V \mathbb{E}\left[\sigma_S \middle| \hat{S}_T = K_d \right] \right) = 0.$$

Regarding $\sigma_S (S_t; K_d, T) = \mathbb{E}\left[\sigma_S \middle| \hat{S}_T = K_d \right] = \sqrt{\mathbb{E}\left[\sigma_S^2 \middle| \hat{S}_T = K_d \right]}$, we get the desired result. □

3.3. Type III

The fixed exchange rate foreign equity call has a payoff given by

$$V_0 \max (S_T - K_f, 0), \tag{10}$$

where V_0 is the some predetermined fixed exchange rate and K_f is the foreign strike price.

Theorem 3.3. *Suppose that $C_q^{(3)}$ is the price of a European quanto call option at time t in domestic currency with the payoff* (10). *Then the local volatility is expressed by*

$$\sigma_S (S_t; K_f, T)$$
$$= \frac{\rho\sigma_V \left(C_q^{(3)} - K_f \frac{\partial C_q^{(3)}}{\partial K_f} \right) \pm \sqrt{\rho^2 \sigma_V^2 \left(C_q^{(3)} - K_f \frac{\partial C_q^{(3)}}{\partial K_f} \right)^2 + 2K_f^2 \frac{\partial^2 C_q^{(3)}}{\partial K^2} \left\{ \frac{\partial C_q^{(3)}}{\partial T} + r_f K \frac{\partial C_q^{(3)}}{\partial K_f} - (r_f - r_d) C_q^{(3)} \right\}}}{K_f^2 \frac{\partial^2 C_q^{(3)}}{\partial K_f^2}}.$$

Proof. The proof is identical to that of Theorem 3.1 in [5]. □

3.4. Type IV

The equity-linked foreign exchange call has a payoff given by

$$S_T \max (V_T - K_e, 0), \tag{11}$$

where K_e is the strike price on the exchange rate.

Theorem 3.4. *Suppose that $C_q^{(4)}$ is the price of a European quanto call option at time t in domestic currency with the payoff* (11). *Then the local volatility is expressed by*

$$\sigma_S (S_t; K_e, T) = \frac{r_f - r_d}{\rho\sigma_V} - \frac{\frac{\partial C_q^{(4)}}{\partial T} - \frac{1}{2}\sigma_V^2 K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2}}{\rho \sigma_V K_e \frac{\partial C_q^{(4)}}{\partial K_e}}.$$

Proof. We may write $C_q^{(4)}$ as

$$C_q^{(4)}\left(S_t; K_e, T\right) = \mathbb{E}\left[e^{-r_d(T-t)} \max\left(\hat{S}_T - K_e S_T, 0\right)\right], \tag{12}$$

where $\hat{S}_T = V_T S_T$. It may be rewritten the payoff (11) as an exchange option with $\hat{S}_t$ and $K_e S_t$ in domestic currency.

Differentiating (12) with respect to K_e, it gives

$$\frac{\partial C_q^{(4)}}{\partial K_e} = -\mathbb{E}\left[e^{-r_d(T-t)} S_T H\left(\hat{S}_T - K_e S_T\right)\right]. \tag{13}$$

Differentiating again (13) with respect to K_e, it gives

$$\frac{\partial^2 C_q^{(4)}}{\partial K_e^2} = \mathbb{E}\left[e^{-r_d(T-t)} S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right].$$

Also, differentiating (12) with respect to T, it gives

$$\frac{\partial C_q^{(4)}}{\partial T} = -r_d C_q^{(4)} + e^{-r_d(T-t)} \frac{\partial}{\partial T} \mathbb{E}\left[\max\left(\hat{S}_T - K_e S_T, 0\right)\right].$$

Applying the Itô formula to this payoff, we have

$$\begin{aligned}
&d \max\left(\hat{S}_T - K_e S_T, 0\right) \\
&= \frac{\partial}{\partial \hat{S}_T} \max\left(\hat{S}_T - K_e S_T, 0\right) d\hat{S}_T + \frac{1}{2} \frac{\partial^2}{\partial \hat{S}_T^2} \max\left(\hat{S}_T - K_e S_T, 0\right)\left(d\hat{S}_T\right)^2 \\
&\quad + \frac{\partial}{\partial S_T} \max\left(\hat{S}_T - K_e S_T, 0\right) dS_T + \frac{1}{2} \frac{\partial^2}{\partial S_T^2} \max\left(\hat{S}_T - K_e S_T, 0\right)\left(dS_T\right)^2 \\
&\quad + \frac{\partial^2}{\partial \hat{S}_T \partial S_T} \max\left(\hat{S}_T - K_e S_T, 0\right) d\hat{S}_T dS_T \\
&= H\left(\hat{S}_T - K_e S_T\right)\left(r_d \hat{S}_T dT + \sigma_S \hat{S}_T dB_T^d + \sigma_V \hat{S}_T dW_T^d\right) \\
&\quad + \frac{1}{2} \delta\left(\hat{S}_T - K_e S_T\right)\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V\right) \hat{S}_T^2 dT \\
&\quad - K_e H\left(\hat{S}_T - K_e S_T\right)\left\{\left(r_f - \rho\sigma_S\sigma_V\right) S_T dT + \sigma_S S_T dB_T^d\right\} \\
&\quad + \frac{1}{2} K_e^2 \delta\left(\hat{S}_T - K_e S_T\right) \sigma_S^2 S_T^2 dT \\
&\quad - K_e \delta\left(\hat{S}_T - K_e S_T\right)\left(\sigma_S^2 + \rho\sigma_S\sigma_V\right) \hat{S}_T S_T dT
\end{aligned}$$

from (2) and (8).

Now, taking the expectation on both sides, it follows that

$$\begin{aligned}
&d\mathbb{E}\left[\max\left(\hat{S}_T - K_e S_T, 0\right)\right] \\
&= r_d \mathbb{E}\left[\left(\hat{S}_T - K_e S_T\right) H\left(\hat{S}_T - K_e S_T\right)\right] dT + r_d K_e \mathbb{E}\left[S_T H\left(\hat{S}_T - K_e S_T\right)\right] dT
\end{aligned}$$

$$+\frac{1}{2}\mathbb{E}\left[\sigma_S^2\hat{S}_T^2\delta\left(S_T-K_eS_T\right)\right]dT+\frac{1}{2}\sigma_V^2\mathbb{E}\left[\hat{S}_T^2\delta\left(S_T-K_eS_T\right)\right]dT$$
$$+\rho\sigma_V\mathbb{E}\left[\sigma_S\hat{S}_T^2\delta\left(S_T-K_eS_T\right)\right]dT-r_fK_e\mathbb{E}\left[S_TH\left(\hat{S}_T-K_eS_T\right)\right]dT$$
$$+\rho\sigma_VK_e\mathbb{E}\left[\sigma_SS_TH\left(\hat{S}_T-K_eS_T\right)\right]dT+\frac{1}{2}K_e^2\mathbb{E}\left[\sigma_S^2S_T^2\delta\left(\hat{S}_T-K_eS_T\right)\right]dT$$
$$-K_e\mathbb{E}\left[\sigma_S^2\hat{S}_TS_T\delta\left(\hat{S}_T-K_eS_T\right)\right]dT-\rho\sigma_VK_e\mathbb{E}\left[\sigma_S\hat{S}_TS_T\delta\left(\hat{S}_T-K_eS_T\right)\right]dT,$$

and hence,

$$\frac{\partial}{\partial T}\mathbb{E}\left[\max\left(\hat{S}_T-K_eS_T,0\right)\right]$$
$$=r_d\mathbb{E}\left[\max\left(\hat{S}_T-K_eS_T,0\right)\right]+r_dK_e\mathbb{E}\left[H\left(\hat{S}_T-K_eS_T\right)\right]$$
$$+\frac{1}{2}\mathbb{E}\left[\sigma_S^2\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\right]+\frac{1}{2}\sigma_V^2\mathbb{E}\left[\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\right]$$
$$+\rho\sigma_V\mathbb{E}\left[\sigma_S\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\right]-r_fK_e\mathbb{E}\left[S_TH\left(\hat{S}_T-K_eS_T\right)\right]$$
$$+\rho\sigma_VK_e\mathbb{E}\left[\sigma_SS_TH\left(\hat{S}_T-K_eS_T\right)\right]+\frac{1}{2}K_e^2\mathbb{E}\left[\sigma_S^2S_T^2\delta\left(\hat{S}_T-K_eS_T\right)\right]$$
$$-K_e\mathbb{E}\left[\sigma_S^2\hat{S}_TS_T\delta\left(\hat{S}_T-K_eS_T\right)\right]-\rho\sigma_VK_e\mathbb{E}\left[\sigma_S\hat{S}_TS_T\delta\left(\hat{S}_T-K_eS_T\right)\right].$$

Finally, we obtain

$$\begin{aligned}\frac{\partial C_q^{(4)}}{\partial T}=&-r_dC_q^{(4)}+r_dC_q^{(4)}-r_dK_e\frac{\partial C_q^{(4)}}{\partial K_e}\\&+\frac{1}{2}e^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_S^2\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\\&+\frac{1}{2}\sigma_V^2e^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\\&+\rho\sigma_Ve^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_S\hat{S}_T^2\delta\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\\&+r_fK_e\frac{\partial C_q^{(4)}}{\partial K_e}\\&+\rho\sigma_VK_ee^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_SS_TH\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\\&+\frac{1}{2}K_e^2e^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_S^2S_T^2\delta\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\\&-K_ee^{-r_d(T-t)}\mathbb{E}\left[\mathbb{E}\left[\sigma_S^2\hat{S}_TS_T\delta\left(\hat{S}_T-K_eS_T\right)\middle|\frac{\hat{S}_T}{S_T}=K_e\right]\right]\end{aligned}$$

$$
\begin{aligned}
&- \rho\sigma_V K_e e^{-r_d(T-t)} \mathbb{E}\left[\mathbb{E}\left[\sigma_S \hat{S}_T S_T \delta\left(\hat{S}_T - K_e S_T\right) \middle| \frac{\hat{S}_T}{S_T} = K_e\right]\right] \\
= \; &- r_d K_e \frac{\partial C_q^{(4)}}{\partial K_e} \\
&+ \frac{1}{2} K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
&+ \frac{1}{2} \sigma_V^2 K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
&+ \rho\sigma_V K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
&+ r_f K_e \frac{\partial C_q^{(4)}}{\partial K_e} \\
&+ \rho\sigma_V K_e e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T H\left(\hat{S}_T - K_e S_T\right)\right] \\
&+ \frac{1}{2} K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
&- K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
&- \rho\sigma_V K_e^2 e^{-r_d(T-t)} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \mathbb{E}\left[S_T^2 \delta\left(\hat{S}_T - K_e S_T\right)\right] \\
= \; &- r_d K_e \frac{\partial C_q^{(4)}}{\partial K_e} + \frac{1}{2} K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] + \frac{1}{2} \sigma_V^2 K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \\
&+ \rho\sigma_V K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] + r_f K_e \frac{\partial C_q^{(4)}}{\partial K_e} \\
&- \rho\sigma_V K_e \frac{\partial C_q^{(4)}}{\partial K_e} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \\
&+ \frac{1}{2} K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] - K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \mathbb{E}\left[\sigma_S^2 \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \\
&- \rho\sigma_V K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} \mathbb{E}\left[\sigma_S \middle| \frac{\hat{S}_T}{S_T} = K_e\right] \\
= \; &- r_d K_e \frac{\partial C_q^{(4)}}{\partial K_e} + \frac{1}{2} \sigma_V^2 K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} + r_f K_e \frac{\partial C_q^{(4)}}{\partial K_e}
\end{aligned}
$$

$$- \rho\sigma_V K_e \frac{\partial C_q^{(4)}}{\partial K_e} \mathbb{E}\left[\sigma_S \left| \frac{\hat{S}_T}{S_T} = K_e \right.\right],$$

which follows that

$$\frac{\partial C_q^{(4)}}{\partial T} + \left(r_d - r_f + \rho\sigma_V \mathbb{E}\left[\sigma_S \left| \frac{\hat{S}_T}{S_T} = K_e \right.\right]\right) K_e \frac{\partial C_q^{(4)}}{\partial K_e} - \frac{1}{2}\sigma_V^2 K_e^2 \frac{\partial^2 C_q^{(4)}}{\partial K_e^2} = 0.$$

Regarding $\sigma_S\left(S_t; K_e, T\right) = \mathbb{E}\left[\sigma_S \left| \frac{\hat{S}_T}{S_T} = K_e \right.\right]$, we get the desired result. □

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