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SOME RESULTS ON COMPLEX DIFFERENTIAL-DIFFERENCE ANALOGUE OF BRÜCK CONJECTURE

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Abstract. In this paper, we utilize the Nevanlinna theory and uniqueness theory of meromorphic function to investigate the differential-difference analogue of Brück conjecture. In other words, we consider $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $f'(z)$ share one value or one small function, and then obtain the precise expression of transcendental entire function $f(z)$ under certain conditions, where $\eta \in \mathbb{C} \setminus \{0\}$ is a constant such that $f(z+\eta) - f(z) \not\equiv 0$.

1. Introduction and results

In this paper, we assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory [9, 11]. In addition, we use notations $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of the zero-sequence and the order of growth of meromorphic function $f(z)$ respectively.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a be a complex number in the extended complex plane. We say that $f(z)$ and $g(z)$ share a CM (IM) provided that $f(z)$ and $g(z)$ have the same a-points counting multiplicities (ignoring multiplicities).

In 1996, Brück [2] posed a well-known conjecture.

Conjecture (See [2]). *Let $f(z)$ be a nonconstant entire function with hyper-order $\sigma_2(f) < \infty$, and $\sigma_2(f)$ be not a positive integer. If $f(z)$ and $f'(z)$ share the finite value a CM, then*

$$\frac{f'(z) - a}{f(z) - a} = c,$$

where c is a nonzero constant.

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The conjecture has been verified in the special cases when $a = 0$ (see [2]), or when $f(z)$ is an entire function of finite order (see [7]).

Recently, some results on difference analogues of Brück conjecture were considered in [3, 4, 10, 12, 13, 14]. Here, we recall the following results.

Theorem A (See [10]). *Let $f(z)$ be a meromorphic function of $\sigma(f) < 2$, and η be a non-zero constant. If $f(z)$ and $f(z+\eta)$ share the finite value a and ∞ CM, then*

$$\frac{f(z+\eta)-a}{f(z)-a} = \tau$$

for some constant τ.

In [10], Heittokangas et al. gave the example $f(z) = e^{z^2} + 1$ which shows that $\sigma(f) < 2$ can not be relaxed to $\sigma(f) \leq 2$.

It is well-known that $\Delta_\eta f(z) = f(z+\eta) - f(z)$ (where $\eta \in \mathbb{C}\backslash\{0\}$ is a constant such that $f(z+\eta) - f(z) \not\equiv 0$) is regarded as the difference counterpart of $f'(z)$. For a transcendental entire function $f(z)$ of finite order which has a finite Borel exceptional value, Chen and Yi [4] and Chen [3] considered the problem that $\Delta_\eta f(z)$ and $f(z)$ shared one finite value CM and obtained the following results.

Theorem B (See [4]). *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a, and let η be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta_\eta f(z)$ and $f(z)$ share a CM, then*

$$a = 0 \text{ and } \frac{f(z+\eta)-f(z)}{f(z)} = c$$

for some constant c.

Theorem C (See [3]). *Let $f(z)$ be a transcendental entire function of finite order that is of a finite Borel exceptional value α, and $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share $a(\neq \alpha)$ CM, then*

$$\frac{\Delta_\eta f(z) - a}{f(z) - a} = \frac{a}{a-\alpha}.$$

Most recently, Liu and Dong [15] considered the differential-difference analogue of Brück conjecture and obtained the following result.

Theorem D (See [15]). *Suppose that $f(z)$ is an entire solution of equation*

$$f'(z) - a(z) = e^{P(z)}(f(z+c) - a(z)),$$

where $c \in \mathbb{C} \setminus \{0\}$ is a constant, $P(z)$ is a polynomial and $a(z)$ is an entire function with $\sigma(a) < \sigma(f)$. If $\lambda(f-a) < \sigma(f)$, then $\sigma(f) = 1 + \deg P(z)$.

In this paper, we consider $\Delta_\eta f(z)$ and $f'(z)$ share one value or one small function, and obtain more precise results than Theorem D in the following.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order, $\eta \in \mathbb{C} \setminus \{0\}$ be a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $a(z)$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f-a) < \sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, then one of the following two cases holds*:

(i) *If $a(z) \not\equiv 0$, then*

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = 1 \quad \text{and} \quad f(z) = a(z) + H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$ and $c \in \mathbb{C} \setminus \{0\}$ is a constant satisfying $e^{c\eta} = 1 + c$.

(ii) *If $a(z) \equiv 0$, then*

$$\frac{\Delta_\eta f(z)}{f'(z)} = A \quad \text{and} \quad f(z) = H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$, $A,\ c \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function of finite order, η be a non-zero constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $b(z)$ be an entire function such that $\sigma(b) < 1$ and $\lambda(f-b) < \sigma(f)$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, where $a(z)$ is an entire function satisfying $\sigma(a) < 1$ and $a(z) \not\equiv b'(z)$, then*

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = A \quad \text{and} \quad f(z) = b(z) + H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$, $A,\ c \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$.

Remark 1.1. From the conditions of Theorem 1.2, we see that $a(z) \not\equiv b'(z)$, if $a(z) \equiv b(z)$, then $a(z) \not\equiv 0$, which is the case (i) of Theorem 1.1. In Theorem 1.2, if $b(z) \equiv b$ and $a(z) \equiv a$ or $b'(z) \not\equiv 0$ and $a(z) \equiv 0$, the following corollaries can be obtained.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order which has a finite Borel exceptional b, η be a non-zero constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$. If $\Delta_\eta f(z)$ and $f'(z)$ share $a(\neq 0)$ CM, then*

$$\frac{\Delta_\eta f(z) - a}{f'(z) - a} = A \quad \text{and} \quad f(z) = b + H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$, $A,\ c \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$.

Remark 1.2. From Corollary 1.1, if $a = b(\neq 0)$, which is also a special case in Theorem 1.1 when $a(z) \equiv a(\neq 0)$.

Corollary 1.2. *Let $f(z)$ be a transcendental entire function of finite order, η be a non-zero constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$, $b(z)$ be a* *nonconstant entire function such that* $\sigma(b) < 1$ *and* $\lambda(f-b) < \sigma(f)$. *If* $\Delta_\eta f(z)$ *and* $f'(z)$ *share* 0 *CM, then*

$$\frac{\Delta_\eta f(z)}{f'(z)} = A \quad \text{and} \quad f(z) = b(z) + H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ *is an entire function with* $\lambda(H) = \sigma(H) < 1$, A, $c \in \mathbb{C} \setminus \{0\}$ *are constants satisfying* $e^{c\eta} = 1 + Ac$.

Remark 1.3. From the conditions of Corollary 1.2, we know that $b'(z) \not\equiv 0$. If $b'(z) \equiv 0$, namely, $b(z)$ is a constant, then Corollary 1.2 is still valid according to the following Lemma 2.5.

Example 1.1. Suppose that $f(z) = z + e^{cz}$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant. Then $\lambda(f-z) < \sigma(f)$. Let $\eta = 1$ and c satisfy $e^c = 1 + c$, we see that $\Delta_\eta f(z) = 1 + ce^{cz} = f'(z)$. Then $\frac{\Delta_\eta f(z) - z}{f'(z) - z} = 1$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share z CM.

Example 1.2. Suppose that $f(z) = e^{cz}$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant. Then $\lambda(f) < \sigma(f)$. Let $\eta = \log 2$ and c satisfy $2^c = 1 + 2c$, we see that $\Delta_\eta f(z) = 2ce^{cz} = 2f'(z)$. Then $\frac{\Delta_\eta f(z)}{f'(z)} = 2$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share 0 CM.

Example 1.3. Suppose that $f(z) = z^2 + e^{cz}$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant. Then $\lambda(f - z^2) < \sigma(f)$. Let $\eta = 1$ and c satisfy $e^c = 1 + \frac{1}{2}c$, we see that $\Delta_\eta f(z) = 2z + 1 + \frac{1}{2}ce^{cz}$ and $f'(z) = 2z + ce^{cz}$. Then $\frac{\Delta_\eta f(z) - 2(z+1)}{f'(z) - 2(z+1)} = \frac{1}{2}$, that is, $\Delta_\eta f(z)$ and $f'(z)$ share $2(z+1)(\not\equiv 2z)$ CM.

2. Lemmas for the proof of Theorems

Lemma 2.1 (See [16, p. 77])**.** *Suppose that* $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ *and* $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 1)$ *are entire functions satisfying*:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$.

(ii) *The order of* $f_j(z)$ *is less than the order of* $e^{g_k(z)}$ *for* $1 \leq j \leq n+1$, $1 \leq k \leq n$. *And furthermore, the order of* $f_j(z)$ *is less than the order of* $e^{g_h(z) - g_k(z)}$ *for* $n \geq 2$ *and* $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.

Then $f_j(z) \equiv 0$, $(j = 1, 2, \ldots, n+1)$.

***ε*-set.** Following Hayman [8, pp. 75–76], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.2 (See [1, Lemma 3.3]). *Let $f(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1$. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{f'(z+c)}{f(z+c)} \to 0 \quad \text{and} \quad \frac{f(z+c)}{f(z)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large $z \notin E$, the function $f(z)$ has no zeros or poles on $|\zeta - z| \leq h$.

Lemma 2.3 (See [6, Corollary 2]). *Let $f(z)$ be a transcendental meromorphic function of finite order σ, k, j $(k > j \geq 0)$ be integers. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.4 (See [5, Theorem 8.2]). *Let $f(z)$ be a meromorphic function of finite order σ, η be a non-zero complex number, and $\varepsilon > 0$ be given real constants, then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left| \frac{f(z+\eta)}{f(z)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.5. *Let $f(z)$ be a transcendental entire function of finite order which has a finite Borel exceptional b, $\eta \in \mathbb{C} \setminus \{0\}$ be a constant such that $\Delta_\eta f(z) = f(z+\eta) - f(z) \not\equiv 0$. If $\Delta_\eta f(z)$ and $f'(z)$ share 0 CM, then*

$$\frac{\Delta_\eta f(z)}{f'(z)} = A \quad \text{and} \quad f(z) = b + H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$, A, $c \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$.

Proof. Since $f(z)$ has a Borel exceptional b, by the Hadamard's factorization theorem [16, Theorem 2.5], we obtain

$$f(z) = b + h(z)e^{Q(z)}, \tag{2.1}$$

where $h(z)(\not\equiv 0)$ is an entire function, $Q(z)$ is a polynomial such that

$$\sigma(h) = \lambda(h) = \lambda(f-b) < \sigma(f) = \deg Q(z). \tag{2.2}$$

Furthermore, $\Delta_\eta f(z)$ and $f'(z)$ share 0 CM, we have

$$\frac{\Delta_\eta f(z)}{f'(z)} = e^{P(z)}, \tag{2.3}$$

where $P(z)$ is a polynomial. It follows from (2.2) and (2.3) that

$$\deg P(z) \leq \deg Q(z). \tag{2.4}$$

Substituting (2.1) into (2.3) yields

$$h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) = (h'(z) + h(z)Q'(z))e^{P(z)}. \tag{2.5}$$

In what follows, we assume that $P(z) \not\equiv 0$ and $\deg P(z) = p$, we discuss two cases: $1 \leq \deg P(z) = \deg Q(z)$ and $0 \leq \deg P(z) < \deg Q(z)$. Denote

$$P(z) = a_p z^p + a_{p-1}z^{p-1} + \cdots + a_0, \quad Q(z) = b_q z^q + b_{q-1}z^{q-1} + \cdots + b_0, \tag{2.6}$$

where $a_p(\neq 0), \ldots, a_0,\ b_q(\neq 0), \ldots, b_0$ are constants, $q = \sigma(f) \geq 1$ is an integer.

Case 1. $1 \leq p = q$. Since $\deg(Q(z+\eta) - Q(z)) = q - 1$, $h(z) \not\equiv 0$ and $\sigma(h) < q$, then $h'(z) + h(z)Q'(z) \not\equiv 0$, we see that the order of growth of the left side of (2.5) is less than q, and the order of growth of the right side of (2.5) is q. This is a contradiction.

Case 2. $0 \leq p < q$. If $0 \leq p < q-1$, then (2.5) can be rewritten as

$$e^{Q(z+\eta)-Q(z)} = \left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}. \tag{2.7}$$

If $\sigma(h) < 1$, since $\deg(Q(z+\eta) - Q(z)) = q - 1 \geq 1$ and $\deg P(z) < q-1$, we know that the order of growth of the left side of (2.7) is $q-1$, and the order of growth of the right side of (2.7) is less than $q-1$, a contradiction. Then we have $\sigma(h) \geq 1$.

By Lemma 2.3, for any given $\varepsilon_1 > 0$, there exists a set $E_1 \subset (1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{h'(z)}{h(z)}\right| \leq |z|^{\sigma(h)-1+\varepsilon_1}. \tag{2.8}$$

By Lemma 2.4, for any given $\varepsilon_2 > 0$, there exists a set $E_2 \subset (1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we have

$$\exp\{-r^{\sigma(h)-1+\varepsilon_2}\} \leq \left|\frac{h(z+\eta)}{h(z)}\right| \leq \exp\{r^{\sigma(h)-1+\varepsilon_2}\}. \tag{2.9}$$

Set $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ $\left(0 < \varepsilon < \min\left\{\frac{q-\sigma(h)}{3}, \frac{q-1-p}{3}\right\}\right)$, there exists $r_0 > 0$ such that for all z satisfying $|z| = r > r_0$, we have

$$r^{q-1-\varepsilon} \leq |Q'(z)| \leq r^{q-1+\varepsilon} \quad \text{and} \quad |e^{P(z)}| \leq \exp\{r^{p+\varepsilon}\}. \tag{2.10}$$

From (2.7), we see that $\left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}$ is an entire function. Then for all z satisfying $|z| = r > r_0$ and $|z| = r \notin [0,1] \cup E_1 \cup E_2$, by (2.8)-(2.10), we have

$$\begin{aligned}
&\left|\left[1 + \left(\frac{h'(z)}{h(z)} + Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right| \\
\leq\ &\left[1 + \left(\left|\frac{h'(z)}{h(z)}\right| + |Q'(z)|\right)|e^{P(z)}|\right]\left|\frac{h(z)}{h(z+\eta)}\right| \\
\leq\ &(1 + (r^{\sigma(h)-1+\varepsilon} + r^{q-1+\varepsilon})\exp\{r^{p+\varepsilon}\})\exp\{r^{\sigma(h)-1+\varepsilon}\} \\
\leq\ &r^{\sigma(h)+q-2+2\varepsilon}\exp\{r^{p+\varepsilon} + r^{\sigma(h)-1+\varepsilon}\} < \exp\{r^{q-1}\};
\end{aligned}$$

that is,

$$\begin{aligned}
&T\left(r, \left[1+\left(\frac{h'(z)}{h(z)}+Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right)\\
&= m\left(r, \left[1+\left(\frac{h'(z)}{h(z)}+Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right)\\
&< r^{q-1}.
\end{aligned}$$

The above inequality yields that

$$\sigma\left(\left[1+\left(\frac{h'(z)}{h(z)}+Q'(z)\right)e^{P(z)}\right]\frac{h(z)}{h(z+\eta)}\right) < q-1.$$

It follows from $\deg(Q(z+\eta)-Q(z)) = q-1$ that (2.7) is a contradiction.

Then we must have $q-1 = p \geq 0$. We claim that $q-1 = p = 0$, otherwise $q-1 = p \geq 1$. It follows from (2.6) that

$$P(z) = a_{q-1}z^{q-1} + P_{q-2}(z), \quad Q(z+\eta)-Q(z) = q\eta b_q z^{q-1} + Q_{q-2}(z), \tag{2.11}$$

where $a_{q-1}(\neq 0)$, $b_q(\neq 0)$ are constants, $P_{q-2}(z)$, $Q_{q-2}(z)$ are polynomials, $\deg P_{q-2}(z) \leq q-2$, $\deg Q_{q-2}(z) \leq q-2$. In what follows, we consider two subcases: 2.1, $a_{q-1} = q\eta b_q$; 2.2, $a_{q-1} \neq q\eta b_q$.

Subcase 2.1. If $a_{q-1} = q\eta b_q$, then (2.5) can be rewritten as

$$e^{-P(z)} = \frac{h(z+\eta)}{h(z)}e^{Q(z+\eta)-Q(z)-P(z)} - \left(\frac{h'(z)}{h(z)}+Q'(z)\right). \tag{2.12}$$

It follows from $a_{q-1} = q\eta b_q$ that $\deg(Q(z+\eta)-Q(z)-P(z)) = \deg(Q_{q-2}(z)-P_{q-2}(z)) \leq q-2$. Using the same method as above, we can obtain that

$$\sigma\left(\frac{h(z+\eta)}{h(z)}e^{Q(z+\eta)-Q(z)-P(z)} - \left(\frac{h'(z)}{h(z)}+Q'(z)\right)\right) < q-1.$$

It follows from $\deg(-P(z)) = q-1 \geq 1$ that (2.12) is a contradiction.

Subcase 2.2. If $a_{q-1} \neq q\eta b_q$, it follows from (2.5) and (2.11) that

$$\left(\frac{h'(z)}{h(z)}+Q'(z)\right)e^{a_{q-1}z^{q-1}} = \frac{h(z+\eta)}{h(z)}e^{q\eta b_q z^{q-1}+Q_{q-2}(z)-P_{q-2}(z)} - e^{-P_{q-2}(z)}. \tag{2.13}$$

Without loss of generality, we assume that $q|\eta b_q| \leq |a_{q-1}|$. Set $\arg a_{q-1} = \theta_1$ and $\arg(\eta b_q) = \theta_2$. For the above given ε and for all z satisfying $|z| = r > r_1$ and $|z| = r \notin [0,1]\cup E_1 \cup E_2$, $z = re^{i\theta_0}$, where θ_0 is a real constant such that $\cos((q-1)\theta_0+\theta_1) = 1$, by (2.8)-(2.10), we have

$$\begin{aligned}
\left|\left(\frac{h'(z)}{h(z)}+Q'(z)\right)e^{a_{q-1}z^{q-1}}\right| &\geq \left(|Q'(z)| - \left|\frac{h'(z)}{h(z)}\right|\right)|e^{a_{q-1}z^{q-1}}|\\
&\geq (r^{q-1-\varepsilon} - r^{\sigma(h)-1+\varepsilon})\exp\{|a_{q-1}|r^{q-1}\}\\
&\geq r^{q-1-2\varepsilon}(1+o(1))\exp\{|a_{q-1}|r^{q-1}\}\\
&\geq \exp\{|a_{q-1}|r^{q-1}\},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{h(z+\eta)}{h(z)}e^{q\eta b_q z^{q-1}+Q_{q-2}(z)-P_{q-2}(z)}-e^{-P_{q-2}(z)}\right| \\
\le\ &\left|\frac{h(z+\eta)}{h(z)}\right| |e^{q\eta b_q z^{q-1}+Q_{q-2}(z)-P_{q-2}(z)}|+|e^{-P_{q-2}(z)}| \\
\le\ &\exp\{r^{\sigma(h)-1+\varepsilon}\}\exp\{q|\eta b_q|\cos((q-1)\theta_0+\theta_2)r^{q-1}+O(r^{q-2})\} \\
\le\ &\exp\{q|\eta b_q|\cos((q-1)\theta_0+\theta_2)r^{q-1}+o(r^{q-1})\},
\end{aligned}$$

that is,

$$\exp\{|a_{q-1}|r^{q-1}\}\le\exp\{q|\eta b_q|\cos((q-1)\theta_0+\theta_2)r^{q-1}+o(r^{q-1})\}. \tag{2.14}$$

We claim that $q|\eta b_q|\cos((q-1)\theta_0+\theta_2)<|a_{q-1}|$. In fact, if $q|\eta b_q|=|a_{q-1}|$, it follows from $a_{q-1}\neq q\eta b_q$ that $\cos((q-1)\theta_0+\theta_2)\neq 1$, then $\cos((q-1)\theta_0+\theta_2)<1$. Thus $q|\eta b_q|\cos((q-1)\theta_0+\theta_2)<q|\eta b_q|=|a_{q-1}|$. If $q|\eta b_q|<|a_{q-1}|$, then $q|\eta b_q|\cos((q-1)\theta_0+\theta_2)\le q|\eta b_q|<|a_{q-1}|$.

For any given ε_3 $\left(0<\varepsilon_3<\frac{|a_{q-1}|-q|\eta b_q|\cos((q-1)\theta_0+\theta_2)}{3}\right)$, it follows from (2.14) that

$$\begin{aligned}
\exp\{|a_{q-1}|r^{q-1}\}&\le\exp\{q|\eta b_q|\cos((q-1)\theta_0+\theta_2)r^{q-1}+o(r^{q-1})\} \\
&<\exp\{(|a_{q-1}|-\varepsilon_3)r^{q-1}\}.
\end{aligned}$$

This is a contradiction.

Thus, we must have $q-1=p=0$, that is $p=0$ and $q=1$. Then $e^{P(z)}$ is a nonzero constant and $f(z)=b+H(z)e^{cz}$, where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H)=\sigma(H)<1$, $c\in\mathbb{C}\setminus\{0\}$ is a constant. Set $e^{P(z)}\equiv A$, then (2.5) can be rewritten as

$$\frac{h(z+\eta)}{h(z)}e^{c\eta}=1+\left(\frac{h'(z)}{h(z)}+c\right)A, \tag{2.15}$$

where A,c,η are non-zero constants. If $h(z)(\not\equiv 0)$ is a polynomial, then

$$\frac{h'(z)}{h(z)}\to 0,\quad \frac{h(z+\eta)}{h(z)}\to 1,\quad z\to\infty. \tag{2.16}$$

It follows from (2.15) and (2.16) that $e^{c\eta}=1+Ac$. If $h(z)(\not\equiv 0)$ is a transcendental entire function with $\sigma(h)<1$, by Lemma 2.2, we also have $e^{c\eta}=1+Ac$.

If $P(z)\equiv 0$, then $e^{P(z)}\equiv 1$. Using the same method as in the proof of Case 2, we obtain that $f(z)=b+H(z)e^{cz}$, where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H)=\sigma(H)<1$, $c\in\mathbb{C}\setminus\{0\}$ is a constant satisfying $e^{c\eta}=1+c$. □

3. Proofs of Theorems

Proof of Theorem 1.1. Now we suppose that $a(z)\not\equiv 0$. By the Hadamard's factorization theorem [16, Theorem 2.5] and $\lambda(f-a)<\sigma(f)$, we obtain

$$f(z)=a(z)+h(z)e^{Q(z)}, \tag{3.1}$$

where $h(z)$ is a non-zero entire function, $Q(z)$ is a polynomial, and $h(z),\ Q(z)$ satisfy

$$\sigma(h) = \lambda(h) = \lambda(f-a) < \sigma(f) = \deg Q(z). \tag{3.2}$$

Since $\Delta_\eta f(z)$ and $f'(z)$ share $a(z)$ CM, we have

$$\frac{\Delta_\eta f(z) - a(z)}{f'(z) - a(z)} = e^{P(z)}, \tag{3.3}$$

where $P(z)$ is a polynomial. It follows from (3.2) and (3.3) that

$$\deg P(z) \le \deg Q(z). \tag{3.4}$$

Substituting (3.1) into (3.3) yields

$$h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) + c(z)e^{-Q(z)} = (h'(z) + h(z)Q'(z) + d(z)e^{-Q(z)})e^{P(z)}, \tag{3.5}$$

where $c(z) = a(z+\eta) - 2a(z)$ and $d(z) = a'(z) - a(z)$. Since $\sigma(a) < 1$, we see that $\max\{\sigma(c),\ \sigma(d)\} < 1$.

In what follows, we consider two cases: $1 \le \deg P(z) < \deg Q(z)$ and $\deg P(z) = \deg Q(z)$. Set

$$P(z) = a_p z^p + a_{p-1}z^{p-1} + \cdots + a_0, \quad Q(z) = b_q z^q + b_{q-1}z^{q-1} + \cdots + b_0,$$

where $a_p(\neq 0), \ldots, a_0,\ b_q(\neq 0), \ldots, b_0$ are constants, p, q are positive integers.

Case 1**.** Suppose that $1 \le p < q$, then (3.5) can be rewritten as

$$h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) - (h'(z) + h(z)Q'(z))e^{P(z)} = (d(z)e^{P(z)} - c(z))e^{-Q(z)}. \tag{3.6}$$

It follows from $a(z) \not\equiv 0$ and $\sigma(a) < 1$ that $d(z) = a'(z) - a(z) \not\equiv 0$. If $d(z)e^{P(z)} - c(z) \equiv 0$, then we have $e^{P(z)} = \frac{c(z)}{d(z)}$. By $\max\{\sigma(c),\ \sigma(d)\} < 1$, we see that $\sigma(e^{P(z)}) \le \max\{\sigma(c),\ \sigma(d)\} < 1$, which contradicts with $\sigma(e^{P(z)}) = \deg P(z) = p \ge 1$. Hence, we must have $d(z)e^{P(z)} - c(z) \not\equiv 0$. Since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q-1$ and $\sigma(e^{P(z)}) = \deg P(z) = p < q$, we see that the order of growth of the left side of (3.6) is less than q, and the order of growth of the right side of (3.6) is q. This is a contradiction.

Case 2**.** Suppose that $p = q$. For a_q and b_q, we consider three subcases: 2.1, $a_q = b_q$; 2.2, $a_q = -b_q$; 2.3, $a_q \neq b_q$ and $a_q \neq -b_q$.

Subcase 2.1**.** Suppose that $a_q = b_q$, then (3.5) can be rewritten as

$$\begin{aligned} &(h'(z) + h(z)Q'(z))e^{P(z)} - c(z)e^{-Q(z)} \\ &= h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) - d(z)e^{P(z)-Q(z)}. \end{aligned} \tag{3.7}$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q-1$, $\max\{\sigma(c),\ \sigma(d)\} < 1 \le q$ and $\deg(P(z) - Q(z)) \le q-1$, we have $\sigma(h'(z) + h(z)Q'(z)) < q$ and $\sigma(h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) - d(z)e^{P(z)-Q(z)}) < q$.

Note that $e^{P(z)}$, $e^{-Q(z)}$ and $e^{P(z)+Q(z)}$ are of regular growth, and $\sigma(e^{P(z)}) = \sigma(e^{-Q(z)}) = \sigma(e^{P(z)+Q(z)}) = q$, it follows from Lemma 2.1 and (3.7) that

$$h'(z) + h(z)Q'(z) \equiv 0 \quad \text{and} \quad c(z) = a(z+\eta) - 2a(z) \equiv 0.$$

If $h'(z) + h(z)Q'(z) \equiv 0$, then $h(z) \equiv 0$ or $h(z) = ce^{-Q(z)}$, $c \in \mathbb{C} \setminus \{0\}$, it contradicts with $h(z) \not\equiv 0$ and $\sigma(h) < q$. If $a(z+\eta) - 2a(z) \equiv 0$, since $a(z) \not\equiv 0$, suppose that $a(z)$ is a polynomial, then $\frac{a(z+\eta)}{a(z)} \equiv 2$, it contradicts with $\lim_{z\to\infty} \frac{a(z+\eta)}{a(z)} = 1$. If $a(z+\eta) - 2a(z) \equiv 0$, since $a(z) \not\equiv 0$, suppose that $a(z)$ is a transcendental entire function and $\sigma(a) < 1$, by Lemma 2.2, we can also obtain a contradiction. Then we see that the above two identities are absurd.

Subcase 2.2. Suppose that $a_q = -b_q$, then (3.5) can be rewritten as

$$\begin{aligned} &[(h'(z) + h(z)Q'(z))e^{P(z)+Q(z)} - c(z)]e^{-Q(z)} + d(z)e^{P(z)-Q(z)} \\ &= h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z). \end{aligned} \tag{3.8}$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q - 1$, $\max\{\sigma(c),\ \sigma(d)\} < 1 \le q$ and $\deg(P(z)+Q(z)) \le q-1$, we have $\sigma((h'(z)+h(z)Q'(z))e^{P(z)+Q(z)} - c(z)) < q$ and $\sigma(h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z)) < q$.

Note that $e^{-Q(z)}$, $e^{P(z)-Q(z)}$ and $e^{-P(z)}$ are of regular growth, and $\sigma(e^{-Q(z)}) = \sigma(e^{-P(z)}) = \sigma(e^{P(z)-Q(z)}) = q$, it follows from Lemma 2.1 and (3.8) that

$$d(z) = a'(z) - a(z) \equiv 0.$$

If $a'(z) - a(z) \equiv 0$, then $a(z) \equiv 0$ or $a(z) = ce^z$, $c \in \mathbb{C} \setminus \{0\}$, it contradicts with $a(z) \not\equiv 0$ and $\sigma(a) < 1$. Then we see that the above identity is absurd.

Subcase 2.3. Suppose that $a_q \neq b_q$ and $a_q \neq -b_q$, then (3.5) can be rewritten as

$$\begin{aligned} &(h'(z) + h(z)Q'(z))e^{P(z)} - c(z)e^{-Q(z)} + d(z)e^{P(z)-Q(z)} \\ &= h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z). \end{aligned} \tag{3.9}$$

Since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q - 1$ and $\max\{\sigma(c),\ \sigma(d)\} < 1 \le q$, we have $\sigma(h'(z) + h(z)Q'(z)) < q$ and $\sigma(h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z)) < q$.

Note that $e^{\pm P(z)}$, $e^{\pm Q(z)}$ and $e^{P(z)\pm Q(z)}$ are of regular growth, and $\sigma(e^{\pm P(z)}) = \sigma(e^{\pm Q(z)}) = \sigma(e^{P(z)\pm Q(z)}) = q$, it follows from Lemma 2.1 and (3.9) that

$$h'(z)+h(z)Q'(z) \equiv 0,\ c(z) = a(z+\eta)-2a(z) \equiv 0 \ \text{ and } \ d(z) = a'(z)-a(z) \equiv 0.$$

From the above conclusions of Subcase 2.1 and Subcase 2.2, we see that the above three identities are also absurd.

Thus, $P(z)$ can only be a constant, so is $e^{P(z)}$. Set $e^{P(z)} \equiv A$, where A is a non-zero constant. Then (3.5) can be rewritten as

$$h(z+\eta)e^{Q(z+\eta)-Q(z)} - h(z) - A(h'(z) + h(z)Q'(z)) = (Ad(z) - c(z))e^{-Q(z)}. \tag{3.10}$$

If $Ad(z) - c(z) \not\equiv 0$, since $\sigma(h) < q$, $\deg(Q(z+\eta) - Q(z)) = q - 1$ and $\max\{\sigma(c),\ \sigma(d)\} < 1 \le q$, we see that the order of growth of the left side of (3.10) is less than q, and the order of growth of the right side of (3.10) is q, a contradiction. Then $Ad(z) - c(z) \equiv 0$. Since $a(z) \not\equiv 0$, then we have

$A\frac{a'(z)-a(z)}{a(z)} \equiv \frac{a(z+\eta)-2a(z)}{a(z)}$, that is $A\left(\frac{a'(z)}{a(z)}-1\right) \equiv \frac{a(z+\eta)}{a(z)}-2$. If $a(z)$ is a non-zero polynomial, then $\frac{a'(z)}{a(z)} \to 0,\ \frac{a(z+\eta)}{a(z)} \to 1,\ z \to \infty$, then we have $A = 1$. If $a(z)$ is a transcendental entire function with $\sigma(a) < 1$, from Lemma 2.2, we also obtain that $A = 1$. Then (3.10) can be rewritten as

$$e^{Q(z+\eta)-Q(z)} = \left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}. \tag{3.11}$$

We claim that $q = 1$. In fact, if it is not true, then $q \geq 2$. If $\sigma(h) < 1$, since $\deg(Q(z+\eta) - Q(z)) = q - 1 \geq 1$, we see that the order of growth of the left side of (3.11) is $q - 1 \geq 1$, and the order of growth of the right side of (3.11) is less than 1, a contradiction. Then we have $\sigma(h) \geq 1$. From (3.11), we see that $\left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}$ is an entire function. Then for all z satisfying $|z| = r > r_2$ and $|z| = r \notin [0, 1] \cup E_1 \cup E_2$, for any given $\varepsilon \left(0 < \varepsilon < \frac{q-\sigma(h)}{3}\right)$, from (2.8)-(2.10), we have

$$\begin{aligned}
&\left|\left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}\right| \\
\leq\ &\left(1 + \left|\frac{h'(z)}{h(z)}\right| + |Q'(z)|\right)\left|\frac{h(z)}{h(z+\eta)}\right| \\
\leq\ &(1 + r^{\sigma(h)-1+\varepsilon} + r^{q-1+\varepsilon})\exp\{r^{\sigma(h)-1+\varepsilon}\} \\
\leq\ &r^{\sigma(h)+q-2+2\varepsilon}\exp\{r^{\sigma(h)-1+\varepsilon}\} < \exp\{r^{q-1}\},
\end{aligned}$$

that is,

$$\begin{aligned}
&T\left(r, \left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}\right) \\
=\ &m\left(r, \left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}\right) \\
<\ &r^{q-1}.
\end{aligned}$$

It follows from the above inequality that

$$\sigma\left(\left(1 + \frac{h'(z)}{h(z)} + Q'(z)\right)\frac{h(z)}{h(z+\eta)}\right) < q - 1.$$

Since $\deg(Q(z+\eta) - Q(z)) = q - 1$, we see that (3.11) is absurd. Then we must have $q = 1$, then

$$f(z) = a(z) + H(z)e^{cz},$$

where $c \in \mathbb{C} \setminus \{0\}$ is a constant and $H(z)$ ($\not\equiv 0$) is an entire function with $\lambda(H) = \sigma(H) < 1$. It follows from (3.11) that

$$\frac{h(z+\eta)}{h(z)}e^{c\eta} = 1 + c + \frac{h'(z)}{h(z)}. \tag{3.12}$$

If $h(z)(\not\equiv 0)$ is a polynomial, then

$$\frac{h'(z)}{h(z)} \to 0, \quad \frac{h(z+\eta)}{h(z)} \to 1, \quad z \to \infty. \tag{3.13}$$

It follows from (3.12) and (3.13) that $e^{c\eta} = 1+c$. If $h(z)(\not\equiv 0)$ is a transcendental entire function with $\sigma(h) < 1$, from Lemma 2.2, we also have $e^{c\eta} = 1 + c$.

If $a(z) \equiv 0$, by Lemma 2.5, set $b = 0$, then

$$\frac{\Delta_\eta f(z)}{f'(z)} = A \quad \text{and} \quad f(z) = H(z)e^{cz},$$

where $H(z)(\not\equiv 0)$ is an entire function with $\lambda(H) = \sigma(H) < 1$, $A, c, \eta \in \mathbb{C} \setminus \{0\}$ are constants satisfying $e^{c\eta} = 1 + Ac$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Using the same method as in the proof of Theorem 1.1, the conclusion of Theorem 1.2 follows immediately. We omit the proof here. □

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