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A HYBRID PROJECTION METHOD FOR COMMON ZERO OF MONOTONE OPERATORS IN HILBERT SPACES

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Abstract. The purpose of this paper is to introduce some strong convergence theorems for the problem of finding a common zero of a finite family of monotone operators and the problem of finding a common fixed point of a finite family of nonexpansive in Hilbert spaces by hybrid projection method.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We use the symbols $\rightharpoonup$ and $\rightarrow$ to denote the weak convergence and strong convergence, respectively.

Consider the problem

$$\text{find } x \in H \text{ such that } 0 \in A_i(x) \text{ for all } i = 1, 2, \ldots, N, \tag{1}$$

where H is a real Hilbert space, and $A_i : D(A_i) \subset H \longrightarrow 2^H$ are monotone operators. We denote the set of solution of this problem by

$$S = \{x \in H : \ 0 \in A_i(x), \ \forall i = 1, 2, \ldots, N\}.$$

One of the classical methods for solving equation $0 \in A(x)$ with A is a maximal monotone operator in Hilbert space H, is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_{r_n}^A(x_n) \text{ for all } n \in \mathbb{N}, \tag{2}$$

where $\{r_n\}$ is a sequence of positive real numbers and $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent of A. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (2) and others proved strong convergence theorems by imposing assumptions on A.

Note that, algorithm (2), can be rewritten as

$$x_{n+1} - x_n + r_n A(x_{n+1}) \ni 0 \text{ for all } n \in \mathbb{N}. \tag{3}$$

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This algorithm was first introduced by Martinet [7]. If $\psi:\ H \longrightarrow \mathbb{R}\cup\{\infty\}$ is a proper lower semicontinuous convex function, the algorithm reduces to

$$x_{n+1} = \text{argmin}_{y\in H}\left\{\psi(y) + \frac{1}{2c}\|x_n - y\|^2\right\} \text{ for all } n \in \mathbb{N}.$$

Moreover, Rockafellar [9] has given a more practical method which is an inexact variant of the method:

$$x_n + e_n \ni x_{n+1} + c_n A x_{n+1} \quad \text{for all } n \in \mathbb{N}, \tag{4}$$

where $\{e_n\}$ is regarded as an error sequence and $\{c_n\}$ is a sequence of positive regularization parameters. Note that the algorithm (4) can be rewritten as

$$x_{n+1} = J^A_{r_n}(x_n + e_n) \quad \text{for all } n \in \mathbb{N}, \tag{5}$$

This method is called inexact proximal point algorithm. It was shown in Rockafellar [9] that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

Further, Rockafellar [9] posed an open question of whether the sequence generated by (2) converges strongly or not. In 1991, Güler [4] gave an example showing that Rockafellar's proximal point algorithm does not converges strongly. An example of the authors Bauschke, Matoušková and Reich [2] also showed that the proximal algorithm only converges weakly but not in norm.

In 2000, Solodov and Svaiter [10] proposed the following algorithm: Choose any $x_0 \in H$ and $\sigma \in [0, 1)$. At iteration n, having x_n, choose $\mu_n > 0$ and find (y_n, v_n) an inexact solution of

$$0 \in A(x) + \mu_n(x - x_n),$$

with tolerance σ. Define the sequence $\{x_n\}$ by

$$\begin{aligned} &C_n = \{z \in H :\ \langle z - y_n, v_n\rangle \le 0\},\\ &Q_n = \{z \in H :\ \langle z - x_n, x_0 - x_n\rangle \le 0\},\\ &x_{n+1} = P_{C_n\cap Q_n}x_0. \end{aligned}$$

They prove that if the sequence of the regularization parameters $\mu_n \ge c > 0$, then $\{x_n\}$ converges strongly to $x^* \in A^{-1}0$.

To find a fixed point of a nonexpansive mapping T on the closed and convex subset C of H, that is, find an element $p \in F(T) = \{x \in C :\ Tx = x\}$, Nakajo and Takahashi [8] also considered the sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$\begin{aligned} &y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\ &C_n = \{z \in C :\ \|z - y_n\| \le \|z - x_n\|\},\\ &Q_n = \{z \in C :\ \langle z - x_n, x_0 - x_n\rangle \le 0\},\\ &x_{n+1} = P_{C_n\cap Q_n}x_0,\ n \ge 0, \end{aligned} \tag{6}$$

where $\{\alpha_n\} \subset [0, a]$, with $a \in [0, 1)$. They proved that the sequence $\{x_n\}$ generated by (6) converges strongly to $P_{F(T)}x_0$.

Further, some generalized hybrid projection methods have been introduced for γ-strictly pseudocontractive mapping T (see, [5], [12], ...) or families of hemi-relatively and weak relatively nonexpansive mappings (see, [6], [11], ...).

In this paper, base on hybrid projection method, we introduce some new iterative methods to find a common zero of a finite family of monotone operators or a common fixed point of a finite family of nonexpansive mappings in a real Hilbert space. The results in this paper are the extension of the results of Solodov and Svaiter in [10], Nakajo and Takahashi [8].

2. Preliminaries

Let C be a nonempty, closed and convex subset of H. We know that for each $x \in H$, there is unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\|, \tag{7}$$

and the mapping $P_C: H \longrightarrow C$ define by (7) is called metric projection from H onto C. Moreover, we have

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall x \in H, \ y \in C. \tag{8}$$

Recall that, a mapping $T: C \longrightarrow C$ is said to be nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of fixed point of T by $F(T)$, i.e., $F(T) = \{x \in C: Tx = x\}$.

For an operator $A: H \longrightarrow 2^H$, we define its domain, range and graph as follows:

$$D(A) = \{x \in H: Ax \neq \emptyset\},$$
$$R(A) = \cup\{Az: z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in H \times H: x \in D(A), \ y \in Ax\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y, \text{ if and only if } y \in Ax.$$

The operator A is said to be monotone if, for each $x, y \in D(A)$, we have $\langle u - v, x - y\rangle \geq 0$ for all $u \in Ax$ and $v \in Ay$. We denote by I the identity operator on H. A monotone operator A is said to be maximal monotone if there is no proper monotone extension of A or $R(I + \lambda A) = H$ for all $\lambda > 0$. If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A: R(I + \lambda A) \longrightarrow D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1},$$

it is called the resolvent of A.

A monotone operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. We know that for a monotone operator A which satisfies the range condition, $A^{-1}0 = F(J_\lambda^A)$ for all $\lambda > 0$.

Remark 2.1. If A is a maximal monotone operator, then A satisfies the range condition.

The following lemmas will be needed in the sequel for the proof of main results in this paper.

Lemma 2.2 ([5]). *Let H be a real Hilbert space. For all $x, y \in H$ and $t \in [0,1]$, we have*

$$\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2.$$

Lemma 2.3. *Let H be a real Hilbert space and let $\{x_n\}$ be a sequence in H. Then, we have the following statements*:

i) *If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$.*
ii) *If $x_n \rightharpoonup x$ as $n \to \infty$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

Proof. i) We have

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x, x_n\rangle + \|x\|^2 \to 0 \quad \text{as } n \to \infty,$$

thus $x_n \to x$ as $n \to \infty$.

ii) We have

$$\|x_n\| \cdot \|x\| \geq \langle x_n, x\rangle \to \|x\|^2 \quad \text{as } n \to \infty$$

which implies that $\|x\| \leq \|x_n\|$, when n large enough. So, $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$.

This completes the proof. $\square$

Lemma 2.4 ([1]). *Let $A: D(A) \longrightarrow 2^H$ be a monotone operator. Then $\lambda,\ \mu > 0$, and $x \in H$, we have*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \big(1 - \frac{\mu}{\lambda}\big)J_\lambda^A x\right).$$

Lemma 2.5. *Let $A: D(A) \longrightarrow 2^H$ be a monotone operator. For $r \geq s > 0$, we have*

$$\|x - J_s^A x\| \leq 2\|x - J_r^A x\|$$

for all $x \in R(I + rA) \cap R(I + sA)$.

Proof. From Lemma 2.4, we have

$$\begin{aligned}
\|x - J_s^A x\| &\leq \|x - J_r^A x\| + \|J_r^A x - J_s^A x\| \\
&= \|x - J_r^A x\| + \|J_s^A(\frac{s}{r}x + (1 - \frac{s}{r})J_r^A x) - J_s^A x\| \\
&\leq \|x - J_r^A x\| + (1 - \frac{s}{r})\|x - J_r^A x\| \\
&\leq 2\|x - J_r^A x\|.
\end{aligned}$$

This completes the proof. $\square$

Lemma 2.6 ([3]). *Assume T be a nonexpansive self-map of a closed and convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$.*

3. Main results

The first, we have the following theorem:

Theorem 3.1. *Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let $A_i: D(A_i) \subset H \longrightarrow 2^H$, $i = 1, 2, \dots, N$, be monotone operators such that $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$ and $\overline{D(A_i)} \subset C \subset \cap_{r>0} R(I + rA_i)$ for all $i = 1, 2, \dots, N$. Let $\{\beta_n^i\}$ and $\{r_n^i\}$, $i = 1, 2, \dots, N$ be sequences of positive real numbers such that $\{\beta_n^i\} \subset (\alpha, \beta)$, with α, $\beta \in (0,1)$ and $\min_{i=1,2,\dots,N}\{\inf_n\{r_n^i\}\} \geq r > 0$. Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and*

$$
\begin{aligned}
&y_n^0 = x_n,\\
&y_n^i = \beta_n^i y_n^{i-1} + (1-\beta_n^i) J_{i,n} y_n^{i-1},\ J_{i,n} = J_{r_n^i}^{A_i},\ i = 1, 2, \dots, N\\
&C_n = \{z \in C:\ \|y_n^N - z\| \leq \|x_n - z\|\},\\
&Q_n = \{z \in C:\ \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \geq 0,
\end{aligned}
\tag{9}
$$

converges strongly to $P_S x_0$.

Proof. **Step 1.** C_n and Q_n are closed and convex subsets of C.

Indeed, we rewrite C_n and Q_n in the forms

$$
\begin{aligned}
C_n &= C \cap \{z \in H : \langle x_n - y_n^N, z\rangle \leq \frac{1}{2}(\|x_n\|^2 - \|y_n^N\|^2)\},\\
Q_n &= C \cap \{z \in H : \langle x_0 - x_n, z\rangle \leq \langle x_n, x_0 - x_n\rangle\},
\end{aligned}
$$

respectively. By C is closed and convex and $\{z \in H : \langle x_n - y_n^N, z\rangle \leq \frac{1}{2}(\|x_n\|^2 - \|y_n^N\|^2)\}$, $\{z \in H : \langle x_0 - x_n, z\rangle \leq \langle x_n, x_0 - x_n\rangle\}$ are closed halfspaces of H, so C_n and Q_n are closed and convex subsets of C for all $n \geq 0$. It follows that, $C_n \cap Q_n$ is closed and convex in H. Hence, the sequence $\{x_n\}$ is well-defined.

Step 2. $S \subset C_n \cap Q_n$ for all $n \geq 0$.

For each $u \in S$, we have

$$
\begin{aligned}
\|y_n^N - u\| &= \|\beta_n^N y_n^{N-1} + (1-\beta_n^N) J_{i,n} y_n^{N-1}\|\\
&\leq \beta_n^N \|y_n^{N-1} - u\| + (1-\beta_n^N)\|J_{i,n} y_n^{N-1} - J_{i,n} u\|\\
&\leq \beta_n^N \|y_n^{N-1} - u\| + (1-\beta_n^N)\|y_n^{N-1} - u\|\\
&= \|y_n^{N-1} - u\|
\end{aligned}
$$

$$\vdots$$
$$\leq \|y_n^0 - u\| = \|x_n - u\|.$$

By the definition of C_n, we get $u \in C_n$. Hence,

$$S \subset C_n \text{ for all } n \geq 0. \tag{10}$$

Now, we will show that $S \subset Q_n$ for all $n \geq 0$. Indeed, obviously $S \subset Q_0$. So $u \in C_0 \cap Q_0$ and from $x_1 = P_{C_0 \cap Q_0} x_0$, we obtain that

$$\langle x_1 - u, x_0 - x_1 \rangle \geq 0.$$

Thus, $u \in Q_1$. By induction, we get $u \in Q_n$ for all $n \geq 0$. Hence,

$$S \subset Q_n \text{ for all } n \geq 0. \tag{11}$$

From (10) and (11), we have $S \subset C_n \cap Q_n$.

Step 3. $\{x_n\}$ is bounded.

Let $x^* = P_S x_0 \in C_n \cap Q_n$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_0 - x^*\| \text{ for all } n \geq 0, \tag{12}$$

which implies that the sequence $\{x_n\}$ is bounded.

Step 4. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} = P_{C_n \cap Q_n} x_0$, $x_{n+1} \in Q_n$. By the definition of Q_n, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0, \tag{13}$$

which implies that

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|.$$

So, $\{\|x_n - x_0\|\}$ is nondecresing sequence, which combine with the boundedness of the sequence $\{x_n\}$, therefore, there exists the finite limit $\lim_{n\to\infty} \|x_n - x_0\| = d$.

From (13), we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, x_n - x_0 \rangle + \|x_n - x_0\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to d^2 - d^2 = 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{14}$$

Step 5. $\lim_{n\to\infty} \|x_n - y_n^N\| = 0$.

Indeed, by $x_{n+1} \in C_n$, we get

$$\|x_{n+1} - y_n^N\| \leq \|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty,$$

which combine with (14) and the following estimate

$$\|x_n - y_n^N\| \leq \|x_{x+1} - y_n^N\| + \|x_{n+1} - x_n\|,$$

therefore

$$\lim_{n\to\infty} \|x_n - y_n^N\| = 0. \tag{15}$$

Step 6. $\lim_{n\to\infty} \|y_n^{i-1} - J_{i,n} y_n^{i-1}\| = 0$ for all $i = 1, 2, \ldots, N$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \|y_n^{i-1} - J_{i,n} y_n^{i-1}\| = \lim_{k\to\infty} \|y_{n_k}^{i-1} - J_{i,n_k} y_{n_k}^{i-1}\|,$$

and let $\{x_{n_{k_l}}\}$ be a subsequence of $\{x_{n_k}\}$ such that

$$\limsup_{k\to\infty} \|x_{n_k} - x^*\| = \lim_{l\to\infty} \|x_{n_{k_l}} - x^*\|.$$

We have

$$\begin{aligned}
\|x_{n_{k_l}} - x^*\| &\le \|x_{n_{k_l}} - y_{n_{k_l}}^N\| + \|y_{n_{k_l}}^N - x^*\| \\
&\le \|x_{n_{k_l}} - y_{n_{k_l}}^N\| + \|y_{n_{k_l}}^{N-1} - x^*\| \\
&\ \ \vdots \\
&\le \|x_{n_{k_l}} - y_{n_{k_l}}^N\| + \|y_{n_{k_l}}^0 - x^*\| \\
&= \|x_{n_{k_l}} - y_{n_{k_l}}^N\| + \|x_{n_{k_l}} - x^*\|.
\end{aligned}$$

Therefore,

$$\lim_{l\to\infty} \|x_{n_{k_l}} - x^*\| = \lim_{l\to\infty} \|y_{n_{k_l}}^i - x^*\|, \ i = 1, 2, \ldots, N.$$

Next, by Lemma 2.2 and (9),

$$\begin{aligned}
\|y_{n_{k_l}}^i - x^*\|^2 &= \beta_{n_{k_l}}^i \|y_{n_{k_l}}^{i-1} - x^*\|^2 + (1 - \beta_{n_{k_l}}^i)\|J_{i,n_{k_l}} y_{n_{k_l}}^{i-1} - x^*\|^2 \\
&\quad - (1 - \beta_{n_{k_l}}^i)\beta_{n_{k_l}}^i \|y_{n_{k_l}}^{i-1} - J_{i,n_{k_l}} y_{n_{k_l}}^{i-1}\|^2 \\
&\le \|y_{n_{k_l}}^{i-1} - x^*\|^2 - (1 - \beta_{n_{k_l}}^i)\beta_{n_{k_l}}^i \|y_{n_{k_l}}^{i-1} - J_{i,n_{k_l}} y_{n_{k_l}}^{i-1}\|^2 \\
&\le \|x_{n_{k_l}} - x^*\|^2 - (1 - \beta_{n_{k_l}}^i)\beta_{n_{k_l}}^i \|y_{n_{k_l}}^{i-1} - J_{i,n_{k_l}} y_{n_{k_l}}^{i-1}\|^2.
\end{aligned}$$

Hence,

$$\alpha(1-\beta)\|y_{n_{k_l}}^{i-1} - J_{i,n_{k_l}} y_{n_{k_l}}^{i-1}\|^2 \le \|x_{n_{k_l}} - x^*\|^2 - \|y_{n_{k_l}}^i - x^*\|^2 \to 0,$$

as $l \to \infty$ for all $i = 1, 2, \ldots, N$. So, $\|y_{n_{k_l}}^{i-1} - J_{i,n_{k_l}} y_{n_{k_l}}^{i-1}\| \to 0$, which implies that

$$\limsup_{n\to\infty} \|y_n^{i-1} - J_{i,n} y_n^{i-1}\| = 0.$$

Thus, $\lim_{n\to\infty} \|y_n^{i-1} - J_{i,n} y_n^{i-1}\| = 0$ for all $i = 1, 2, \ldots, N$.

Step 7. $\lim_{n\to\infty} x_n = P_S x_0$.

We show that $\|x_n - J_{i,n}x_n\| \to 0$ for all $i = 1, 2, \ldots, N$. Indeed, in the case that $i = 1$, $\|x_n - J_{1,n}x_n\| = \|y_n^0 - J_{1,n}y_n^0\| \to 0$. In the case that $i = 2$, we have

$$\begin{aligned}\|x_n - J_{2,n}x_n\| &\le \|x_n - y_n^1\| + \|y_n^1 - J_{2,n}y_n^1\| + \|J_{2,n}y_n^1 - J_{2,n}x_n\| \\ &\le 2\|x_n - y_n^1\| + \|y_n^1 - J_{2,n}y_n^1\| \\ &\le 2\|y_n^0 - J_{1,n}y_n^0\| + \|y_n^1 - J_{2,n}y_n^1\| \to 0.\end{aligned}$$

So, we get $\|x_n - J_{2,n}x_n\| \to 0$. Similarly, we obtain that $\|x_n - J_{i,n}x_n\| \to 0$ for all $i = 3, 4, \ldots, N$. By Lemma 2.5, $\|x_n - J_r^{A_i}x_n\| \le 2\|x_n - J_{i,n}x_n\|$. So, $\|x_n - J_r^{A_i}x_n\| \to 0$ for all $i = 1, 2, \ldots, N$.

Since, $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^\dagger$ as $k \to \infty$. From $\|x_{n_k} - J_r^{A_i}x_{n_k}\| \to 0$ for all $i = 1, 2, \ldots, N$ and Lemma 2.6, we get $x^\dagger \in S$.

By $x^* = P_S x_0$, $x^\dagger \in S$, (12) and Lemma 2.3, we have

$$\begin{aligned}\|x_0 - x^*\| \le \|x_0 - x^\dagger\| &\le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \\ &\le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|x_0 - x^*\|.\end{aligned}$$

From the definition of x^*, it follows that $x^\dagger = x^*$. Thus, $\|x_{n_k} - x_0\| \to \|x^* - x_0\|$ and by Lemma 2.3, we get $x_{n_k} \to x^*$ as $k \to \infty$. Form the uniqueness of x^*, we obtain that $x_n \to x^*$ as $n \to \infty$.

This completes the proof. □

Remark 3.2. If $N = 1$, we can choose the sequence $\{\beta_n\} \subset [0, a)$, with $a \in [0, 1)$. So, we have the following corollary.

Corollary 3.3. *Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let $A : D(A) \subset H \longrightarrow 2^H$ be a monotone operator such that $S = A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{\beta_n\}$ and $\{r_n\}$ be sequences of positive real numbers such that $\{\beta_n\} \subset [0, a)$, with $a \in [0, 1)$ and $\inf_n\{r_n\} \ge r > 0$. Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and*

$$\begin{aligned}&y_n = \beta_n x_n + (1 - \beta_n)J_{r_n}^A x_n, \\ &C_n = \{z \in H : \ \|y_n - z\| \le \|x_n - z\|\}, \\ &Q_n = \{z \in H : \ \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ &x_{n+1} = P_{C_n \cap Q_n} x_0, \ n \ge 0,\end{aligned} \tag{16}$$

converges strongly to $P_S x_0$.

Remark 3.4. Theorem 3.1 is more general than the result of Solodov and Svaiter in [10].

Next, we give strong convergence theorems to find a common fixed point of nonexpansive mapping. By the careful analysis of the proof of Theorem 3.1, we can obtain the following results for the problem of finding a common fixed point of a finite family of nonexpasive mappings. Because its proof is much simpler than that of Theorem 3.1, we omit it proof.

Theorem 3.5. *Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let $T_i: C \longrightarrow C$, $i = 1, 2, \ldots, N$, be nonexpansive mappings from C into itself such that $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\beta_n^i\}$ be sequences of positive real numbers such that $\{\beta_n^i\} \subset (\alpha, \beta)$ with $\alpha,\ \beta \in (0,1)$ for all $i = 1, 2, \ldots, N$. Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and*

$$\begin{aligned} & y_n^0 = x_n, \\ & y_n^i = \beta_n^i y_n^{i-1} + (1-\beta_n^i) T_i y_n^{i-1},\ i = 1, 2, \ldots, N \\ & C_n = \{z \in C:\ \|y_n^N - z\| \leq \|x_n - z\|\}, \\ & Q_n = \{z \in C:\ \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ & x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \geq 0, \end{aligned} \tag{17}$$

converges strongly to $P_S x_0$.

In the case $N = 1$, we have the following corollary:

Corollary 3.6 (Theorem 3.4, in [8])**.** *Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $T: C \longrightarrow C$ be a nonexpansive mapping from C into itself such that $S = F(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of positive real numbers such that $\{\beta_n\} \subset [0, a)$, with $a \in [0, 1)$. Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and*

$$\begin{aligned} & y_n = \beta_n x_n + (1-\beta_n) T x_n, \\ & C_n = \{z \in C:\ \|y_n - z\| \leq \|x_n - z\|\}, \\ & Q_n = \{z \in C:\ \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ & x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \geq 0, \end{aligned} \tag{18}$$

converges strongly to $P_S x_0$.

Remark 3.7. The Theorem 3.5 is more general than the result of Nakajo and Takahashi in [8].

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