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SLANT CURVES IN 3-DIMENSIONAL ALMOST f-KENMOTSU MANIFOLDS

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Abstract. In this paper, we study slant curves in a 3-dimensional almost f-Kenmotsu manifold with proper mean curvature vector field.

1. Introduction

Euclidean submanifolds $M^m \subset \mathbb{R}^n$ with *proper mean curvature vector field* $\triangle H = \lambda H$, $\lambda \in \mathbb{R}$ have been studied extensively (see [8] and references therein).

Arroyo, Barros and Garay ([1], [3]) studied curves and surfaces in the 3-sphere $\mathbb{S}^3$ with proper mean curvature vector field. Chen studied surfaces in hyperbolic 3-space $\mathbb{H}^3$ with proper mean curvature vector fields in [9].

On the other hand, as the generalization of Legendre curve, the notion of slant curves was introduced in [10].

A unit speed curve γ in an almost contact metric 3-manifold $(M; \varphi, \xi, \eta, g)$ is said to be *slant* if its tangent vector field makes constant *contact angle* θ with ξ, *i.e.*, $\cos\theta := \eta(\gamma')$ is constant along γ.

In our previous paper [10], we studied slant curves in Sasakian 3-manifolds. In [11], we have shown that biharmonic curves in Sasakian space forms are slant.

Călin and Crasmareanu [5] studied slant curves in 3-dimensional normal almost contact geometry. Moreover, Călin, Crasmareanu and Munteanu [6] studied slant curves with proper mean curvature vector field in three-dimensional f-Kenmotsu manifolds. In particular, they have given explicit parametrization of slant curves in the hyperbolic 3-space equipped with natural homogeneous normal almost contact metric structure (Kenmotsu structure of constant curvature). The present authors studied almost Legendre curves in normal almost contact metric 3-manifolds with proper mean curvature vector field [12]. Suh, Lee and the second named author studied Legendre curves in Sasakian 3-manifolds whose mean curvature vector field satisfies C-parallel or C-proper condition [13].

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In this paper, we study slant curve in a 3-dimensional almost f-Kenmotsu manifold. As a generalization of the class of f-Kenmotsu manifolds, the notion of almost f-Kenmotsu manifold was introduced in Section 2.2. An almost f-Kenmotsu manifold is f-Kenmotsu manifold if and only if it is normal.

In Section 3.1, we determine the torsion of slant curve in a 3-dimensional almost f-Kenmotsu manifold. In Section 3.2, we obtain the necessary and sufficient conditions for a non-geodesic slant curve in 3-dimensional almost f-Kenmotsu manifolds to have proper mean curvature vector field.

2. Almost contact manifolds

2.1. Almost contact manifolds

Let M be a manifold of odd dimension $m = 2n+1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to the existence of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

From these conditions one can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all X, $Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. Such a metric is called an *associated metric* of the almost contact manifold $M = (M, \varphi, \xi, \eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X)$$

for all $X \in \mathfrak{X}(M)$. A structure (φ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold*.

The *fundamental* 2-*form* Φ of $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X,Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

On the direct product manifold $M \times \mathbb{R}$ of an almost contact metric manifold and the real line $\mathbb{R}$, any tangent vector field can be represented as the form $(X, f\mathrm{d}/\mathrm{d}t)$, where $X \in \mathfrak{X}(M)$ and f is a function on $M \times \mathbb{R}$ and t is the Cartesian coordinate on the real line $\mathbb{R}$.

Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda\mathrm{d}/\mathrm{d}t) = (\varphi X - \lambda\xi, \eta(X)\mathrm{d}/\mathrm{d}t).$$

If J is integrable, then M is said to be *normal*.

Equivalently, M is normal if and only if

$$[\varphi,\varphi](X,Y)+2\mathrm{d}\eta(X,Y)\xi=0,$$

where $[\varphi,\varphi]$ is the *Nijenhuis torsion* of φ defined by

$$[\varphi,\varphi](X,Y)=[\varphi X,\varphi Y]+\varphi^2[X,Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]$$

for any $X,Y\in\mathfrak{X}(M)$.

For more details on almost contact metric manifolds, we refer to Blair's monograph [4].

2.2. Almost f-Kenmotsu manifolds

For an arbitrary almost contact metric 3-manifold M, we have [14]:

$$(\nabla_X\varphi)Y=g(\varphi\nabla_X\xi,Y)\xi-\eta(Y)\varphi\nabla_X\xi, \tag{2.1}$$

where ∇ is the Levi-Civita connection on M. Moreover, we have

$$\mathrm{d}\eta=\eta\wedge\nabla_\xi\eta+\alpha\Phi,\quad \mathrm{d}\Phi=2f\eta\wedge\Phi,$$

where α and f are the functions defined by

$$\alpha=\frac{1}{2}\mathrm{Trace}\,(\varphi\nabla\xi),\quad f=\frac{1}{2}\mathrm{Trace}\,(\nabla\xi)=\frac{1}{2}\mathrm{div}\,\xi. \tag{2.2}$$

Now assume that M is an almost f-Kenmotsu 3-manifold. Then we have

$$\nabla_X\xi=f(X-\eta(X)\xi)+h\varphi X, \tag{2.3}$$

where $h=\pounds_\xi\varphi/2$ and $f\in C^\infty(M)$ is strictly positive.

From this equation we have

$$\varphi\nabla_X\xi=f\varphi X+\varphi h\varphi X.$$

Inserting this into (2.1), we get

$$(\nabla_X\varphi)Y=g(\varphi(fI+h\varphi)X,Y)\xi-\eta(Y)\varphi(fI+h\varphi)X. \tag{2.4}$$

For a 3-dimensional f-Kenmotsu manifold M, using the equations (2.3), (2.4) and $h=0$, we have

$$(\nabla_X\varphi)Y=f(g(\varphi X,Y)\xi-\eta(Y)\varphi X),$$

$$\nabla_X\xi=f(X-\eta(X)\xi).$$

If f is a positive constant β, we get an *almost β-Kenmotsu manifold*. In particular, if $h=0$, then it is a *β-Kenmotsu manifold*. 1-Kenmotsu manifold is called *Kenmotsu manifold*.

2.3. Frenet frame field

Now let $\gamma(s)$ be a unit curve in the oriented Riemannian 3-manifold $(M^3, g, \mathrm{d}v_g)$ with non-vanishing acceleration $\nabla_{\gamma'}\gamma'$. Then we put $\kappa := |\nabla_{\gamma'}\gamma'|$. We can take a unit normal vector field N by the formula $\nabla_{\gamma'}\gamma' = \kappa N$. Next define a unit vector field B by $B = T \times N$. Here $T = \gamma'$. In this way we obtain an orthonormal frame field $\mathcal{F} = (T, N, B)$ along γ which is *positively oriented*, that is, $\mathrm{d}v_g(T, N, B) = 1$. The orthonormal frame field $\mathcal{F}$ is called the *Frenet frame field* and satisfies

$$\nabla_{\gamma'}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \tag{2.5}$$

for some function τ. The functions κ and τ are called the *curvature* and *torsion* of γ, respectively. The ordinary differential equation (2.5) is called the *Frenet-Serret formula* of γ. The unit vector fields T, N and B are called the *tangent vector field*, *principal normal vector field* and *binormal vector field* of γ, respectively.

3. Slant curves in almost f-Kenmotsu manifolds

In this section, we consider slant curves in almost f-Kenmotsu manifolds.

Let γ be a non-geodesic curve in an almost contact metric 3-manifold M. Differentiating the formula $g(T, \xi) = \cos\theta$ along γ with respect to the Levi-Civita connection ∇, then it follows that

$$-\theta' \sin\theta = g(\kappa N, \xi) + g(T, \nabla_T \xi) = \kappa\, \eta(N) + f \sin^2\theta + g(T, h\varphi T).$$

This equation implies the following result.

Proposition 3.1. *A Frenet curve γ is a slant curve in an almost f-Kenmotsu manifold M then γ satisfies*

$$\eta(N) = -\frac{1}{\kappa}\{f \sin^2\theta + g(\gamma', h\varphi\gamma')\}. \tag{3.1}$$

Using the Frenet frame field $\{T, N, B\}$, we express

$$\xi = (\cos\theta)T - \frac{1}{\kappa}(f \sin^2\theta + g(T, h\varphi T))N + \eta(B)B.$$

Since ξ is a unitary vector field, we get

$$\eta(B) = \frac{1}{\kappa}\sqrt{\kappa^2 \sin^2\theta - (f \sin^2\theta + b)^2}.$$

Hence we get:

Remark 1. For slant curve γ the decomposition of ξ is

$$\xi = \cos\theta T - \frac{1}{\kappa}(f \sin^2\theta + b)N + \Big(\frac{1}{\kappa}\sqrt{\kappa^2 \sin^2\theta - (f \sin^2\theta + b)^2}\Big)B,$$

where $b = g(h\varphi\gamma', \gamma')$.

3.1. The curvature and torsion

We suppose that γ is non-geodesic slant curve then γ can not be an integral curve of ξ. In general, we find an orthonormal frame field in almost contact metric 3-manifold M along γ

$$e_1 = T = \gamma', \quad e_2 = \frac{\varphi\gamma'}{|\sin\theta|}, \quad e_3 = \frac{\xi - \cos\theta\gamma'}{|\sin\theta|}. \tag{3.2}$$

Also $\xi = \cos\theta e_1 + |\sin\theta|e_3$. Thus, we put $a = g(h\gamma', \gamma')$, $b = g(h\varphi\gamma', \gamma')$ then

$$h\gamma' = ae_1 + \frac{b}{|\sin\theta|}e_2 - a\frac{\cos\theta}{|\sin\theta|}e_3, \tag{3.3}$$

$$h\varphi\gamma' = be_1 - a|\sin\theta|e_2 - \frac{\cos\theta}{|\sin\theta|}(b + a\cos\theta)e_3. \tag{3.4}$$

From the equation (2.3), we get

$$\nabla_{\gamma'}\xi = (b + f\sin^2\theta)e_1 - a|\sin\theta|e_2 - \frac{\cos\theta}{|\sin\theta|}(f\sin^2\theta + b + a\cos\theta)e_3.$$

Then we have

$$\begin{cases} \nabla_{\gamma'}e_1 = \delta|\sin\theta|e_2 - \dfrac{1}{|\sin\theta|}(b + f\sin^2\theta)e_3, \\ \nabla_{\gamma'}e_2 = -\delta|\sin\theta|e_1 + (a + \delta\cos\theta)e_3, \\ \nabla_{\gamma'}e_3 = \dfrac{1}{|\sin\theta|}(b + f\sin^2\theta)e_1 - (a + \delta\cos\theta)e_2, \end{cases} \tag{3.5}$$

where $\delta = g(\nabla_{\gamma'}\gamma', \varphi\gamma')/\sin^2\theta$, $a = g(h\gamma', \gamma')$, $b = g(h\varphi\gamma', \gamma')$.

From the first equation of (3.5), we get

$$\kappa = \sqrt{\delta^2\sin^2\theta + \frac{1}{\sin^2\theta}(b + f\sin^2\theta)^2}. \tag{3.6}$$

From the above equation, we have:

Proposition 3.2. *Let γ be a slant curve in 3-dimensional almost f-Kenmotsu manifolds. Then γ is a geodesic if and only if γ satisfies*

$$g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0 \quad \textit{and} \quad g(h\varphi\gamma', \gamma') + f\sin^2\theta = 0,$$

where γ is non-parallel to ξ.

Thus the principal normal vector field $N = \frac{1}{\kappa}\{\delta|\sin\theta|e_2 - \frac{1}{|\sin\theta|}(b + f\sin^2\theta)e_3\}$. Differentiating N and using (3.5) we get

$$\begin{aligned} \nabla_{\gamma'}N = & -\frac{1}{\kappa}\{\delta^2\sin^2\theta + \frac{1}{\sin^2\theta}(b + f\sin^2\theta)^2\}e_1 \\ & + \{-\frac{\kappa'}{\kappa^2}\delta|\sin\theta| + \frac{1}{\kappa}\delta'|\sin\theta| + \frac{1}{\kappa}\frac{1}{|\sin\theta|}(b + f\sin^2\theta)(a + \delta\cos\theta)\}e_2 \\ & + \{\frac{\kappa'}{\kappa^2}\frac{1}{|\sin\theta|}(b + f\sin^2\theta) - \frac{1}{\kappa}\frac{1}{|\sin\theta|}(b' + f'\sin^2\theta) \end{aligned}$$

$$+ \frac{1}{\kappa}\delta|\sin\theta|(a+\delta\cos\theta)\}e_3.$$

Differentiating (3.6), $\kappa' = \frac{1}{\kappa}\{\delta\delta'\sin^2\theta + \frac{1}{\sin^2\theta}(b+f\sin^2\theta)(b'+f'\sin^2\theta)\}$. Hence we obtain

$$\tau = \frac{1}{\kappa^2}\{\delta'(b+f\sin^2\theta) - \delta(b'+f'\sin^2\theta)\} + (a+\delta\cos\theta), \tag{3.7}$$

thus binormal vector field $B = \frac{1}{\kappa}\{\frac{1}{|\sin\theta|}(b+f\sin^2\theta)e_2 + \delta \mid \sin\theta \mid e_3\}$.

From (3.6) we get $\delta = \frac{1}{\sin^2\theta}\sqrt{\kappa^2\sin^2\theta - (b+f\sin^2\theta)^2}$. Differentiating (3.6) we have $\delta' = \frac{1}{\delta\sin^4\theta}\{\kappa\kappa'\sin^2\theta - (b+f\sin^2\theta)(b'+f'\sin^\theta)\}$. Hence we have:

Theorem 3.1. *Let γ be a non-geodesic slant curve in 3-dimensional almost f-Kenmotsu manifolds. Then*

$$\begin{aligned}\tau = {} & \frac{1}{\kappa\sqrt{\kappa^2\sin^2\theta - (b+f\sin^2\theta)^2}}\{\kappa'(b+f\sin^2\theta) - \kappa(b'+f'\sin^2\theta)\} \\ & + \frac{\cos\theta}{\sin^2\theta}\sqrt{\kappa^2\sin^2\theta - (b+f\sin^2\theta)^2} + a,\end{aligned} \tag{3.8}$$

where $a = g(h\gamma', \gamma')$ and $b = g(h\varphi\gamma', \gamma')$.

For a non-geodesic slant curve in f-Kenmotsu manifolds, since $a = b = 0$, we have the following

Corollary 3.1 ([6])**.** *Let γ be a non-geodesic slant curve with $\theta \neq 0, \pi$ such that N is non-parallel to ξ in 3-dimensional f-Kenmotsu manifolds. Then its torsion is*:

$$\tau = \frac{\cos\theta}{|\sin\theta|}\sqrt{\kappa^2 - f^2\sin^2\theta} - \frac{\kappa|\sin\theta|}{\sqrt{\kappa^2 - f^2\sin^2\theta}}\gamma'(f/\kappa).$$

In case f is a non-zero constant, the following statements hold: *a non-geodesic slant curve with constant curvature κ has a constant torsion τ and so, is a helix.*

3.2. Proper mean curvature vector field

The mean curvature vector field H with respect to the Levi-Civita connection ∇ of a curve γ in 3-dimensional oriented Riemannian 3-manifold is defined by

$$H = \nabla_{\dot\gamma}\dot\gamma = \kappa N.$$

Using (2.5), we have:

Lemma 3.1. *Let (M, g) be an oriented Riemannian 3-manifold and γ a unit speed curve. Then we have*

$$\nabla_{\gamma'}H = -\kappa^2 T + \kappa' N + \kappa\tau B, \tag{3.9}$$

$$\nabla_{\gamma'}\nabla_{\gamma'}H = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B. \tag{3.10}$$

Definition 3.1. In 3-dimensional oriented Riemannian manifolds M^3 , a vector field X along a unit speed curve γ is said to be *parallel* if $\nabla_{\gamma'} X = 0$.

Using the Lemma 3.1, we get:

Proposition 3.3. *Let (M, g) be an oriented Riemannian 3-manifold and γ a unit speed curve. Then γ has parallel mean curvature vector field if and only if γ is a geodesic.*

We define the Laplace-Beltrami operator Δ of $\gamma^* TM$,

$$\Delta = -\nabla_{\gamma'} \nabla_{\gamma'}.$$

For a curve γ in an oriented Riemannian 3-manifold M with Levi-Civita connection ∇,

$$\Delta H = -\nabla_{\gamma'} \nabla_{\gamma'} \nabla_{\gamma'} \gamma'.$$

γ has a proper mean curvature vector field if and only if γ is a helix satisfying $\lambda = \kappa^2 + \tau^2$.

From Theorem 3.1 we have:

Theorem 3.2. *A non-geodesic slant curve γ in 3-dimensional almost f-Kenmotsu manifolds has proper mean curvature vector field if and only if γ is a helix satisfying*

$$\begin{aligned}\lambda = {} & \frac{\kappa^2}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^4\theta}(b + f\sin^2\theta)^2 - \frac{2\cos\theta}{\sin^2\theta}(b' + f'\sin^2\theta) \\ & + \frac{(b' + f'\sin^2\theta)^2}{\kappa^2\sin^2\theta - (b + f\sin^2\theta)^2} \\ & + a\{\frac{2\cos\theta}{\sin^2\theta}\sqrt{\kappa^2\sin^2\theta - (b + f\sin^2\theta)^2} - \frac{2(b' + f'\sin^2\theta)}{\sqrt{\kappa^2\sin^2\theta - (b + f\sin^2\theta)^2}}\} + a^2.\end{aligned}$$

For 3-dimensional f-Kenmotsu manifolds, since $a = b = 0$, we have the following.

Corollary 3.2 ([6]). *A non-geodesic slant curve in 3-dimensional f-Kenmotsu manifolds has proper mean curvature vector field if and only if γ is a helix satisfying*

$$\lambda = \frac{\kappa^2}{\sin^2\theta} - f^2\cos^2\theta + \frac{f'^2\sin^2\theta}{\kappa^2 - f^2\sin^2\theta} - 2f'\cos\theta.$$

Corollary 3.3 ([6]). *Let γ be a slant curve in the hyperbolic space $H^3(-1)$ as the warped product K_1^3 with constant sectional curvature -1. If γ has proper mean curvature vector field. Then γ is*

$$\gamma(s) = \left(s\cos\theta, \tfrac{\sin\theta e^{-s\cos\theta}}{c^2+\cos^2\theta}(c\sin(cs) - \cos\theta\cos(cs),\ -c\cos(cs) - \cos\theta\sin(cs))\right),$$

where c is a constant.

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