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THE EXISTENCE OF S-ASYMPTOTICALLY ω-PERIODIC MILD SOLUTIONS FOR SOME DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. We study the existence and uniqueness of S-asymptotically ω-periodic mild solutions for some partial functional integrodifferential equations with infinite delay and nonlocal conditions.

1. Introduction

The aim of this paper is to study the existence of S-asymptotically ω-periodic mild solutions of the following partial functional integrodifferential equation with infinite delay and nonlocal conditions,

$$(1)\qquad \begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u_t),\ t \geq 0,\\ u_0 &= g(u) + \phi, \quad \phi \in \mathfrak{B}, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup on a Banach space X. For $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(A) \subset D(B(t))$. The history u_t is given by $u_t : (-\infty, 0] \to X, u_t(\theta) = u(t+\theta), \theta \in (-\infty, 0]$. f and g satisfy the hypotheses (H1)-(H3). Here (H1)-(H3) and g defined by $g(t_1, \ldots, t_p, u_{t_1}, \ldots, u_{t_p}) = \Sigma_{i=1}^p c_i u_{t_i}$ will be introduced later.

Caicedo et al. [4] studied the existence of S-asymptotically ω-periodic mild solutions for a class of the equation

$$(2)\qquad \begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + \hat{f}(t,u(t)),\ t \geq 0,\\ u(0) &= u_0 \in X, \end{aligned}$$

where $\hat{f}$ is a suitable function for any given conditions.

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Caicedo and Cuevas [3] investigated the existence and uniqueness of S-asymptotically ω-periodic mild solutions to the abstract partial integrodifferential equation with infinite delay

$$\frac{d}{dt}D(t,u_t) = AD(t,u_t) + \int_0^t B(t-s)D(s,u_s)ds + f(t,u_t),\ t \geq 0,$$
$$u_0 = \phi \in \mathfrak{B},$$

where $D(t,\phi) = \phi(0) + f(t,\phi)$ and u_t belongs to an abstract phase space $\mathfrak{B}$ by Hale and Kato [11].

Eq. (2) has received much attention in recent years. Properties of the solution of Eq. (2) have been studied from different points of view. Eq. (2) with a nonlocal initial condition has been of great interest for many researchers. The existence of mild solutions for various forms of Eq. (2) on a finite interval has been investigated in many papers. The existence of almost periodic, asymptotically almost periodic, almost automorphic, pseudo almost periodic and pseudo almost automorhic solutions to differential equations is among the most attractive topics in mathematical analysis due to their possible applications in areas such as physics, economics, mathematical biology, engineering, etc [9].

Hui-Sheng Ding et al. [10] investigated the existence of asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)),\ t \geq 0,$$
$$u(0) = u_0 + g(u).$$

Consider the following partial functional integrodifferential equation with delay

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u_t),\ t \geq 0, \tag{3}$$
$$u_0 = \phi,$$

where ϕ is delay.

Recently, Choi et al. [5] investigated the existence and uniqueness of an S-asymptotically ω-periodic mild solution to Eq. (3).

Stimulated by the above study, additionally with nonlocal initial conditions and infinite delay, we prove the existence and uniqueness of an S-asymptotically ω-periodic mild solution to Eq. (1). The Cauchy problems with nonlocal conditions have been investigated in many papers ([1, 2, 10, 15]).

It is demonstrated there that nonlocal Cauchy problems have better effects in applications than the traditional Cauchy problems with initial datum $u(t_0) = u_{t_0}$. In many studies of nonlocal Cauchy problems, the function g is given by $g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p))$. For example ([1, 2]),

$$g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p)) = \Sigma_{i=1}^p c_i u(t_i)$$

(for some given constants c_i) is used to describe the diffusion phenomenon of a small amount of a gas in a transparent tube. In these cases, $u(t_0) + g(t_1, \dots t_p, u(t_1), \dots, u(t_p))$ allows the measurement at $t = t_0, t_1, \dots, t_p$ rather than just such a g is completely determinded on $[t_0 + \delta, t_0 + T]$ for some small $\delta > 0$, i.e., such a g ignores $t = t_0$.

2. Preliminaries

To define S-asymptotically ω-periodic functions, it is very convenient to introduce the following notations: For a Banach space X,

$$\begin{aligned} C(\mathbb{R}_+, X) &= \{x : \mathbb{R}_+ \to X : x \text{ is continuous}\}, \\ C_b(\mathbb{R}_+, X) &= \{x \in C(\mathbb{R}_+, X) : \sup_{t \geq 0} ||x(t)|| < \infty\}, \\ C_0(\mathbb{R}_+, X) &= \{x \in C_b(\mathbb{R}_+, X) : \lim_{t \to \infty} ||x(t)|| = 0\}, \\ C_\omega(\mathbb{R}_+, X) &= \{x \in C_b(\mathbb{R}_+, X) : x \text{ is } \omega\text{-periodic}\}, \end{aligned}$$

endowed with the norm of the uniform convergence.

We consider Eq. (2). In the following Y denotes the Banach space $D(A)$ equipped with the graph norm defined by $||y||_Y = ||Ay|| + ||y||,\ y \in Y$.

Definition 2.1 ([8]). A family $\{R(t) : t \geq 0\}$ of continuous linear operators on X is called a resolvent operator for Eq. (1) if the following conditions are fulfilled.

(1) For each $x \in X$, $R(0)x = x$ and $R(\cdot)x \in C(\mathbb{R}_+, X)$.
(2) The map $R : \mathbb{R}_+ \to \mathcal{L}(D(A))$ is strongly continuous.
(3) For each $y \in D(A)$, the function $t \to R(t)y$ is continuously differentiable and

$$\frac{d}{dt} R(t)y = AR(t)y + \int_0^t B(t-s)R(s)yds = R(t)Ay + \int_0^t R(t-s)B(s)yds, t \geq 0.$$

We assume that there exists a unique resolvent operator for Eq. (2).

In this system, we employ the axiomatic definition of the phase space $\mathfrak{B}$ which plays a fundamental role in the study of qualitative theory of such equations. Specifically, $\mathfrak{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $|| \cdot ||_{\mathfrak{B}}$ and verifying the following axioms.

(HA) $x : (-\infty, \sigma + a) \to X,\ a > 0,\ \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathfrak{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

(1) $x_t \in \mathfrak{B}$;
(2) $||x(t)|| \leq H||x_t||_{\mathfrak{B}}$;
(3) $||x_t||_{\mathfrak{B}} \leq K(t - \sigma) \sup\{||x(s)|| | \sigma \leq s \leq t\} + M(t - \sigma)||x_\sigma||_{\mathfrak{B}}$, where $H > 0$ is a constant, $K,\ M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A_1) For the function $x(\cdot)$ in (HA), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

(B) The space $\mathfrak{B}$ is complete.

(C_2) If $(\psi^n)_{n\in N}$ is a uniformly bounded sequence of continuous functions with compact support and $\psi^n \to \psi$ for $n \to \infty$ in the compact-open topology, then $\psi \in \mathfrak{B}$ and $||\psi^n - \psi||_{\mathfrak{B}} \to 0$ as $n \to \infty$.

We introduce the space $\mathfrak{B}_0 = \{\psi \in \mathfrak{B} | \psi(0) = 0\}$ and the operator $S(t) : \mathfrak{B} \to \mathfrak{B}$ given by

$$[S(t)\psi](\theta) = \begin{cases} T(t+\theta)\psi(0), & -t \le \theta \le 0, \\ \psi(t+\theta), & -\infty < \theta \le -t. \end{cases}$$

It is well known that $(S(t))_{t\le 0}$ is a C_0-semigroup [14]. The phase space $\mathfrak{B}$ is said to be a *fading memory space* if $||S(t)\psi||_{\mathfrak{B}} \to 0$ as $t \to \infty$ for every $\psi \in \mathfrak{B}_0$.

Remark. Since $\mathfrak{B}$ verifies axiom (C_2), the space $C_b((-\infty, 0], X)$ consisting of continuous and bounded functions $\psi : (-\infty, 0] \to X$ is continuously included in $\mathfrak{B}$. Thus there exists a constant $Q \ge 0$ such that $||\psi||_{\mathfrak{B}} \le Q||\psi||_\infty$ for every $\psi \in C_b((-\infty, 0], X)$ [14]. Moreover, $\mathfrak{B}$ is a fading memory space, then K, M are bounded functions (see [13, Proposition 7.1.5]).

Additionally, the following hypothesis needs.

(H) There are positive constants M, μ such that $||R(t)|| \le Me^{-\mu t}$ for all $t \ge 0$.

We introduce some definitions well known from our references.

Definition 2.2. A function $f \in C(\mathbb{R}, X)$ is called *almost periodic* if for every $\epsilon > 0$, if there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathbb{H}(\epsilon, f)$, such that

$$||f(t+\tau) - f(t)|| < \epsilon \quad \text{for every } t \in \mathbb{R}, \text{ and all } \tau \in \mathbb{H}(\epsilon, f).$$

Many authors have furthermore generalized the notion of almost periodicity in different directions.

Definition 2.3. A function $f \in C(\mathbb{R}_+, X)$ is said to be *S-asymptotically ω-periodic* if there exists an $\omega > 0$ such that

$$\lim_{t\to\infty} ||f(t+\omega) - f(t)|| = 0.$$

In this case, we say that ω is an asymptotic period of f and f is S-asymptotically ω-periodic. Denote by $SAP_\omega(X)$ the set of such functions. It is clear that $SAP_\omega(X)$ is a Banach space.

Definition 2.4. A function $f \in C_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* if there exist $g \in AP(\mathbb{R}, X)$ and $\phi \in C_0(\mathbb{R}_+, X)$ such that $f = g + \phi$. Also, f is said to be a *asymptotically ω-periodic* when $g \in C_\omega(\mathbb{R}_+, X)$.

Let W be a Banach space.

Definition 2.5. A function $F \in C(\mathbb{R}_+ \times W, X)$ is called *uniformly S-asymptotically ω-periodic* on bounded sets if $F(\cdot, x)$ are bounded for each $x \in W$, and for every $\epsilon > 0$ and all bounded set $K \subset W$ there exists $L_{K,\epsilon} \ge 0$ such that

$$||F(t+\omega, x) - F(t, x)||_W \le \epsilon, t \le L_{K,\epsilon}, x \in K.$$

Definition 2.6. A function $F \in C(\mathbb{R}_+ \times W, X)$ is called *asymptotically uniformly continuous* on bounded sets if for every $\epsilon > 0$ and all bounded set $K \subset W$ there exist constants $L_{K,\epsilon} \geq 0$ and $\delta_{K,\epsilon} > 0$ such that

$$||F(t,x) - F(t,y)||_W \leq \epsilon, t \geq L_{K,\epsilon},$$

when $||x - y||_W \leq \delta_{K,\epsilon}, x, y \in K$.

3. Existence results for S-asymptotically ω-periodic mild solution

First, we introduce some Lemmas and Theorems.

Lemma 3.1 ([12])**.** *Assume that $f \in C(\mathbb{R}_+ \times W, X)$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u \in SAP_\omega(W)$, then the function $t \to f(t, u(t))$ belongs to $SAP_\omega(X)$.*

Lemma 3.2 ([4])**.** *If $u \in SAP_\omega(W)$, then $v(t) = \int_0^t R(t-s)u(s)ds$ belongs to $SAP_\omega(X)$.*

Definition 3.3. A function $u \in C(\mathbb{R}_+, X)$ is called a *mild solution* of Eq. (2) if u satisfies

$$\begin{aligned} u(t) &= R(t)u_0 + \int_0^t R(t-s)\hat{f}(s, u(s))ds,\ t \geq 0, \\ u(0) &= u_0 \in X. \end{aligned}$$

Theorem 3.4 ([4])**.** *Assume that $\hat{f} : \mathbb{R}_+ \times X \to X$ is uniformly S-asymptotically ω-periodic on bounded sets. Also, f satisfies the Lipschitz condition*

$$||\hat{f}(t,x) - \hat{f}(t,y)|| \leq L||x - y|| \quad \textit{for all}\, x, y \in X, t \geq 0.$$

If $\frac{LM}{\mu} < 1$, then the Eq. (2) *has a unique S-asymptotically ω-periodic mild solution.*

The following lemma is immediately obtained from our definitions.

Lemma 3.5 ([4])**.** *Let $u : [-r, \infty) \to X$ be a continuous function. If $u|_{\mathbb{R}_+} \in SAP_\omega(X)$, then the function $t \to u_t$ belongs to $SAP_\omega(X)$.*

Definition 3.6. A function $u \in C([-r, \infty); X)$ is called a *mild solution* of Eq. (3) if u satisfies

$$\begin{aligned} u(t) &= R(t)\phi(0) + \int_0^t R(t-s)f(s, u_s)ds,\ t \geq 0, \\ u_0 &= \phi. \end{aligned}$$

The following result is an immediate consequence of Theorem 3.4 and Lemma 3.5.

Theorem 3.7 ([4])**.** *Assume that $f : \mathbb{R}_+ \times C([-r,0],X) \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Also, f satisfies the Lipschitz condition*:

$$||f(t,\phi_1) - f(t,\phi_2)|| \le L||\phi_1 - \phi_2||_\infty \text{ for all } \phi_1, \phi_2 \in C([-r,0],X), t \ge 0.$$

If $u \in SAP_\omega(X)$ and $\frac{LM}{\mu} < 1$, then Eq. (3) *has a unique S-asymptotically ω-periodic mild solution.*

For a fading memory space, the following property holds.

Lemma 3.8 ([12])**.** *Assume that $\mathfrak{B}$ is a fading memory space. Let $u : \mathbb{R} \to X$ be a continuous function with $u_0 \in \mathfrak{B}$ and $u|_{\mathbb{R}_+} \in SAP_\omega(X)$. Then the function $t \to u_t$ belongs to $SAP_\omega(\mathfrak{B})$.*

Next, we assume that $\phi \in \mathfrak{B}$ and $f : \mathbb{R}_+ \times \mathfrak{B} \to X$ is a continuous function.

Definition 3.9. A function $u \in C(\mathbb{R}, X)$ is called a *mild solution* of Eq. (3) if u satisfies

$$\begin{aligned} u(t) &= R(t)\phi(0) + \int_0^t R(t-s)f(s,u_s)ds,\ t \ge 0, \\ u_0 &= \phi \in \mathfrak{B}. \end{aligned}$$

The following result was obtained by Caicedo et al. [4] without a nonlocal initial condition. For completeness, we give a proof in detail by using Theorem 3.4 and Lemma 3.8.

Theorem 3.10. *Assume that $f : \mathbb{R}_+ \times \mathfrak{B} \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets and also, satisfies the Lipschitz condition*:

$$||f(t,\phi_1) - f(t,\phi_2)|| \le L||\phi_1 - \phi_2||_{\mathfrak{B}}.$$

If $u \in SAP_\omega(X)$ and $\frac{LMQ}{\mu} < 1$, then Eq. (3) *has a unique S-asymptotically ω-periodic mild solution.*

Proof. Define the operator $\Gamma : SAP_\omega(X) \to SAP_\omega(X)$ given by

$$\Gamma u(t) = R(t)\phi(0) + \int_0^t B(t-s)f(s,u_s)ds, t \ge 0.$$

Then

$$v(t) = \int_0^t R(t-s)f(s,u_s)ds$$

belongs to $SAP_\omega(X)$ by Lemma 3.8. By the similar calculation of the proof in Theorem 3.5, for $x, y \in SAP_\omega(X)$, we have

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\le \int_0^t B(t-s)[f(s,x_s) - f(s,y_s)]ds \\ &\le LM \int_0^t e^{-\mu(t-s)}||x_s - y_s||_{\mathfrak{B}} ds \end{aligned}$$

$$\leq LMQ \int_0^t e^{-\mu(t-s)} ||x-y||_\infty ds$$
$$\leq \frac{LMQ}{\mu} ||x-y||_\infty.$$

Therefore Γ is a contraction and there exists a unique fixed point $u \in SAP_\omega(X)$. This function u is an S-asymptotically ω-periodic mild solution of Eq. (3). $\square$

Finally, for additionally nonlocal condition, we are in a position to state and prove our main result which proves the existence of S-asymptotically ω-periodic mild solution with infinite delay and nonlocal initial condition.

Definition 3.11. A function $u \in C(\mathbb{R}, X)$ is called a *mild solution* of Eq. (1) if u satisfies

$$u(t) = R(t)[u_0 + g(u)] + \int_0^t R(t-s) f(s, u_s) ds, \ t \geq 0,$$
$$u_0 = g(u) + \phi, \quad \phi \in \mathfrak{B}.$$

For the proof of our main theorem, we need the following hypotheses:

(H_1) The function $f : \mathbb{R}_+ \times \mathfrak{B} \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and there exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $r \geq 0$ small enough and $||u||, ||v|| \leq r$,

$$||f(t,u) - f(t,v)|| \leq L_f(r) ||u-v||_{\mathfrak{B}}, \ (t,u), (t,v) \in \mathbb{R}_+ \times \mathfrak{B}.$$

(H_2) The function $g : C(\mathbb{R}_+, X) \to X$ satisfies that there exists a function $L_g : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $r \geq 0$

$$||g(u) - g(v)|| \leq L_g(r) ||u - v||, \quad ||u||, ||v|| \leq r.$$

(H_3) There exists a positive constant $\hat{K}$ such that

$$\hat{K} := \sup_{r>0} [\frac{\omega r}{MQ} - \omega r L_g(r) - r L_f(r)] > \omega(||u_0|| + ||g(0)||) + \sup_{s \in \mathbf{R}} ||f(s,0)||.$$

Theorem 3.12. *Assume that $f : \mathbb{R}_+ \times \mathfrak{B} \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Also, f satisfies the Lipschitz condition*:

$$||f(t, \phi_1) - f(t, \phi_2)|| \leq L ||\phi_1 - \phi_2||_{\mathfrak{B}} \quad \textit{for all } \phi_1, \phi_2 \in \mathfrak{B}, t \geq 0.$$

If $u \in SAP_\omega(X)$ and $\frac{LM}{\mu} < 1$, then Eq. (1) *has a unique S-asymptotically ω-periodic mild solution.*

Proof. By (H_3), there exists a constant $r > 0$ such that

$$\frac{\omega r}{MQ} - \omega r L_g(r) - r L_f(r) > \omega(||u_0|| + ||g(0)||) + \sup_{s \in \mathbf{R}} ||f(s,0)||.$$

Let $E = \{u \in SAP_\omega(X) : ||u|| \le r\}$, Then E is a closed subspace of $SAP_\omega(X)$. We define an operator Ψ on E by

$$(\Psi u)(t) = R(t)[u(0) + g(u)] + \int_0^t R(t-s)f(s,u_s)ds,\ t \ge 0.$$

From the same calculation of the proof in Theorem 3.7,

Let

$$F(t) = \int_0^t R(t-s)f(s,u_s)ds,$$

then $F(t)$ belongs to $SAP_\omega(X)$.

On the other hand, since $R(\cdot)$ is exponentially stable, $\lim_{t\to\infty} R(t)[u_0 + g(u)] = 0$.

Thus, $\Psi u \in SAP_\omega(X)$.

Next, for any given $u \in E$, let us show that $\Psi u \in E$. It suffices to prove that $||\Psi u|| \le r$. We can deduce as follows

$$\begin{aligned}
||(\Psi u)(t)|| &\le M||u_0 + g(u)|| + \int_0^t Me^{-\omega(t-s)}[||f(s,0)|| + ||f(s,u_s) - f(s,0)||]ds \\
&\le M[||u_0|| + ||g(0)|| + ||g(u) - g(0)||] + \frac{M}{\omega}[\sup_{s\in\mathbb{R}} ||f(s,0)|| + L_f(r)\cdot r] \\
&\le M[||u_0|| + ||g(0)|| + L_g(r)\cdot r] + \frac{M}{\omega}[\sup_{s\in\mathbb{R}} ||f(s,0)|| + L_f(r)\cdot r] \\
&< r
\end{aligned}$$

for all $t \ge 0$. Thus $||\Psi u|| \le r$.

From given condition, we know $\frac{\omega r}{MQ} - \omega r L_g(r) - rL_f(r) > 0$, i.e., $\frac{\omega r}{MQ} > \omega r L_g(r) + rL_f(r)$. Therefore $M \cdot L_g(r) + \frac{MQ}{\omega} L_f(r) < 1$.

For $u, v \in E$ and $t \ge 0$, we have

$$\begin{aligned}
||(\Psi u)(t) - (\Psi v)(t)|| &\le ||R(t)[g(u) - g(v)|| \\
&\quad + \int_0^t ||R(t-s)|| ||[f(s,u_s) - f(s,v_s)]|| ds \\
&\le M \cdot L_g(r)||u - v||_\infty \\
&\quad + \int_0^t Me^{-\omega(t-s)} L_f(r)||u_s - v_s||_{\mathfrak{B}} ds \\
&\le M \cdot L_g(r)||u - v||_\infty + \frac{MQ}{\omega} \cdot L_f(r)||u - v||_\infty \\
&= [M \cdot L_g(r) + \frac{MQ}{\omega} \cdot L_f(r)]||u - v||_\infty.
\end{aligned}$$

Hence,

$$||(\Psi u)(t) - (\Psi v)(t)|| \le [M \cdot L_g(r) + \frac{MQ}{\omega} \cdot L_f(r)]||u - v||_\infty.$$

By the given condition, Ψ is a contraction from E to E. So Ψ has a unique fixed point in E, which means there exists an S-asymptotically ω-periodic mild solution to Eq. (1). $\square$

With similar calculation of the proof in Theorem 3.12, we can also obtain the following result.

Consider the following system

$$
\begin{aligned}
&u(t) = R(t)[u_0 + g(u)] + \int_0^t R(t-s)f(s, u_s)ds, \ t \geq 0, \\
&u_0 = g(u) + \phi, \phi \in C([-r, 0], X).
\end{aligned} \tag{4}
$$

We define the mild solution of equation with finite delay and nonlocal conditions as follows.

Definition 3.13. A function $u \in C([-r, \infty), X)$ is called a *mild solution* of Eq. (4) if u satisfies

$$
\begin{aligned}
&u(t) = R(t)[u_0 + g(u)] + \int_0^t R(t-s)f(s, u_s)ds, \ t \geq 0, \\
&u_0 = g(u) + \phi.
\end{aligned}
$$

Corollary 3.14. *Assume that $f : \mathbb{R}_+ \times C([-r, 0], X) \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Also, f satisfies the Lipschitz condition*:

$$||f(t, \phi_1) - f(t, \phi_2)|| \leq L||\phi_1 - \phi_2||_\infty \quad \textit{for all } \phi_1, \phi_2 \in C([-r, 0], X), \ t \geq 0.$$

If $u \in SAP_\omega(X)$ and $\frac{LM}{\mu} < 1$, then Eq. (4) has a unique S-asymptotically ω-periodic mild solution.

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