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COINCIDENCE THEOREMS FOR COMPARABLE GENERALIZED NON LINEAR CONTRACTIONS IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we prove some coincidence point theorems involving φ-contraction in ordered partial metric spaces. We also extend newly introduced notion of g-comparability of a pair of maps for linear contraction in ordered metric spaces to non-linear contraction in ordered partial metric spaces. Thus, our results extend, modify and generalize some recent well known coincidence point theorems of ordered metric spaces.

1. Introduction

The Banach contraction mapping theorem, is the limelight result for finding the existence and uniqueness of fixed points of certain mappings, in the framework of metric spaces. Matthews [16] extended the Banach contraction mapping theorem to the partial matric spaces for applications in program verification. Subsequently several authors (see for instance, [7, 9, 21, 24, 32]) obtained many useful fixed point results in this direction. The existence of several connection between partial metrics and topological aspects of domain theory has been pointed by many authors see [9, 10, 15, 16, 25, 26, 27].

On the other hand fixed point theorem for monotone mapping was initiated by Turinci [28, 29] in 1986. Later Ran and Reuring [32] proved slightly more natural version of this corresponding fixed point theorem of Turinci for continuous monotone mappings with some application to matrix equations. In this continuation, Nieto and Rodriguez-Lopez [18, 19] generalized the theorem for increasing mappings and analogously proved a fixed point theorem for decreasing mapping in ordered metric setting which has been generalized by many authors [1, 6, 5, 11, 20, 22, 30, 31] in the recent years.

Most recently, Alam et al. [3, 4] extended the foregoing results to generalized nonlinear φ-contractions in ordered metric setting. Also, in light of the

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g-monotonicity condition of a pair of maps, Alam et al. [2] introduced the notion of g-comparability, and proved the existence and uniqueness results on coincidence points for linear contraction in partially ordered metric spaces.

Our aim in this paper is to utilise the notion of g-comparability of a pair of maps for non-linear contraction and generalize the recent coincidence theorems in ordered metric spaces to ordered partial metric spaces.

2. Preliminaries

Definition 2.1. A partial metric on a set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:
(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Note that the self-distance of any point need not be zero, hence the idea of generalizing metrics so that a metric on a non-empty set X is precisely metric p on X such that for any $x \in X$, $p(x,x) = 0$.

Similar to the case of metric space, a partial metric space is a pair (X,p) consisting of a non-empty set X and a partial metric p on X.

Example 2.1. Let a function $p : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $p(x,y) = \max\{x,y\}$ for any $x, y \in R^+$. Then, $(\mathbb{R}^+, p)$ is a partial metric space where the self-distance for any point $x \in \mathbb{R}^+$ is its value itself.

Example 2.2. If $X := \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$, then $p : X \times X \to \mathbb{R}^+$ defined by $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric on X.

Each partial metric p on X generates a T_0 topology T_p on X, which has as a base the family of open p-balls $B_p(x,\epsilon), x \in X, \epsilon > 0$, where

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

If p is a partial metric on X, then the function $p^s : X * X \to \mathbb{R}^+$ defined by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2.2. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then
(a) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$,
(b) $\{x_n\}$ is a Cauchy sequence if there exists (and is finite) $\lim\limits_{n,m\to\infty} p(x_n,x_m)$.

Definition 2.3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to T_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Remark 2.1. It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 2.1 ([15, 21])**.** *Let (X,p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*

(b) *(X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x)=0$ if and only if*

$$p(x,x)=\lim_{n\to\infty} p^s(x_n,x)=\lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 2.4. A triplet $(X,p,\preceq)$ is called an ordered partial metric space if (X,p) is a partial metric space and $(X,\preceq)$ is an ordered set.

Definition 2.5 ([11])**.** Let $(X,\preceq)$ be an ordered set and (f,g) a pair of self mappings on X. We say that

(a) f is g-increasing if for any $x,y\in X$

$$g(x)\preceq g(y)\Rightarrow f(x)\preceq f(y),$$

(b) f is g-decreasing if for any $x,y\in X$

$$g(x)\preceq g(y)\Rightarrow f(x)\succeq f(y),$$

(c) f is g -monotone if f is either g-increasing or g-decreasing.

Proposition 2.1. *Let f and g be a pair of self-mappings defined on an ordered set $(X,\preceq)$. If f is g-monotone and $g(x)=g(y)$, then $f(x)=f(y)$.*

Proof. As $g(x)=g(y)$, on using reflexivity of $\preceq$, we have $g(x)\preceq g(y)$ and $g(x)\succeq g(y)$. Suppose that f is g-increasing (resp. g-decreasing), we have $f(x)\preceq f(y)$ and $f(x)\succeq f(y)$ (resp. $f(x)\succeq f(y)$ and $f(x)\preceq f(y)$), which, in both cases (owing to the antisymmetric property of $\preceq$) gives rise to $f(x)=f(y)$. □

Definition 2.6 ([13])**.** Let X be a non-empty set and f and g two self-mappings on X. Then

(a) an element $x\in X$ is called a coincidence point of f and g if

$$g(x)=f(x),$$

(b) if $x\in X$ is a coincidence point of f and g and $u\in X$ such that $u=g(x)=f(x)$, then u is called a point of coincidence of f and g,

(c) if $x\in X$ is a coincidence point of f and g such that $u=g(x)=f(x)$, then u is called a common fixed point of f and g.

Definition 2.7. Let $(X,p,\preceq)$ be an ordered partial metric space and f a self-mapping on X. We say that

(i) $(X,p,\preceq)$ has the f-ICU (increasing-convergence-upper bound) property if f-image of every increasing convergent sequence $\{x_n\}$ in X is bounded above by f-image of its limit (as an upper bound), i.e.,

$$x_n\uparrow x\Rightarrow f(x_n)\preceq f(x),\ \forall n\in N\cup\{0\},$$

(ii) $(X, p, \preceq)$ has the f-DCL (decreasing-convergence-lower bound) property if f-image of every decreasing convergent sequence $\{x_n\}$ in X is bounded below by f-image of its limit (as a lower bound), i.e.,

$$x_n \downarrow x \Rightarrow f(x_n) \succeq f(x), \forall n \in N \cup \{0\},$$

(iii) $(X, p, \preceq)$ has the f-MCB (monotone-convergence-bounded) property if it has the f-ICU property as well as f-DCL property. Notice that under the restriction $f = I$, the identity mapping on X, Definition 2.7 reduces to Definition 12 of Alam et al. [4].

Definition 2.8. Let $(X, p, \preceq)$ be an ordered partial metric space and f a self-mapping on X. We say that $(X, p, \preceq)$ has the f-TCC property if every termwise monotone convergent sequence $\{x_n\}$ in X has a subsequence, whose f-image is termwise bounded by f-image of the limit of $\{x_n\}$ (as a c-bound), i.e., $x_n \updownarrow x \Rightarrow \exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $f(x_{n_k}) \prec\succ f(x)$, $\forall k \in N \cup \{0\}$.

Notice that under the restriction $f = I$, the identity mapping on X, Definition 2.8 reduces to Definition 2.5 of Alam et al. [3].

Definition 2.9 ([2])**.** Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X. We say f is g-comparable (or weakly g-monotone or $(g, \prec\succ)$-preserving) if for any $x, y \in X$,

$$g(x) \prec\succ g(y) \Rightarrow f(x) \prec\succ f(y).$$

Notice that on setting $g = I$, the identity mapping on X, Definition 2.9 reduces to Definition 3.1 of Alam et al. [2].

Definition 2.10. Let (X, p) be a partial metric space and f and g two self-mappings on X. Then the pair (f, g) is said to be partial compatible if the following conditions hold:

(a) $p(x, x) = 0 \Rightarrow p(gx, gx) = 0$,

(b) $\lim_{n\to\infty} p(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fx_n \to t$ and $gx_n \to t$ for some $t \in X$.

It is clear that Definition 2.7 extend and generalizes the notion of compatibility introduced by Jungck [13].

The following family of control functions is essentially due to Boyad and Wong [8] $\Psi = \{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t$ for each $t > 0$ and φ is right-upper semicontinuous$\}$.

Mukherjea [17] introduced the following family of control functions:

$\Theta = \{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t$ for each $t > 0$ and φ is right continuous$\}$.

The following family of control functions found in literature is more natural.

$\Im = \{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t$ for each $t > 0$ and φ is continuous$\}$.

The following family of control functions is due to Lakshmikantham and Ciric [14].

$\Phi = \{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t$ for each $t > 0$ and $\lim_{r\to\ t^+} \varphi(t) < t$ for each $t > 0\}$.

The following family of control functions is indicated by Boyd and Wong [8] but was later used in Jotic [12].

$\Omega = \{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t$ for each $t > 0$ and $\limsup_{r \to\ t^+} \varphi(t) < t$ for each $t > 0\}$.

Proposition 2.2 ([4]). *The class Ω enlarges the classes Ψ, Θ, $\Im$ and Φ under the following inclusion relation*:

$$\Im \subset \Theta \subset \Psi \subset \Omega \text{ and } \Im \subset \Theta \subset \Phi \subset \Omega.$$

Lemma 2.2 ([4]). *Let $\varphi \in \Omega$. If $\{x_n\} \subset (0, \infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n)$, $\forall n \in N$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main results

We prove our main result as follows:

Theorem 3.1. *Let $(X, p, \preceq)$ be an ordered partial metric space and f, g be two self mappings on X. Suppose that the following hold*:

(a) $f(X) \subseteq g(X)$,

(b) *f is g-increasing,*

(c) *there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$,*

(d) *there exists $\varphi \in \Omega$ such that*

$$p(fx, fy) \preceq \varphi(p(gx, gy)),\ \forall x, y \in X \text{ with } g(x) \prec\succ g(y), \tag{1}$$

(e) *(X, p) is complete,*

(f) *(f, g) is partial compatible pair,*

(g) *f and g continuous mappings, or alternately*

(g′) *g is continuous and $(X, p, \preceq)$ has f-ICU property.*

Then (f, g) have a coincidence point, that is there exists $x \in X$ such that $f(x) = g(x)$. Moreover, we have $p(x, x) = p(fx, fx) = p(gx, gx) = 0$.

Proof. According to assumption (d) the contractivity condition $p(fx, fy) \leq \varphi(p(gx, gy))$ holds for any $x, y \in X$ under two possibilities:

$$\text{either } g(x) \preceq g(y) \text{ or } g(x) \succeq g(y).$$

If it is satisfied for first possibility, then by the symmetry of partial metric space it must be satisfied for second possibility and vice-versa. Therefore on applying the given contractivity condition these two possibilities are same and hence we use only first to prove our result.

In light of assumption (c) if $g(x_0) = f(x_0)$, then x_0 is coincidence point of f and g and hence the proof. Otherwise if $g(x_0) \neq f(x_0)$, then we have $g(x_0) \prec f(x_0)$.

So in light of assumption (a) (i.e., $f(X) \subseteq g(X)$). We can choose $\{x_1\}$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$ we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process, we can define a sequence $\{x_n\} \subset X$ such that

$$g(x_{n+1}) = f(x_n),\ \forall n \geq 0. \tag{2}$$

Now, we claim that gx_n is an increasing sequence, i.e.,

$$g(x_n) \preceq g(x_{n+1}),\ \forall n \geq 0. \tag{3}$$

We prove this assertion by mathematical induction. On using (2) with $n = 0$ and in light of (c), we have $g(x_0) \preceq f(x_0) = g(x_1)$. Thus (3) holds for $n = 0$. Suppose that (3) holds for $n = r > 0$, i.e.,

$$g(x_r) \preceq g(x_{r+1}) \tag{4}$$

then we have to show that (3) holds for $n = r + 1$. To verify this we use (2), (4) and in light of assumption (b), we have

$$g(x_{r+1}) = f(x_r) \preceq f(x_{r+1}) = g(x_{r+2}).$$

Thus, by induction (3) holds for all $n \geq 0$. Suppose that there exists $n \in N$ such that $p(fx_n, fx_{n+1}) = 0$ which implies that $fx_n = fx_{n+1}$, i.e., $gx_{n+1} = fx_{n+1}$, then x_{n+1} is a coincidence point of f and g, so we are through. On the other hand we can assume that $fx_n \neq fx_{n+1},\ \forall n \in N \cup \{0\}$, i.e.,

$$p(fx_n, fx_{n+1}) > 0,\ \forall n \geq 0. \tag{5}$$

We will show that

$$p(fx_n, fx_{n+1}) \leq \varphi(p(fx_{n-1}, fx_n)),\ \forall n \in N, \tag{6}$$

using (3) and in light of the assumption (d) with $x = x_n$, $y = x_{n+1}$ we get

$$p(fx_n, fx_{n+1}) \leq \varphi(p(gx_n, gx_{n+1})) = \varphi(p(fx_{n-1}, fx_n)).$$

Since φ is nondecreasing repeating n-times, we get

$$p(fx_n, fx_{n+1}) \leq \varphi^n(p(fx_0, fx_1)). \tag{7}$$

Letting $n \to \infty, \varphi^n(t) \to 0$ for all $t > 0$, we obtain

$$\lim_{n\to\infty} p(fx_n, fx_{n+1}) = 0. \tag{8}$$

On the other hand, we have

$$\begin{aligned} p^s(fx_n, fx_{n+1}) &= 2p(fx_n, fx_{n+1}) - p(fx_n, fx_n) - p(fx_{n+1}, fx_{n+1}) \\ &\leq 2p(fx_n, fx_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in this inequality, using (8), we get

$$\lim_{n\to\infty} p^s(fx_n, fx_{n+1}) = 0. \tag{9}$$

Next, we shall show that $\{fx_n\}$ is a Cauchy sequence in the metric space (X, p^s). On contrary suppose that $\{fx_n\}$ is not a Cauchy sequence in (X, p^s). Then there exists $\epsilon > 0$ such that for each positive integer k, there exists two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $n_k > m_k > k$, and

$$p^s(fx_{m_k}, fx_{n_k}) \geq \epsilon. \tag{10}$$

Since $p^s(x,y) \leq 2p(x,y)$ for all $x,y \in X$, from (10) for all $k \geq 0$, we have $n_k > m_k > k$ and $p^s(fx_{m_k}, fx_{n_k}) \geq \epsilon/2$. Without loss of generality, we can assume that

$$n_k > m_k > k,\ p(fx_{m_k}, fx_{n_k}) \geq \epsilon/2,\ p(fx_{m_k}, fx_{n_k-1}) < \epsilon/2. \tag{11}$$

From(11) and triangular inequality of partial metric space, we have

$$\begin{aligned}\epsilon/2 &\leq p(fx_{m_k}, fx_{n_k})\\ &\leq p(fx_{m_k}, fx_{n_k+1}) + p(fx_{n_k+1}, fx_{n_k}) - p(fx_{n_k+1}, fx_{n_k+1})\\ &< \epsilon/2 + p(fx_{n_k+1}, fx_{n_k}).\end{aligned}$$

Letting $k \to \infty$ and using (8), we get

$$\lim_{k\to\infty} p(fx_{m_k}, fx_{n_k}) = \epsilon/2. \tag{12}$$

Again using triangular inequality and in light of our contrary supposition that $\{fx_n\}$ is not Cauchy, we obtain the following sequences tend to $\epsilon/2$ when $k \to \infty$.

$$\begin{aligned}\lim_{k\to\infty} p(fx_{m_k+1}, fx_{n_k}) &= \lim_{k\to\infty} p(fx_{m_k}, fx_{n_k+1})\\ &= \lim_{k\to\infty} p(fx_{m_k+1}, fx_{n_k+1}) = \epsilon/2.\end{aligned} \tag{13}$$

On the other hand, we have

$$p(fx_{n_k}, fx_{m_k}) \leq p(fx_{n_k}, fx_{n_k+1}) + p(fx_{n_k+1}, fx_{m_k}) - p(fx_{n_k+1}, fx_{n_k+1}).$$

Owing to assumption (d), $p(fx_{n_k+1}, fx_{m_k}) \leq \varphi(p(fx_{n_k}, fx_{m_k-1}))$, we have

$$p(fx_{m_k}, fx_{n_k} \leq p(fx_{n_k}, fx_{n_k+1}) + \varphi(p(fx_{n_k}, fx_{m_k-1})) - p(fx_{n_k+1}, fx_{m_k+1}).$$

Letting $k \to \infty$, using (12), (13) and continuity of φ, we have

$$\epsilon/2 \leq \varphi(\epsilon/2) < \epsilon/2,$$

a contraction.

Thus our supposition that $\{fx_n\}$ is not Cauchy sequence was wrong. Therefore $\{fx_n\}$ is a Cauchy sequence in the metric space (X, p^s) and

$$\lim_{m,n\to\infty} p^s(fx_n, fx_m) = 0. \tag{14}$$

Now, since (X,p) is complete, from Lemma 2.1, (X,p^s) is a complete metric space. Therefore, the sequence $\{fx_n\}$ converges to some $x \in X$, that is

$$\lim_{n\to\infty} p^s(fx_n, x) = \lim_{n\to\infty} p^s(gx_{n+1}, x) = 0.$$

In light of property (b) of Lemma 2.1, we have

$$p(x,x) = \lim_{n\to\infty} p(fx_n, x) = \lim_{n\to\infty} p(gx_{n+1}, x) = \lim_{n,m\to\infty} p(fx_n, fx_m). \tag{15}$$

On the other hand, from property (p_2) of a partial metric space, we have

$$p(fx_n, fx_n) \leq p(fx_n, fx_{n+1}),\ \forall n \in N.$$

Letting $n \to \infty$ in the above inequality and using (7), we have

$$\lim_{n\to\infty} p(fx_n, fx_n) = 0.$$

Therefore, from the definition of p^s and using (14), we get

$$\lim_{m,n\to\infty} p(fx_n, fx_m) = 0.$$

Thus, from (15), we have

$$p(x,x) = \lim_{n\to\infty} p(fx_n, x) = \lim_{m,n\to\infty} p(fx_n, fx_m) = 0. \tag{16}$$

Now, since f is continuous, and using (16), we have

$$\lim_{n\to\infty} p(f(fx_n), fx)) = p(fx, fx),\ \forall n \in N. \tag{17}$$

Using the triangular inequality, we have

$$\begin{aligned} p(fx, gx) \le\ & p(fx, ffx_n) + p(ffx_n, gfx_{n+1}) + p(gfx_{n+1}, gx) \\ & - p(ffx_n, ffx_n) - p(gfx_{n+1}, gfx_{n+1}) \\ =\ & p(fx, ffx_n) + p(fgx_{n+1}, gfx_{n+1}) + p(gfx_{n+1}, gx) \\ & - p(ffx_n, ffx_n) - p(gfx_{n+1}, gfx_{n+1}). \end{aligned} \tag{18}$$

Now, letting $n \to \infty$ in (18) and using (16), (17), assumption f, and continuity of g, we have

$$p(fx, gx) \le p(fx, fx) + 0 + p(gx, gx) - p(fx, fx) - p(gx, gx) = 0.$$

Also, we know that $p(fx, gx) \ge 0$. So we conclude that $p(fx, gx) = 0$, i.e., $fx = gx$. Thus $x \in X$ is a coincidence point of f and g.

Alternately, suppose that g is continuous and $(X, p, \preceq)$ has f-ICU property, On account of (3) and (16), we have $fx_n \uparrow x$ which gives rise

$$g(fx_{n+1}) \preceq gx,\ \forall n \ge 0. \tag{19}$$

Using (19) and assumption (d), we have $p(ffx_{n+1}, fx) \le \varphi(p(gfx_{n+1}, gx))$, $\forall n \ge 0$. Now we claim that

$$p(ffx_{n+1}, fx) \le p(gfx_{n+1}, gx),\ \forall n \in N. \tag{20}$$

In order to verify this, two different possibilities arising here. We resolve them by partitioning N such that $N = N^0 \cup N^+$ and $N^0 \cap N^+ = \phi$ verifying that,

(c$_1$) $p(gfx_{n+1}, gx) = 0$, $\forall n \in N^0$,

(c$_2$) $p(gfx_{n+1}, gx) > 0$, $\forall n \in N^+$.

In case (c$_1$), $p(gfx_{n+1}, gx) = 0$, i.e., $g(fx_{n+1}) = gx$, using Proposition 2.1, we have $f(fx_{n+1}) = fx$. Thus (20) holds for all $n \in N^0$. In case (c$_2$), owing to definition of Ω we have $p(ffx_{n+1}, fx) \le \varphi(p(gfx_{n+1}, gx)) < p(gfx_{n+1}, gx), \forall n \in N^+$. Finally (20) holds for all $n \in N$.

Also from property (p_2) of partial metric spaces

$$p(fgx_{n+1}, fgx_{n+1}) \le p(gfx_{n+1}, fgx_{n+1}),\ \forall n \in N.$$

Letting $n \to \infty$, in light of the assumption (f) we have

$$\lim_{n\to\infty} p(fgx_{n+1}, fgx_{n+1}) = 0, \forall n \in N. \tag{21}$$

On using triangular inequality, (16), (19), (20) and (21), we have

$$\begin{aligned} p(fx, gx) &\leq p(fx, ffx_n) + p(ffx_n, gfx_{n+1}) + p(gfx_{n+1}, gx) \\ &\quad - p(ffx_n, ffx_n) - p(gfx_{n+1}, gfx_{n+1}) \\ &= p(fx, ffx_n) + p(fgx_{n+1}, gfx_{n+1}) + p(gfx_{n+1}, gx) \\ &\quad - p(fgx_{n+1}, fgx_{n+1}) - p(gfx_{n+1}, gfx_{n+1}) \\ &= 2p(gfx_{n+1}, gx) + p(fgx_{n+1}, gfx_{n+1}) \\ &\quad - p(fgx_{n+1}, fgx_{n+1}) - p(gfx_{n+1}, gfx_{n+1}). \end{aligned}$$

Letting $n \to \infty$, we have $p(fx, gx) \leq 2p(gx, gx) + 0 - p(gx, gx) = 0$.

So we have $fx = gx$. Thus $x \in X$ is a coincidence point of f and g. This completes the proof. □

Corollary 3.1. *Theorem 3.1 remains true if we replace condition* (c) *and* (g′) *by the conditions* (c′) *and* (g″) *respectively* (*besides retaining the rest of the assumptions*):

(c′) *there exists* $x_0 \in X$ *such that* $g(x_0) \succeq f(x_0)$,

(g″) g *is continuous and* $(X, p, \preceq)$ *has* f*-DCL property.*

Corollary 3.2. *Theorem 3.1 and Corollary 3.1 remains true if we replace condition* (c) *and* (g′) *by the conditions* (c″) *and* (g‴) *respectively* (*besides retaining the rest of assumptions*):

(c″) *there exists* $x_0 \in X$ *such that* $g(x_0) \prec\succ f(x_0)$,

(g‴) g *is continuous and* $(X, p, \preceq)$ *has* f*-MCB property.*

Theorem 3.2. *Let* $(X, p, \preceq)$ *be an ordered partial metric space and* f, g *be two self mappings on* X. *Suppose that the followings hold*:

(a) $f(X) \subseteq g(X)$,

(b) f *is* g*-comparable,*

(c) *there exists* $x_0 \in X$ *such that* $g(x_0) \prec\succ f(x_0)$,

(d) *there exists* $\varphi \in \Omega$ *such that*

$$p(fx, fy) \preceq \varphi(p(gx, gy)), \forall x, y \in X \text{ with } g(x) \prec\succ g(y),$$

(e) (X, p) *is complete,*

(f) (f, g) *is partial compatible pair,*

(g) f *and* g *are continuous mappings, or alternately*

(g′) g *is continuous and* $(X, p, \preceq)$ *has* f*-TCC property.*

Then (f, g) *have a coincidence point, that is there exists* $x \in X$ *such that* $f(x) = g(x)$. *Moreover, we have* $p(x, x) = p(fx, fx) = p(gx, gx) = 0$.

Proof. The proof of this theorem starts along the lines of the proof of Theorem 3.3 proved in [2] and runs up to the lines where by induction the following hold:

$$g(x_{r+1}) = f(x_r) \prec\succ f(x_{r+1}) = g(x_{r+2}).$$

Now following the lines of the proof of our earlier Theorem 3.1 in light of assumptions (a)-(g) the proof is accomplished.

Again, alternately suppose that g is continuous and $(X, p, \preceq)$ has f-TCC property. As $f(x_n) \updownarrow z$, $\exists$ a subsequence $\{y_{n_k}\}$ of $\{fx_n\}$ such that

$$g(y_{n_k}) \prec\succ g(z), \forall k \in N \cup \{0\}. \tag{22}$$

Now $\{fx_n\} \subset f(X)$ and $\{y_{n_k}\} \subset f(x_{n_k})$, so $\exists\{x_{n_k}\} \subset X$ such that $y_{n_k} = f(x_{n_k+1})$, which implies that

$$g(fx_{n_k+1}) \prec\succ g(z), \forall k \in N \cup \{0\}. \tag{23}$$

Since $fx_{n_k+1} \to z$, so equations (6)-(19) also hold for $\{x_{n_k}\}$ instead of $\{x_n\}$. On using (20), and proceeding on the lines of the proof of the Theorem 3.1, this theorem can be proved. $\square$

4. Uniqueness results

Now, we prove results related to uniqueness of a point of coincidence and coincidence point corresponding to previous results. For a pair f and g of self-mappings on a non empty set X, we classify the following sets:
$C(f, g) = \{x \in X : gx = fx\}$, i.e., the set of all coincidence points of f and g,
$\bar{C}(f, g) = \{\bar{x} \in X :$ there exists an $x \in X$ such that $\bar{x} = gx = fx$ $\}$, i.e., the set of all points of coincidence of f and g.

Theorem 4.1. *In addition to the hypotheses* (a)-(d) *along with* g' *of Theorem 3.1* (*resp. Corollary 3.1, Corollary 3.2 and Theorem 3.2*), *If the following condition holds*:
(u_0) *for all* $x, y \in X$, $\exists z \in X$ *such that* $g(x) \prec\succ g(z)$ *and* $g(y) \prec\succ g(z)$, *then* f *and* g *have a unique point of coincidence.*

Proof. The proof of this theorem runs along the lines of proof of Theorem 5 of [4] and we conclude that

$$\lim_{n\to\infty} p(gx, gz_n) = 0, \forall n \geq 0. \tag{24}$$

Similarly one can prove that

$$\lim_{n\to\infty} p(gy, gz_n) = 0, \forall n \geq 0. \tag{25}$$

Also, from property p_2 of partial metric space, we have

$$p(gz_n, gz_n) \leq p(gy, gz_n).$$

Letting $n \to \infty$, we obtain

$$p(gz_n, gz_n) = 0. \tag{26}$$

On using triangular inequality, (24), (25), and (26) we obtain

$$p(\bar{x}, \bar{y}) = p(gx, gy) \leq p(gx, gz_n) + p(gz_n, gy) - p(gz_n, gz_n) \to 0$$

as $n \to \infty$, this implies that

$$\bar{x} = \bar{y}.$$

Thus, f and g have a unique point of coincidence. $\square$

Example 4.1. Let $X{=}[0\ \infty)$. Then $(X, p, \preceq)$ is an ordered partial metric space equipped with the natural partial order and partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $f, g : X \to X$ by $f(x) = x$, $g(x) = 3x$ and $\varphi : [0\ \infty) \to [0\ \infty)$ such that $\varphi(t) = \frac{7}{9}t$ then f is g-increasing. Also, for $x, y \in X$ with $y \leq x$, we have

$$p(fx, fy) = x < \frac{7}{9}(3x) = \frac{7}{9}p(gx, gy) = \varphi(p(gx, gy)),$$

i.e., f and g satisfy the contractivity condition (d). It is easy to show that all the other conditions mentioned in Theorem 3.1 are also satisfied. Notice that if condition (u_0) of Theorem 4.1 holds then f and g have unique point of coincidence.

Remark 4.1. If we suppose $X{=}[-\frac{1}{2}\frac{1}{2}]$, $f(x) = x^4$ and $g(x) = x^2$, then f is g-comparable with TCC property of the ordered partial metric space. Also, in light of the above example it is easy to show that f and g satisfy the contractivity condition and all the other conditions mentioned in Theorem 3.2. Notice that under the assumption (u_0) of Theorem 4.1, f and g have unique point of coincidence.

References

[1] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), no. 1, 109–116.

[2] A. Alam and M. Imdad, *Comparable linear contractions in ordered metric spaces*, Fixed Point Theory, in press.

[3] ———, *Monotone generalized contractions in ordered metric spaces*, Bull. Korean Math. Soc. **53** (2016), no. 1, 61–81.

[4] A. Alam, A. R. Khan, and M. Imdad, *Some coincidence theorems for genralized nonlinear contractions in ordered metric spaces with applications*, Fixed Point Theory Appl. **2014** (2014), 216, 30 pp.

[5] I. Altun and A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl. **2011** (2011), Article ID 508730, 10 pp.

[6] I. Altun and H. Simsek, *Some fixed point theorems on ordered metric spaces and applications*, Fixed Point Theory Appl. **2010** (2010), Article ID 621469, 17 pp.

[7] I. Altun, F. Sola, and H. Simsek, *Generalized contractions on partial metric spaces.* Topology Appl. **157** (2010), no. 18, 2778–2785.

[8] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

[9] M. Bukatin, R. Kopperman, S. Matthews, and H. Pajoohesh, *Partial metric spaces*, Amer. Math. Monthly **116** (2009), no. 8, 708–718.

[10] M. A. Bukatin and S. Yu. Shorina, *Partial metrics and co-continuous valuations*, In: Nivat M, et al. (eds.) Foundations of software science and computation structures (Lisbon, 1998), 125–139, Lecture Notes in Comput. Sci., 1378, Springer, Berlin, 1998.

[11] L. Ciri'c, N. Cakic, M. Rajovic, and J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl. **2008** (2008), Art. ID 131294, 11 pp.

[12] N. Joti'c, *Some fixed point theorems in metric spaces*, Indian J. Pure Appl. Math. **26** (1995), no. 10, 947–952.

[13] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.

[14] V. Lakshmikantham and L. Ciri'c, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), no. 12, 4341–4349.

[15] S. G. Matthews, *Partial metric topology*, In Proceedings of the 8th Summer Conference on General Topology and Applications, pp. 183–197, Ann. New York Acad. Sci., 728, New York Acad. Sci., New York, 1994.

[16] ———, *An extensional treatment of lazy data on deadlock*, Theoret. Comput. Sci. **151** (1995), no. 1, 195–205.

[17] A. Mukherjea, *Contractions and completely continuous mappings*, Nonlinear Anal. **1** (1977), no. 3, 235–247.

[18] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), no. 3, 223–239.

[19] ———, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica **23** (2007), no. 12, 2205–2212.

[20] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), no. 2, 1241–1252.

[21] S. Oltra and O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste **36** (2004), no. 1-2, 17–26.

[22] S. Radenović and Z. Kadelburg, *Generalized weak contractions in partially ordered metric spaces*, Comput. Math. Appl. **60** (2010), no. 6, 1776–1783.

[23] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1435–1443.

[24] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl. **2010** (2010), Article ID 493298, 6 pp.

[25] I. A. Rus, *Fixed point theory in partial metric spaces*, An. Univ. Vest. Timiş. Ser. Mat.-Inform. **46** (2008), no. 2, 149–160.

[26] J. J. M. M. Rutten, *Weighted colimits and formal balls in generalized metric spaces*, Topology Appl. **89** (1998), no. 1-2, 179–202.

[27] S. Samet, M. Rajovic, R. Lazovic, and R. Stojiljkovic, *Common fixed point results for nonlinear contractions in ordered partial metric spaces*, Fixed Point Theory Appl. **2011** (2011), 71, 14 pp.

[28] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. **117** (1986), no. 1, 100–127.

[29] ———, *Fixed points for monotone iteratively local contractions*, Demonstr. Math. **19** (1986), no. 1, 171–180.

[30] ———, *Nieto-Lopez theorems in ordered metric spaces*, Math. Student **81** (2012), no. 1-4, 219–229.

[31] ———, *Ran-Reurings fixed point results in ordered metric spaces*, Libertas Math. **31** (2011), 49–55.

[32] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol. **6** (2005), no. 2, 229–240.

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