Commun. Korean Math. Soc. **32** (2017), No. 2, pp. 425–433
https://doi.org/10.4134/CKMS.c160106
pISSN: 1225-1763 / eISSN: 2234-3024

BUSY PERIOD DISTRIBUTION OF A BATCH ARRIVAL RETRIAL QUEUE

Jeongsim Kim

Abstract. This paper is concerned with the analysis of the busy period distribution in a batch arrival $M^X/G/1$ retrial queue. The expression for the Laplace-Stieltjes transform of the length of the busy period is well known, but from this expression we cannot compute the moments of the length of the busy period by direct differentiation. This paper provides a direct method of calculation for the first and second moments of the length of the busy period.

1. Introduction

Retrial queues are queueing systems in which arriving customers who find all servers occupied may retry for service again after a random amount of time. Retrial queues have been widely used to model many problems/situations in telephone systems, call centers, telecommunication networks, computer networks and computer systems, and in daily life. For an overview regarding retrial queues, refer to the surveys [8, 10, 11, 12]. For further details, refer to the books [4, 9], and the bibliographies [1, 2, 3].

This paper considers a single server batch arrival retrial queue. The single server batch arrival retrial queues are characterized by the following features: If the server is idle when a batch of customers (called primary customers) arrive from outside the system, then one customer of that batch begins to be served immediately while the other customers join a retrial group, called an orbit. If the server is busy when a batch of customers arrive from outside the system, then all the customers of that batch join the orbit. All the customers in the orbit behave independently of each other. If the server is idle when a customer from the orbit attempts service, this customer receives service immediately. Otherwise the customer comes back to the orbit immediately and repeats the

Received May 6, 2016.
2010 *Mathematics Subject Classification.* 60K25.
Key words and phrases. retrial queue, batch arrivals, busy period.
This work was supported by the intramural research grant of Chungbuk National University in 2015 and Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014R1A1A4A01003813).

retrial process. Thus only the external arrivals take place in batches and the retrials are conducted singly.

In this paper we are concerned with the analysis of the busy period distribution in a batch arrival $M^X/G/1$ retrial queue. For the batch arrival retrial queue, the busy period is defined as the period which starts when a batch of primary customers arrives at an empty system and ends at the next departure epoch when the system is empty again. The busy period distribution for the $M^X/G/1$ retrial queue was studied by Falin [7], who introduced the concept of k-busy period. For the batch arrival retrial queue a k-busy period is defined as the period which starts when a batch of k primary customers arrives at an empty system and ends at the next departure epoch when the system is empty again. Falin [7] derived the expression for the Laplace-Stieltjes transform (LST) of the length of the k-busy period and from this he obtained the LST of the length of the busy period. The expression for the LST of the length of the busy period (which is shown in formula (1)) has practical limitations. For instance, we cannot compute the moments of the length of the busy period by direct differentiation. Falin [7] obtained only the first moment of the length of the busy period. Falin [6] studied the same problem as in Falin [7] for the single arrival $M/G/1$ retrial queue. Artalejo and Lopez-Herrero [5] obtained closed-form expressions for the first and second moments of the length of the busy period in the single arrival $M/G/1$ retrial queue.

In this paper, by a direct method of calculation we obtain explicit expressions for the first and second moments of the length of the busy period in the batch arrival $M^X/G/1$ retrial queue. This paper generalizes the result of Artalejo and Lopez-Herrero [5] to the batch arrival retrial queue.

The paper is organized as follows. In Section 2, we describe our model in detail and present an equation for the LST of the length of the busy period. In Section 3, we obtain explicit formulas for the first and second moments of the length of the busy period.

2. An equation for the length of the busy period

We consider the $M^X/G/1$ retrial queue where customers arrive from outside the system in batches according to a Poisson process with rate λ. The batch sizes are independent and identically distributed (i.i.d.) random variables with a generic random variable B and common distribution $\mathbb{P}(B = k) = b_k$, $k = 1, 2, \ldots$. Let $b(z) = \sum_{k=1}^{\infty} b_k z^k$, $|z| \le 1$, be the probability generating function of the batch size distribution and $b^{(k)}$ be the kth factorial moment of the batch size, i.e., $b^{(k)} = \mathbb{E}[B(B-1)\cdots(B-k+1)] = \frac{d^k}{dz^k} b(z)\big|_{z=1-}$. The service times are i.i.d. random variables with a generic random variable S. Let $\beta(s) = \mathbb{E}[e^{-sS}]$, $s \ge 0$, be the LST of the service time distribution and $\beta^{(k)}$ be the kth moment of the service time, i.e., $\beta^{(k)} = \mathbb{E}[S^k] = (-1)^k \frac{d^k}{ds^k} \beta(s)\big|_{s=0+}$. The inter-retrial time, i.e., the length of the time interval between two consecutive attempts made by a customer in the orbit, is exponentially distributed with mean ν^{-1}.

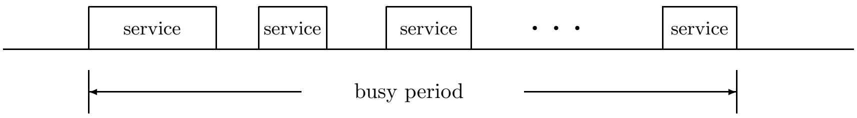

FIGURE 1. A busy period in a retrial queue.

The arrival process, the batch sizes, the service times, and the inter-retrial times are assumed to be mutually independent. The offered load ρ is defined as $\rho = \lambda b^{(1)}\beta^{(1)}$. It is assumed that $\rho < 1$ for stability of the system.

As defined earlier, the busy period is the interval from the moment of a batch arrival of primary customers at an empty system (the server is idle and there is no customer in the orbit) to the moment that the system becomes empty for the first time. The busy period defined in this way consists of alternating service periods and periods during which the server is idle and there are customers in the orbit, as illustrated in Figure 1.

Let L be the length of a busy period and $\pi(s) = \mathbb{E}[e^{-sL}]$ be its LST. Falin [7] obtained the following expression for the LST of the length of the busy period (see equation (10) of [7]):

$$\begin{aligned}\pi(s) =& \frac{s+\lambda}{\lambda} - \frac{\nu}{\lambda}\Big[\int_0^{\pi_\infty(s)} \exp\Big\{-\int_0^u \frac{s+\lambda-\lambda\frac{b(v)}{v}\beta(s+\lambda-\lambda b(v))}{\nu[\beta(s+\lambda-\lambda b(v))-v]}dv\Big\} \\ &\times \frac{1}{\nu[\beta(s+\lambda-\lambda b(u))-u]}du\Big]^{-1}, \end{aligned} \tag{1}$$

where $\pi_\infty(s)$ is the LST of the length of the busy period in the corresponding standard $M^X/G/1$ queue, i.e., for $s \geq 0$, $\pi_\infty(s)$ is the unique solution of the functional equation $z = \beta(s+\lambda-\lambda b(z))$ on the interval $0 \leq z \leq 1$. The above expression provides a theoretical solution, but it has serious practical limitations. For instance, it is even impossible to obtain the moments of L through direct differentiation. Falin [7] obtained only the first moment of the length of the busy period.

By a direct method of calculation we can obtain explicit expressions for the first and second moments of the length of the busy period, as will be shown in Section 3. In order to get this, we will need an equation for the LST of the length of the busy period. At time t, let $N(t)$ be the number of customers in the orbit and $C(t)$ be the number of customers being served (i.e., $C(t) = 1$ or 0 according as the server is busy or idle). Assume that a busy period starts at time 0, i.e., a batch of customers arrive to the empty system at time 0. Hence the length of the busy period, L, is written as

$$L = \inf\{t > 0 : N(t) = C(t) = 0\}.$$

Let $\pi^{(k)}(s) = \mathbb{E}[e^{-sL} \mid N(0) = k-1]$, i.e., $\pi^{(k)}(s)$ is the LST of the length of the k-busy period. Moreover, let

$$\varphi_0^{(k)}(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-st}\mathbb{P}(L > t, N(t) = n, C(t) = 0 \mid N(0) = k-1)dt.$$

Falin [7] derived the following differential equation for $\varphi_0^{(k)}(z,s)$ (see equation (8) of [7]):

$$\begin{aligned}&\pi^{(k)}(s) + \nu(z - \beta(s+\lambda-\lambda b(z)))\frac{\partial}{\partial z}\varphi_0^{(k)}(z,s)\\ &\quad + (s+\lambda-\lambda\tilde{b}(z)\beta(s+\lambda-\lambda b(z)))\varphi_0^{(k)}(z,s) = z^{k-1}\beta(s+\lambda-\lambda b(z)),\end{aligned}$$

where $\tilde{b}(z) = \frac{b(z)}{z} = \sum_{k=0}^{\infty} b_{k+1}z^k$. From this equation and $\pi(s) = \sum_{k=1}^{\infty} b_k\pi^{(k)}(s)$, we obtain the following proposition. This plays a crucial role in obtaining the moments of the length of the busy period.

Proposition 1. *The LST $\pi(s)$ satisfies the following equation*:

$$\begin{aligned}(2)\quad \pi(s) &= \nu(\beta(s+\lambda-\lambda b(z)) - z)\frac{\partial}{\partial z}\Psi(z,s)\\ &\quad - (s+\lambda-\lambda\tilde{b}(z)\beta(s+\lambda-\lambda b(z)))\Psi(z,s) + \tilde{b}(z)\beta(s+\lambda-\lambda b(z)),\end{aligned}$$

where $\Psi(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-st}\mathbb{P}(L > t, N(t) = n, C(t) = 0)dt$.

3. The first two moments of the length of the busy period

In this section we obtain the first and second moments of the length of the busy period by a direct method of calculation with the use of Proposition 1. Let $\pi^{(k)}$ be the kth moment of the length of the busy period, i.e.,

$$\pi^{(k)} = \mathbb{E}[L^k] = (-1)^k \frac{\partial^k}{\partial s^k}\pi(s)\Big|_{s=0+}.$$

In order to obtain the moments of L, we introduce the following moments: for $i = 0, 1, \ldots, j = 0, 1, \ldots$,

$$\Psi^{(i,j)} = (-1)^j \frac{\partial^{i+j}}{\partial z^i \partial s^j}\Psi(z,s)\Big|_{z=1-,s=0+}.$$

Also, we need the following lemma.

Lemma 1. *For $s > 0$, there exists a unique $z(s) \in (0,1)$ such that*

$$z(s) = \beta(s+\lambda-\lambda b(z(s))).$$

Moreover, $\lim_{s\to 0+} z(s) = 1$.

Proof. For fixed $s > 0$, let

$$h(s,z) = z - \beta(s+\lambda-\lambda b(z)),\ 0 \le z \le 1.$$

Then $h(s,0) = -\beta(s+\lambda) < 0$ and $h(s,1) = 1 - \beta(s) > 0$. Hence by the intermediate value theorem, there exists $z \in (0,1)$ such that $h(s,z) = 0$, meaning that $z = \beta(s+\lambda-\lambda b(z))$. We note that since $0 < \frac{d}{dz}\beta(s+\lambda-\lambda b(z)) < 1$ for $0 < z < 1$, $h(s,z)$ is strictly increasing in z on $(0,1)$. Thus the first assertion is proved. Next we prove the second assertion. Since $\beta(\lambda-\lambda b(z))$ is strictly increasing and convex in z on $[0,1)$ and $\frac{d}{dz}\beta(\lambda-\lambda b(z))\big|_{z=1-} = \rho < 1$, we have

$$\beta(\lambda-\lambda b(z)) > z \quad \text{for } 0 \le z < 1.$$

Let z_0 be fixed in $(0,1)$. Then $\lim_{s\to 0+} h(s,z_0) < 0$. This implies that $z_0 < z(s) < 1$ for sufficiently small s. Hence

$$z_0 \le \liminf_{s\to 0+} z(s) \le \limsup_{s\to 0+} z(s) \le 1.$$

Since z_0 is arbitrary, $\lim_{s\to 0+} z(s) = 1$. □

We begin the calculation of $\mathbb{E}[L]$. First, take the partial derivative of (2) with respect to s. This gives

$$\begin{aligned}\pi'(s) &= \nu\beta'(s+\lambda-\lambda b(z))\frac{\partial}{\partial z}\Psi(z,s) + \nu(\beta(s+\lambda-\lambda b(z)) - z)\frac{\partial^2}{\partial z\partial s}\Psi(z,s)\\ &\quad - (1-\lambda\tilde{b}(z)\beta'(s+\lambda-\lambda b(z)))\Psi(z,s)\\ &\quad - (s+\lambda-\lambda\tilde{b}(z)\beta(s+\lambda-\lambda b(z)))\frac{\partial}{\partial s}\Psi(z,s) + \tilde{b}(z)\beta'(s+\lambda-\lambda b(z)). \end{aligned} \tag{3}$$

Putting $z = z(s)$ in (3) and using Lemma 1 yields

$$\begin{aligned}\pi'(s) &= \nu\beta'(s+\lambda-\lambda b(z(s)))\phi_{10}(z(s),s)\\ &\quad - (1-\lambda\tilde{b}(z(s))\beta'(s+\lambda-\lambda b(z(s))))\phi_{00}(z(s),s)\\ &\quad - (s+\lambda-\lambda b(z(s)))\phi_{01}(z(s),s) + \tilde{b}(z(s))\beta'(s+\lambda-\lambda b(z(s))),\end{aligned} \tag{4}$$

where $\phi_{ij}(z,s) = \frac{\partial^{i+j}}{\partial z^i \partial s^j}\Psi(z,s)$. Letting $s \to 0+$ in (4) and using the second assertion of Lemma 1 gives

$$\begin{aligned}\mathbb{E}[L] &= \nu\beta^{(1)}\Psi^{(1,0)} + (1+\lambda\beta^{(1)})\Psi^{(0,0)}\\ &\quad + \lim_{s\to 0+}(s+\lambda-\lambda b(z(s)))\phi_{01}(z(s),s) + \beta^{(1)}.\end{aligned} \tag{5}$$

Now we will show that $\lim_{s\to 0+}(s+\lambda-\lambda b(z(s)))\phi_{01}(z(s),s) = 0$. For $s \ge 0$,

$$\begin{aligned}0 \le -s\phi_{01}(z(s),s) &= -\int_0^s \phi_{01}(z(s),s)du\\ &\le -\int_0^s \phi_{01}(z(s),u)du\\ &= \phi_{00}(z(s),0) - \phi_{00}(z(s),s).\end{aligned}$$

Thus

$$\lim_{s\to 0+} s\phi_{01}(z(s),s) = 0. \tag{6}$$

We note that

$$\lim_{s\to 0+}\frac{s+\lambda-\lambda b(z(s))}{s}=1-\lambda\mathbb{E}[B]\frac{-\mathbb{E}[S]}{1-\rho}=\frac{1}{1-\rho}.$$

From this and (6), we have

$$\lim_{s\to 0+}(s+\lambda-\lambda b(z(s)))\phi_{01}(z(s),s)=0,$$

and so (5) becomes

$$\mathbb{E}[L]=\nu\beta^{(1)}\Psi^{(1,0)}+(1+\lambda\beta^{(1)})\Psi^{(0,0)}+\beta^{(1)}. \tag{7}$$

In order to find $\Psi^{(1,0)}$, take the partial derivative of (2) with respect to z. Following the same procedure as the above, we obtain

$$\Psi^{(1,0)}=\frac{b^{(1)}-1+\rho}{\nu(1-\rho)}(\lambda\Psi^{(0,0)}+1). \tag{8}$$

Substituting (8) into (7), we get

$$\mathbb{E}[L]=\frac{1}{1-\rho}\Psi^{(0,0)}+\frac{b^{(1)}\beta^{(1)}}{1-\rho}. \tag{9}$$

Therefore, in order to obtain $\mathbb{E}[L]$, we only need to find $\Psi^{(0,0)}$. The expression for $\Psi^{(0,0)}$ can be easily obtained from the following lemma.

Lemma 2. *We have*

$$\Psi(z,0)=\frac{1}{\lambda}\exp\left\{\frac{\lambda}{\nu}\int_0^z\frac{1-\tilde{b}(x)\beta(\lambda-\lambda b(x))}{\beta(\lambda-\lambda b(x))-x}dx\right\}-\frac{1}{\lambda}. \tag{10}$$

Proof. By Proposition 1, we get

$$\frac{\partial}{\partial z}\Psi(z,0)+\frac{\lambda}{\nu}\frac{1-\tilde{b}(z)\beta(\lambda-\lambda b(z))}{z-\beta(\lambda-\lambda b(z))}\Psi(z,0)+\frac{1-\tilde{b}(z)\beta(\lambda-\lambda b(z))}{\nu(z-\beta(\lambda-\lambda b(z)))}=0.$$

Let $f(z)=\Psi(z,0)+\frac{1}{\lambda}$. Then

$$\frac{d}{dz}f(z)+\frac{\lambda}{\nu}\frac{1-\tilde{b}(z)\beta(\lambda-\lambda b(z))}{z-\beta(\lambda-\lambda b(z))}f(z)=0.$$

Note that $f(0)=\frac{1}{\lambda}$. Solving the above differential equation, we have

$$f(z)=\frac{1}{\lambda}\exp\left\{\frac{\lambda}{\nu}\int_0^z\frac{1-\tilde{b}(x)\beta(\lambda-\lambda b(x))}{\beta(\lambda-\lambda b(x))-x}dx\right\}.$$

Therefore, (10) follows immediately from $\Psi(z,0)=f(z)-\frac{1}{\lambda}$. □

Putting $z=1$ in (10), we get

$$\Psi^{(0,0)}=\frac{1}{\lambda}\exp\left\{\frac{\lambda}{\nu}\int_0^1\frac{1-\tilde{b}(x)\beta(\lambda-\lambda b(x))}{\beta(\lambda-\lambda b(x))-x}dx\right\}-\frac{1}{\lambda}. \tag{11}$$

Finally, substituting (11) into (9), we obtain the expression for the first moment $\mathbb{E}[L]$:

$$\mathbb{E}[L] = \frac{1}{\lambda(1-\rho)} \exp\Big\{\frac{\lambda}{\nu}\int_0^1 \frac{1-\tilde{b}(x)\beta(\lambda-\lambda b(x))}{\beta(\lambda-\lambda b(x))-x}dx\Big\} - \frac{1}{\lambda}, \tag{12}$$

which is consistent with the result of Falin [7] (refer to Theorem 2 of [7]).

Next, we will obtain the second moment $\mathbb{E}[L^2]$. To do this, take the partial derivative of (2) with respect to s twice. Following the same procedure as that used to derive (7), we have

$$\mathbb{E}[L^2] = \beta^{(2)}(\lambda\Psi^{(0,0)}+1) + \nu\beta^{(2)}\Psi^{(1,0)} + 2(1+\lambda\beta^{(1)})\Psi^{(0,1)} + 2\nu\beta^{(1)}\Psi^{(1,1)}.$$

Substituting (8) into the above equation yields

$$\mathbb{E}[L^2] = \frac{b^{(1)}\beta^{(2)}}{1-\rho}(\lambda\Psi^{(0,0)}+1) + 2(1+\lambda\beta^{(1)})\Psi^{(0,1)} + 2\nu\beta^{(1)}\Psi^{(1,1)}. \tag{13}$$

Thus, in order to obtain $\mathbb{E}[L^2]$, we need to find $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$. First, to find $\Psi^{(1,1)}$, take the partial derivative of (2) with respect to z and then with respect to s. Following the same procedure as that used to derive (7) and using (8), we get

$$\begin{aligned}\Psi^{(1,1)} = {}& \frac{1}{\nu(1-\rho)}\Big\{\frac{(b^{(1)}-1+\rho)(1+\lambda\beta^{(1)}+\lambda\nu b^{(1)}\beta^{(2)})}{\nu(1-\rho)} + \lambda b^{(1)}\beta^{(2)} \\ & - (1-b^{(1)})\beta^{(1)}\Big\}(\lambda\Psi^{(0,0)}+1) + \frac{\lambda(b^{(1)}-1+\rho)}{\nu(1-\rho)}\Psi^{(0,1)} + \frac{\beta^{(1)}}{1-\rho}\Psi^{(2,0)}.\end{aligned} \tag{14}$$

Similarly, take the partial derivative of (2) with respect to z twice and follow the same procedure as the above, to obtain

$$\begin{aligned}\Psi^{(2,0)} = {}& \frac{1}{2\nu(1-\rho)}\Big(\frac{2\lambda(1-\rho-b^{(1)})^2 + \nu\rho b^{(2)} + \nu\lambda^2(b^{(1)})^3\beta^{(2)}}{\nu(1-\rho)} \\ & + 2(1-\rho)(1-b^{(1)}) + b^{(2)}\Big)(\lambda\Psi^{(0,0)}+1).\end{aligned}$$

Substituting this into (14) and then substituting the resulting expression for $\Psi^{(1,1)}$ into (13), we obtain

$$\begin{aligned}\mathbb{E}[L^2] = {}& \frac{1}{\nu(1-\rho)^3}\big\{\nu b^{(1)}\beta^{(2)} + \nu b^{(2)}(\beta^{(1)})^2 - 2\beta^{(1)}(1-\rho-b^{(1)})\big\}(\lambda\Psi^{(0,0)}+1) \\ & + \frac{2}{1-\rho}\Psi^{(0,1)}.\end{aligned} \tag{15}$$

Therefore, in order to obtain $\mathbb{E}[L^2]$, we only need to find $\Psi^{(0,1)}$. The expression for $\Psi^{(0,1)}$ is given by the following lemma.

Lemma 3. *We have*

$$\Psi^{(0,1)} = \int_0^1 \exp\Big\{\frac{\lambda}{\nu}\int_x^1 \frac{1-\tilde{b}(u)\beta(\lambda-\lambda b(u))}{\beta(\lambda-\lambda b(u))-u}du\Big\}g(x)dx, \tag{16}$$

where

$$g(x) = \frac{1}{\nu(\beta(\lambda - \lambda b(x)) - x)} \Big[\frac{1}{\lambda(1-\rho)} \exp\Big\{ \frac{\lambda}{\nu} \int_0^1 \frac{1 - \tilde{b}(u)\beta(\lambda - \lambda b(u))}{\beta(\lambda - \lambda b(u)) - u} du \Big\} + \exp\Big\{ \frac{\lambda}{\nu} \int_0^x \frac{1 - \tilde{b}(u)\beta(\lambda - \lambda b(u))}{\beta(\lambda - \lambda b(u)) - u} du \Big\} \Big(\frac{(1-b(x))\beta'(\lambda-\lambda b(x))}{\beta(\lambda-\lambda b(x))-x} - \frac{1}{\lambda} \Big) \Big].$$

Proof. Recall that $\Psi^{(0,1)} = -\frac{\partial}{\partial s}\Psi(z,s)\Big|_{z=1-,s=0+}$. Let $\psi_0(z,s) = -\frac{\partial}{\partial s}\Psi(z,s)$ and $\psi(z) = \psi_0(z,0)$. Taking the partial derivative of (2) with respect to s and putting $s = 0+$, we obtain

$$\begin{aligned} &\mathbb{E}[L] + \nu(z - \beta(\lambda - \lambda b(z)))\frac{d}{dz}\psi(z) + \nu\beta'(\lambda - \lambda b(z))\frac{\partial}{\partial z}\Psi(z,0) \\ &+ \lambda(1 - \tilde{b}(z)\beta(\lambda - \lambda b(z)))\psi(z) - (1 - \lambda\tilde{b}(z)\beta'(\lambda - \lambda b(z)))\Psi(z,0) \\ &= -\tilde{b}(z)\beta'(\lambda - \lambda b(z)). \end{aligned}$$

Using Lemma 2 and (12), we obtain that the above equation becomes

$$\frac{d}{dz}\psi(z) + \frac{\lambda(1 - \tilde{b}(z)\beta(\lambda - \lambda b(z)))}{\nu(z - \beta(\lambda - \lambda b(z)))}\psi(z) = g(z).$$

Since $\psi(0) = 0$, we have

$$\psi(z) = \int_0^z \exp\Big\{ \frac{\lambda}{\nu} \int_x^z \frac{1 - \tilde{b}(u)\beta(\lambda - \lambda b(u))}{\beta(\lambda - \lambda b(u)) - u} du \Big\} g(x) dx.$$

Since $\Psi^{(0,1)} = \psi(1)$, we have (16). $\square$

Finally, substituting (11) and (16) into (15), we obtain the expression for the second moment $\mathbb{E}[L^2]$. In summary, we have the following theorem.

Theorem 1. *For the $M^X/G/1$ retrial queue, the first and second moments of the length of the busy period are given by*

$$\begin{aligned} \mathbb{E}[L] &= \frac{1}{\lambda(1-\rho)} \exp\Big\{ \frac{\lambda}{\nu} \int_0^1 \frac{1 - \tilde{b}(x)\beta(\lambda - \lambda b(x))}{\beta(\lambda - \lambda b(x)) - x} dx \Big\} - \frac{1}{\lambda}, \\ \mathbb{E}[L^2] &= \frac{1}{\nu(1-\rho)^3} \big\{ \nu b^{(1)}\beta^{(2)} + \nu b^{(2)}(\beta^{(1)})^2 - 2\beta^{(1)}(1 - \rho - b^{(1)}) \big\} \\ &\qquad \times \exp\Big\{ \frac{\lambda}{\nu} \int_0^1 \frac{1 - \tilde{b}(x)\beta(\lambda - \lambda b(x))}{\beta(\lambda - \lambda b(x)) - x} dx \Big\} \\ &\quad + \frac{2}{1-\rho} \int_0^1 \exp\Big\{ \frac{\lambda}{\nu} \int_x^1 \frac{1 - \tilde{b}(u)\beta(\lambda - \lambda b(u))}{\beta(\lambda - \lambda b(u)) - u} du \Big\} g(x) dx, \end{aligned}$$

where $g(x)$ is given by Lemma 3.

In the case of single arrivals, Theorem 1 is reduced to the following corollary.

Corollary 1. *For the $M/G/1$ retrial queue, the first and second moments of the length of the busy period are given by*

$$\mathbb{E}[L] = \frac{1}{\lambda(1-\rho)} \exp\Big\{\frac{\lambda}{\nu}\int_0^1 \frac{1-\beta(\lambda-\lambda x)}{\beta(\lambda-\lambda x)-x}dx\Big\} - \frac{1}{\lambda}, \tag{17}$$

$$\begin{aligned}\mathbb{E}[L^2] = {} & \frac{1}{1-\rho}\exp\Big\{\frac{\lambda}{\nu}\int_0^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du\Big\}\Big[\frac{1}{(1-\rho)^2}\Big(\frac{2\rho\beta^{(1)}}{\nu}+\beta^{(2)}\Big) \\ & - \int_0^1 \frac{2}{\nu(\beta(\lambda-\lambda x)-x)}\Big(\frac{1}{\lambda} - \frac{(1-x)\beta'(\lambda-\lambda x)}{\beta(\lambda-\lambda x)-x} \\ & - \frac{1}{\lambda(1-\rho)}\exp\Big\{\frac{\lambda}{\nu}\int_x^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du\Big\}\Big)dx\Big].\end{aligned} \tag{18}$$

It is noticed that (17) and (18) are consistent with formulas (26) and (38) of Artalejo and Lopez-Herrero [5].

References

[1] J. R. Artalejo, *A classified bibliography of research on retrial queues: Progress in 1990–1999*, Top **7** (1999), no. 2, 187–211.

[2] ———, *Accessible bibliography on retrial queues*, Math. Comput. Modelling **30** (1999), 1–6.

[3] ———, *Accessible bibliography on retrial queues: Progress in 2000–2009*, Math. Comput. Modelling **51** (2010), no. 9-10, 1071–1081.

[4] J. R. Artalejo and A. Gómez-Corral, *Retrial Queueing Systems*, Berlin Heidelberg, Springer, 2008.

[5] J. R. Artalejo and M. J. Lopez-Herrero, *On the busy period of the $M/G/1$ retrial queue*, Naval Res. Logist. **47** (2000), no. 2, 115–127.

[6] G. I. Falin, *A single-line system with secondary orders*, Engrg. Cybernetics Review **17** (1979), 76–83.

[7] ———, *Functioning under nonsteady conditions of a single-channel system with group arrival of requests and repeated calls*, Ukrainian Math. J. **33** (1981), 429–432.

[8] ———, *A survey of retrial queues*, Queueing Syst. **7** (1990), no. 2, 127–168.

[9] G. I. Falin and J. G. C. Templeton, *Retrial Queues*, London, Chapman & Hall, 1997.

[10] J. Kim and B. Kim, *A survey of retrial queueing systems*, Ann. Oper. Res. **247** (2016), no. 1, 1–36.

[11] V. G. Kulkarni and H. M. Liang, *Retrial queues revisited*, In: Frontiers in Queueing: Models and Applications in Science and Engineering (J. H. Dshalalow, ed.), 19–34, CRC Press, Boca Raton, 1997.

[12] T. Yang and J. G. C. Templeton, *A survey on retrial queues*, Queueing Systems **2** (1987), 201–233.

Jeongsim Kim
Department of Mathematics Education
Chungbuk National University
Cheongju 28644, Korea
E-mail address: jeongsimkim@chungbuk.ac.kr