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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS OF MEROMORPHIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and investigate an interesting subclass of meromorphic bi-univalent functions defined on $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. For functions belonging to this class, estimates on the initial coefficients are obtained. The results presented in this paper would generalize and improve some recent works of several earlier authors.

1. Introduction

Let Σ be the family of meromorphic functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \tag{1.1}$$

that are univalent in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse f^{-1} that satisfy

$$f^{-1}(f(z)) = z \ (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \ \ (M < |w| < \infty, \ M > 0)\,.$$

Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} B_n \frac{1}{w^n}, \tag{1.2}$$

where $M < |w| < \infty$. A simple calculation shows that the function f^{-1}, is given by

$$f^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + \cdots\,. \tag{1.3}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

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Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [8] obtained the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ and Duren [1] proved that $|b_n| \leq 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$, $1 \leq k \leq n/2$.

For the coefficients of inverses of meromorphic univalent functions, Springer [10] proved that

$$|B_3| \leq 1 \text{ and } |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}, \quad n = 1, 2, \ldots.$$

In 1977, Kubota [6] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [9] obtained a sharp bounds for the coefficients B_{2n-1}, $1 \leq n \leq 7$.

Several researchers such as (for example) Halim *et al.* [4], Janani and Murugusundaramoorthy [5] and Hamidi *et al.* [2] introduced and investigated new subclasses of meromorphically bi-univalent functions.

Recently T. Panigrahi [7] introduced the following two subclasses of the meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficients $|b_0|$, $|b_1|$ and $|b_2|$ as follows.

Definition 1.1 ([7]). A function $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $M_{\Sigma_{\mathfrak{B}}}(\alpha, \lambda)$, if the following conditions are satisfied:

$$\left|\arg\left\{\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\alpha\pi}{2} \ (0 < \alpha \leq 1, \lambda \geq 1, z \in \Delta)$$

and

$$\left|\arg\left\{\lambda\frac{wg'(w)}{g(w)} + (1-\lambda)\left(1 + \frac{wg''(w)}{g'(w)}\right)\right\}\right| < \frac{\alpha\pi}{2} \ (0 < \alpha \leq 1, \lambda \geq 1, w \in \Delta),$$

where the function g is the inverse of f given by (1.3).

Theorem 1.2. *Let $f(z)$ given by* (1.1) *be in the class $M_{\Sigma_{\mathfrak{B}}}(\alpha, \lambda)$. Then*

$$|b_0| \leq \frac{2\alpha}{\lambda},$$

$$|b_1| \leq \frac{\alpha}{2\lambda - 1}\sqrt{(\alpha-2)^2 + \frac{4\alpha^2}{\lambda^2}}$$

and

$$|b_2| \leq \frac{2\alpha}{3(3\lambda-2)}\left[2\{\frac{6\alpha^2 - \lambda^2(\alpha^2 - 3\alpha + 2)}{3\lambda^2}\} + 3 - 2\alpha\right].$$

Definition 1.3 ([7]). A function $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$, if the following conditions are satisfied:

$$\mathrm{Re}\left\{\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \ (0 \leq \beta < 1, \lambda \geq 1, z \in \Delta)$$

and

$$\operatorname{Re}\left\{\lambda\frac{wg'(w)}{g(w)}+(1-\lambda)\left(1+\frac{wg''(w)}{g'(w)}\right)\right\}>\beta\ (0\leq\beta<1,\lambda\geq 1,w\in\Delta),$$

where the function g is the inverse of f given by (1.3).

Theorem 1.4. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta,\lambda)$. Then*

$$|b_0|\leq\frac{2(1-\beta)}{\lambda},$$

$$|b_1|\leq\frac{(1-\beta)}{2\lambda-1}\sqrt{1+\frac{4(1-\beta)^2}{\lambda^2}}$$

and

$$|b_2|\leq\frac{2(1-\beta)}{3(3\lambda-2)}\left[1+\frac{4(1-\beta)^2}{\lambda^2}\right].$$

The object of the present paper is to introduce a new subclass of the function class $\Sigma_{\mathfrak{B}}$ and obtain estimates on the initial coefficients for functions in this new subclass which improve Theorem 1.2 and Theorem 1.4.

2. Coefficient bounds for the function class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$

In this section, we introduce and investigate the general subclass $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$ $(\lambda\geq 1)$.

Definition 2.1. Let the functions $h,p:\Delta\to\mathbb{C}$ be analytic functions and

$$h(z)=1+\frac{h_1}{z}+\frac{h_2}{z^2}+\frac{h_3}{z^3}+\cdots,\quad p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^2}+\cdots,$$

such that

$$\min\{\operatorname{Re}(h(z)),\operatorname{Re}(p(z))\}>0,\ z\in\Delta.$$

A function $f\in\Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$, if the following conditions are satisfied:

$$f\in\Sigma_{\mathfrak{B}}\ ,\ \lambda\frac{zf'(z)}{f(z)}+(1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right)\in h(\Delta)\ (\lambda\geq 1,\ z\in\Delta)\tag{2.1}$$

and

$$\lambda\frac{wg'(w)}{g(w)}+(1-\lambda)\left(1+\frac{wg''(w)}{g'(w)}\right)\in p(\Delta)\ (\lambda\geq 1,\ w\in\Delta),\tag{2.2}$$

where the function g is the inverse of f given by (1.3).

Remark 2.2. There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the meromorphic function class Σ. For example,

(1) If we let $h(z)=p(z)=\left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha}=1+\frac{2\alpha}{z}+\frac{2\alpha^2}{z^2}+\frac{2\alpha^3}{z^3}+\cdots$ $(0<\alpha\leq 1,\ z\in\Delta)$, it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

Now if $f\in M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$, then

$$f\in\Sigma_{\mathfrak{B}},$$
$$\left|\arg\left\{\lambda\frac{zf'(z)}{f(z)}+(1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right|<\frac{\alpha\pi}{2}\ (0<\alpha\leq 1,\ \lambda\geq 1,\ z\in\Delta)$$

and

$$\left|\arg\left\{\lambda\frac{wg'(w)}{g(w)}+(1-\lambda)\left(1+\frac{wg''(w)}{g'(w)}\right)\right\}\right|<\frac{\alpha\pi}{2}\ (0<\alpha\leq 1,\ \lambda\geq 1,\ w\in\Delta).$$

Therefore in this case, the class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$ reduce to class $M_{\Sigma_{\mathfrak{B}}}(\alpha,\lambda)$ in Definition 1.1.

(2) If we let $h(z)=p(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}=1+\frac{2(1-\beta)}{z}+\frac{2(1-\beta)}{z^2}+\frac{2(1-\beta)}{z^3}+\cdots$ $(0\leq\beta<1,\ z\in\Delta)$, it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

Now if $f\in M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$, then

$$f\in\Sigma_{\mathfrak{B}},\ \operatorname{Re}\left\{\lambda\frac{zf'(z)}{f(z)}+(1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\beta\ (0\leq\beta<1,\lambda\geq 1,z\in\Delta)$$

and

$$\operatorname{Re}\left\{\lambda\frac{wg'(w)}{g(w)}+(1-\lambda)\left(1+\frac{wg''(w)}{g'(w)}\right)\right\}>\beta\ (0\leq\beta<1,\lambda\geq 1,w\in\Delta).$$

Therefore in this case, the class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$ reduce to class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta,\lambda)$ in Definition 1.3.

Now, we derive the estimates of the coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$.

Theorem 2.3. *Let the $f(z)$ given by (1.1) is said to be in the class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda)$. Then*

$$|b_0|\leq\min\left\{\sqrt{\frac{|h_1|^2+|p_1|^2}{2\lambda^2}},\sqrt{\frac{|h_2|+|p_2|}{2\lambda}}\right\},\tag{2.3}$$

$$|b_1|\leq\min\left\{\frac{|h_2|+|p_2|}{4(2\lambda-1)},\frac{1}{(2\lambda-1)}\sqrt{\frac{|h_2|^2+|p_2|^2}{8}+\frac{(|h_1|^2+|p_1|^2)^2}{16\lambda^2}}\right\}\tag{2.4}$$

and

$$|b_2|\leq\frac{1}{3(3\lambda-2)}\left[\frac{|p_1|^3}{\lambda^2}+\frac{2(2\lambda-1)|h_3|+\lambda|p_3|}{5\lambda-2}\right].\tag{2.5}$$

Proof. First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

$$\lambda \frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right) = h(z) \ (z \in \Delta) \tag{2.6}$$

and

$$\lambda \frac{wg'(w)}{g(w)} + (1-\lambda)\left(1 + \frac{wg''(w)}{g'(w)}\right) = p(w) \ (w \in \Delta), \tag{2.7}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 2.1.
Furthermore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots,$$

and

$$p(w) = 1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \frac{p_3}{w^2} + \cdots,$$

respectively. Now, upon equating the coefficients of

$$\begin{aligned} &\lambda \frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 - \frac{\lambda b_0}{z} + \frac{\lambda b_0^2 + 2(1-2\lambda)b_1}{z^2} - \frac{\lambda {b_0}^3 - 3\lambda b_0 b_1 - 3(2-3\lambda)b_2}{z^3} + \cdots \end{aligned} \tag{2.8}$$

with those of $h(z)$ and coefficients of

$$\begin{aligned} &\lambda \frac{wg'(w)}{g(w)} + (1-\lambda)\left(1 + \frac{wg''(w)}{g'(w)}\right) \\ &= 1 + \frac{\lambda b_0}{w} + \frac{\lambda b_0^2 - 2(1-2\lambda)b_1}{w^2} + \frac{\lambda {b_0}^3 - 6(1-2\lambda)b_0 b_1 - 3(2-3\lambda)b_2}{w^3} + \cdots \end{aligned} \tag{2.9}$$

with those of $p(w)$, we get

$$-\lambda b_0 = h_1, \tag{2.10}$$

$$\lambda b_0^2 + 2(1-2\lambda)b_1 = h_2, \tag{2.11}$$

$$-\lambda {b_0}^3 + 3\lambda b_0 b_1 + 3(2-3\lambda)b_2 = h_3, \tag{2.12}$$

$$\lambda b_0 = p_1, \tag{2.13}$$

$$\lambda b_0^2 - 2(1-2\lambda)b_1 = p_2, \tag{2.14}$$

and

$$\lambda {b_0}^3 - 6(1-2\lambda)b_0 b_1 - 3(2-3\lambda)b_2 = p_3. \tag{2.15}$$

From (2.10) and (2.13), we get

$$h_1 = -p_1$$

and

$$2\lambda^2 b_0^2 = h_1^2 + p_1^2. \tag{2.16}$$

Adding (2.11) and (2.14), we get

$$2\lambda b_0^2 = h_2 + p_2. \tag{2.17}$$

Therefore, we find from the equations (2.16) and (2.17) that

$$|b_0|^2 \leq \frac{|h_1|^2 + |p_1|^2}{2\lambda^2},$$

and

$$|b_0|^2 \leq \frac{|h_2| + |p_2|}{2\lambda}$$

respectively. So we get the desired estimate on the coefficient $|b_0|$ as asserted in (2.3).

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (2.14) from (2.11). We thus get

$$4(1 - 2\lambda)b_1 = h_2 - p_2. \tag{2.18}$$

By squaring and adding (2.11) and (2.14), using (2.16) in the computation leads to

$$b_1^2 = \frac{1}{(2\lambda - 1)^2}\left(\frac{h_2^2 + p_2^2}{8} - \frac{(h_1^2 + p_1^2)^2}{16\lambda^2}\right). \tag{2.19}$$

Therefore, we find from the equations (2.18) and (2.19) that

$$|b_1| \leq \frac{|h_2| + |p_2|}{4(2\lambda - 1)}$$

and

$$|b_1| \leq \frac{1}{(2\lambda - 1)}\sqrt{\frac{|h_2|^2 + |p_2|^2}{8} + \frac{(|h_1|^2 + |p_1|^2)^2}{16\lambda^2}}.$$

Finally, to determine the bound on $|b_2|$, consider the sum of (2.12) and (2.15) with $h_1 = -p_1$, we have

$$b_0 b_1 = \frac{h_3 + p_3}{3(5\lambda - 2)}. \tag{2.20}$$

Subtracting (2.15) from (2.12) with $h_1 = -p_1$, we obtain

$$6(2 - 3\lambda)b_2 = h_3 - p_3 + 2\lambda b_0^3 - 3(2 - 3\lambda)b_0 b_1. \tag{2.21}$$

Using (2.16) and (2.20) in (2.21) give to

$$b_2 = \frac{1}{3(2 - 3\lambda)}\left[\frac{p_1^3}{\lambda^2} + \frac{4\lambda - 2}{5\lambda - 2}h_3 - \frac{\lambda}{5\lambda - 2}p_3\right].$$

This evidently completes the proof of Theorem 2.3. □

3. Corollaries and consequences

By setting

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \frac{2\alpha^3}{z^3} + \cdots \ (0 < \alpha \leq 1,\ z \in \Delta),$$

in Theorem 2.3, we conclude the following result.

Corollary 3.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $M_{\Sigma_{\mathfrak{B}}}(\alpha, \lambda)$ $(0 < \alpha \leq 1, \lambda \geq 1)$. *Then*

$$|b_0| \leq \begin{cases} \alpha\sqrt{\frac{2}{\lambda}} \ ; 1 \leq \lambda \leq 2 \\ \frac{2\alpha}{\lambda} \ ; \lambda \geq 2 \end{cases}$$

$$|b_1| \leq \frac{\alpha^2}{2\lambda - 1},$$

and

$$|b_2| \leq \frac{2\alpha^3}{3(3\lambda-2)}[1 + \frac{4}{\lambda^2}].$$

Remark 3.2. Corollary 3.1 is an improvement of estimates obtained in Theorem 1.2. Because

$$\frac{\alpha^2}{2\lambda-1} \leq \frac{\alpha^2}{2\lambda-1}\sqrt{1+\frac{4}{\lambda^2}} \leq \frac{\alpha}{2\lambda-1}\sqrt{(\alpha-2)^2 + \frac{4\alpha^2}{\lambda^2}}$$

and

$$\frac{2\alpha^3}{3(3\lambda-2)}[1 + \frac{4}{\lambda^2}] \leq \frac{2\alpha}{3(3\lambda-2)}\left[2\{\frac{6\alpha^2 - \lambda^2(\alpha^2 - 3\alpha + 2)}{3\lambda^2}\} + 3 - 2\alpha\right].$$

By setting

$$\begin{aligned} h(z) = p(z) &= \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} \\ &= 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \frac{2(1-\beta)}{z^3} + \cdots \ (0 \leq \beta < 1,\ z \in \Delta), \end{aligned}$$

in Theorem 2.3, we conclude the following result.

Corollary 3.3. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$ $(0 \leq \beta < 1, \lambda \geq 1)$. *Then*

$$|b_0| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{\lambda}} \ ; \lambda + 2\beta \leq 2 \\ \frac{2(1-\beta)}{\lambda} \ ; \lambda + 2\beta \geq 2 \end{cases}$$

$$|b_1| \leq \frac{1-\beta}{2\lambda - 1},$$

and

$$|b_2| \leq \frac{2(1-\beta)}{3(3\lambda-2)}[1+\frac{4(1-\beta)^2}{\lambda^2}].$$

Remark 3.4. Corollary 3.3 is an improvement of estimates obtained in Theorem 1.4.

By setting $\lambda = 1$ in Corollary 3.3, we obtain the following result.

Corollary 3.5. *Let the function* $f(z)$ *given by* (1.1) *be meromorphic bi-starlike of order* β $(0 \leq \beta < 1)$ *in* Δ. *Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)} \ ; \beta \leq \frac{1}{2} \\ 2(1-\beta) \ ; \beta \geq \frac{1}{2} \end{cases}$$

and

$$|b_1| \leq (1-\beta).$$

Remark 3.6. The estimate for $|b_0|$ given in Corollary 3.5 is an improvement of estimates obtained by Hamidi *et al.* [3, Theorem 2].

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