Commun. Korean Math. Soc. **32** (2017), No. 2, pp. 353–359
https://doi.org/10.4134/CKMS.c160089
pISSN: 1225-1763 / eISSN: 2234-3024

ON A CLASS OF GENERALIZED TRIANGULAR NORMS

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Abstract. Starting from a t-norm T, it is possible to construct a class of new t-norms, so-called T-generalized t-norm. The purpose of this paper is to describe some properties of this class of generalized t-norms. An algebraic structure as well as a binary relation among t-norms are also investigated. Some open problems are discussed as well.

1. Introduction

Triangular norms are operations which represent conjunctions in fuzzy logic. They were also used in the context of probabilistic metric spaces as a special kind of associative laws defined on the unit interval, see [4]. Geometrically, a triangular norm may be visualized as a surface over the unit square that contains the skew quadrilateral whose vertices are $(0,0,0),(1,0,0),(1,1,1)$ and $(0,1,0)$. For more detail about geometrical illustration of triangular norms, see [4] for instance.

A map $T:[0,1]\times[0,1]\longrightarrow[0,1]$ is called a triangular norm (in short a t-norm), if the following requirements are satisfied, see [1, 2, 3],

(i) $T(x,y)=T(y,x)$ for all $x,y\in[0,1]$, (T is commutative)

(ii) $T(x_1,y_1)\le T(x_2,y_2)$ whenever $x_1\le x_2$ and $y_1\le y_2$, (T is increasing)

(iii) $T\big(x,T(y,z)\big)=T\big(T(x,y),z\big)$ for all $x,y,z\in[0,1]$, (T is associative)

(iv) $T(x,1)=x$ for all $x\in[0,1]$, (T has 1 as identity).

As it is well known, combining (i), (ii) and (iv) we can see that every t-norm T satisfies $T(x,y)\le\min(x,y)$ for all $x,y\in[0,1]$. It follows that $T(x,x)\le x$ for every $x\in[0,1]$, with $T(0,0)=0$ and $T(1,1)=1$.

Typical examples of t-norms include the following:

(1) $M(x,y)=\min(x,y)$; $\Pi(x,y)=xy$; $L(x,y)=\max(x+y-1,0)$;

(2) $W(x,1)=x,\ W(1,y)=y,\ W(x,y)=0$ if $x,y\in[0,1)$;

(3) $N(x,y)=\min(x,y)$ if $x+y\ge1,\ N(x,y)=0$ if $x+y<1$;

(3) $H(0,0)=0,\ H(x,y)=\frac{xy}{x+y-xy}$ if $(x,y)\ne(0,0)$;

Received April 19, 2016.

2010 *Mathematics Subject Classification.* 08A05.

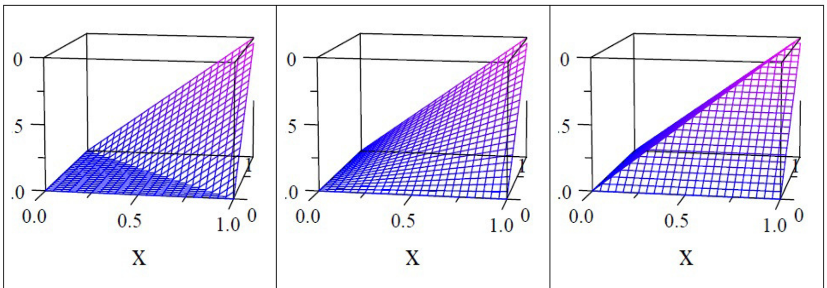
Key words and phrases. t-norm, generalized t-norm, power t-norm, group of t-norms, equivalence relation between t-norms.

and are known in the literature as the minimum or Gödel t-norm, product t-norm, Lukasiewicz t-norm, drastic or weak t-norm, nilpotent t-norm and Hamacher t-norm, respectively.

For two t-norms T and S, we write $T \leq S$ if and only if $T(x,y) \leq S(x,y)$ for all $x, y \in [0,1]$. If $T \leq S$ and there is at least one pair $(a,b) \in [0,1] \times [0,1]$ such that $T(a,b) < S(a,b)$, then we briefly write $T < S$. Every t-norm T satisfies $W \leq T \leq M$ and the above t-norms satisfy the chain of inequalities $W < L < \Pi < H < M$.

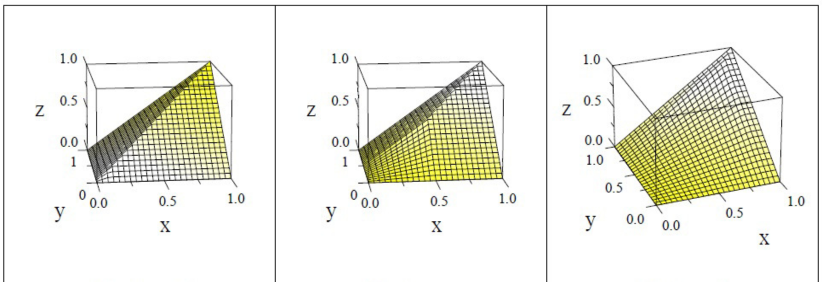
There are many families of t-norms some of which extend the standard t-norms listed above. For instance, we cite [1]

$$H(\gamma; x, y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}, \quad \gamma \geq 0;$$



$H(0; x, y)$ $H(1; x, y)$ $H(100; x, y)$

$$D(\alpha; x, y) = \frac{xy}{\max(x, y, \alpha)}, \quad \alpha \in (0,1),$$



$D(0.1; x, y)$ $D(0.5; x, y)$ $D(0.9; x, y)$

known as the Hamacher and Dubois-Prade (parameterized) t-norms, respectively. Clearly, $H(0; x, y) = H(x, y)$, $H(1; x, y) = \Pi(x, y)$ and

$$D(0; x, y) := \lim_{\alpha \downarrow 0} D(\alpha; x, y) = Z(x, y), \quad D(1; x, y) := \lim_{\alpha \uparrow 1} D(\alpha; x, y) = \Pi(x, y)$$

for all $x, y \in [0,1]$. For other families of t-norms (not needed here) we refer the reader to [1] for instance.

2. On a class of t-norms

For the sake of simplicity, we denote by $C^+\big([0,1]\big)$ the set of all continuous strictly increasing maps $\Phi : [0,1] \longrightarrow [0,1]$. We denote by $id \in C^+\big([0,1]\big)$ the identity map of $[0,1]$. It is well known that, if for some $\Phi \in C^+\big([0,1]\big)$ we set

$$T_\Phi(x,y) = \Phi^{-1}\Big(T\big(\big(\Phi(x),\Phi(y)\big)\Big)$$

for all $x, y \in [0,1]$, then T_Φ defines a t-norm called the T-generalized t-norm associated to Φ. Note that $T_{id} = T$, and note also the following:

• $\big(T_\Phi\big)_\Psi = T_{\Phi\circ\Psi}$ and so $\big(T_\Phi\big)_{\Phi^{-1}} = T$ for all $\Phi,\ \Psi \in C^+\big([0,1]\big)]$.

• If T and S are two t-norms such that $T_\Phi = S_\Phi$ for some $\Phi \in C^+\big([0,1]\big)]$, then $T = S$.

• If T and S are two t-norms such that $T \le S$, then $T_\Phi \le S_\Phi$ for every $\Phi \in C^+\big([0,1]\big)]$.

• $T_\Phi = T_\Psi$ does not ensure $\Phi = \Psi$.

One of the most interesting examples of $\Phi \in C^+\big([0,1]\big)$ is $\Phi(x) = x^p$ for some real number $p > 0$. In this case, we write T_p instead of T_Φ, i.e.,

$$\forall x, y \in [0,1] \qquad T_p(x,y) := \Big(T(x^p, y^p)\Big)^{1/p}.$$

The following properties hold:

• $\big(T_p\big)_q = T_{pq} = \big(T_q\big)_p$ for all $p, q > 0$. In particular, $\big(T_p\big)_{1/p} = T_1 = T$ and so $\big(T_2\big)_{1/2} = \big(T_{1/2}\big)_2 = T$.

• Let T and S be two t-norms. If $T_p = S_p$ for some $p > 0$, then $T = S$, but $T_p = T_q$ does not ensure $p = q$.

• If T and S are two t-norms such that $T < S$, then $T_p < S_p$ for each $p > 0$.

It is easy to see that $M_\Phi = M$ and $W_\Phi = W$ for all $\Phi \in C^+\big([0,1]\big)$ and so $M_p = M$ and $W_p = W$ for every $p > 0$. Further, $\Pi_p = \Pi$ for each $p > 0$. Now, we will illustrate the above with the following examples.

Example 2.1. Let $\Phi \in C^+\big([0,1]\big)$ be as follows

$$\Phi(x) = \frac{2x}{x+1},\ \ \Phi^{-1}(x) = \frac{x}{2-x}.$$

(i) Take $T = \Pi$. Simple computation yields

$$\Pi_\Phi(x,y) = \frac{2xy}{x+y-xy+1} = H(1/2; x, y),$$
$$\Pi_{\Phi^{-1}}(x,y) = \frac{xy}{xy-x-y+2} = H(2; x, y).$$

(ii) Let $T = H$. An elementary computation leads to $H_\Phi = H$. Detail is simple and therefore omitted here.

Example 2.2. Let Φ be as in the previous example. We can reiterate the above by choosing $T = \Pi_\Phi$. If we denote by $\Pi_{\Phi^{n+1}} := \big(\Pi_{\Phi^n}\big)_\Phi$ for all integer $n \geq 0$, a simple mathematical induction shows that

$$\forall x, y \in [0,1] \qquad \Pi_{\Phi^n}(x,y) = \frac{2^n xy}{(2^n - 1)(x + y - xy) + 1} = H(1/2^n; x, y).$$

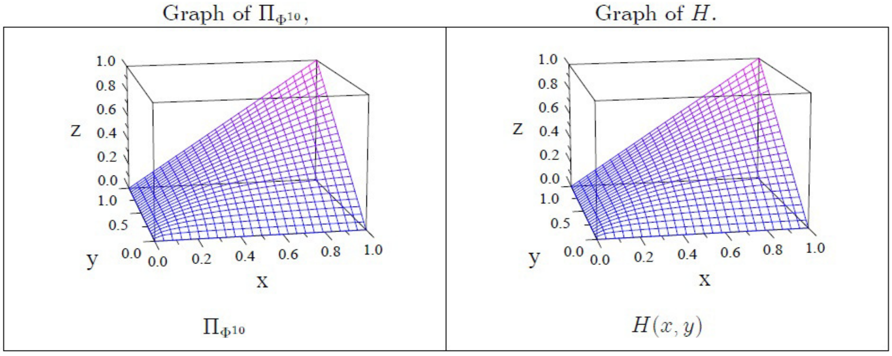
It is worth mentioning that the t-norm $(x,y) \longmapsto \Pi_{\Phi^n}(x,y)$, for fixed $n \geq 0$, is a point-wise approximation of the non-differentiable t-norm $(x,y) \longmapsto H(x,y)$, i.e.,

$$\lim_{n\uparrow\infty} \Pi_{\Phi^n}(x,y) = H(x,y) = \frac{xy}{x + y - xy}$$

for all $x, y \in [0,1]$ with $\Pi_{\Phi^n}(0,0) = H(0,0) = 0$. We omit all discussion about this latter point which is outside the scope of this paper.

The graphs below illustrate the point just made via Example 2.2.

Graph of $\Pi_{\Phi^{10}}$, Graph of H.



3. On an algebraic structure

Let T be a t-norm. We denote by $\mathcal{G}(T)$ the set of all T-generalized t-norms, i.e.,

$$\mathcal{G}(T) = \Big\{T_\Phi, \ \ \Phi \in C^+\big([0,1]\big)\Big\}.$$

Remark that $\mathcal{G}(M) = \{M\}$ and $\mathcal{G}(W) = \{W\}$.

In all cases, we can equip $\mathcal{G}(T)$ with an internal law defined as follows: for $T_\Phi, T_\Psi \in \mathcal{G}(T)$, we set

$$T_\Phi \odot T_\Psi = T_{\Phi \circ \Psi}.$$

With this, we have the next result.

Proposition 3.1. $\big(\mathcal{G}(T), \odot\big)$ *is a group. Its identity element is* $T_{id} = T$ *and the inverse of* T_Φ *is* $T_\Phi^{-1} = T_{\Phi^{-1}}$.

Proof. Since $(\Phi \circ \Psi) \circ \Upsilon = \Phi \circ (\Psi \circ \Upsilon)$ for all $\Phi, \Psi, \Upsilon : [0,1] \longrightarrow [0,1]$ then the associativity axiom for $\odot$ follows. A simple verification leads to $T = T_{id}$ is the identity element and $T_{\Phi^{-1}}$ is the inverse of T_Φ, for the law $\odot$. The proof is so complete. $\square$

Let $\mathcal{P}(T)$ be the set of all power t-norms, i.e., $\mathcal{P}(T) = \{T_p,\ p > 0\}$. Clearly, $\mathcal{P}(T)$ is a subset of $\mathcal{G}(T)$, with $T = T_1 \in \mathcal{P}(T)$. The next result may be stated.

Proposition 3.2. *For all $p, q > 0$ we have*

$$T_p^{-1} = T_{1/p}, \quad T_p \odot T_q = T_q \odot T_p = T_{pq}.$$

Consequently, $\mathcal{P}(T)$ is a subgroup of $\mathcal{G}(T)$ and $\odot$ induces a commutative law on $\mathcal{P}(T)$.

Proof. The relationships just stated follow from the previous proposition with a simple manipulation. Now, $T = T_1 \in \mathcal{P}(T)$ and if $T_p, T_q \in \mathcal{P}(T)$, then we have

$$T_p \odot T_q^{-1} = T_p \odot T_{1/q} = T_{p/q} \in \mathcal{P}(T).$$

The desired result is then obtained. $\square$

It is easy to see that $\big(C^+([0,1]), \circ\big)$ is a group. With this, the following result may be stated.

Theorem 3.3. *Let T be a t-norm. Then the following assertions hold*:

(1) *The map $F : \big(C^+([0,1]), \circ\big) \longrightarrow \big(\mathcal{G}(T), \odot\big)$ defined by $F(\Phi) = T_\Phi$ for all $\Phi \in C^+([0,1])$ is a homomorphism of groups. Further, F is surjective and*

$$Ker\ F = \big\{\Phi \in C^+([0,1]),\ T_\Phi = T\big\}.$$

(2) *The map $f : (0, \infty) \longrightarrow \mathcal{P}(T)$ defined by $f(p) = T_p$ for all $p > 0$ is a homomorphism of groups. Moreover, f is surjective and $Ker\ f = \{p > 0,\ T_p = T\}$.*

Proof. First, remark that F and f are surjective by construction. Further, for all $\Phi, \Psi \in C^+([0,1])$ and all $p, q > 0$ we have

$$F\big(\Phi \circ \Psi\big) = T_{\Phi \circ \Psi} = T_\Phi \odot T_\Psi = F(\Phi) \odot F(\Psi),$$

$$f(pq) = T_{pq} = T_p \odot T_q = f(p) \odot f(q).$$

The desired result follows and the proof is complete. $\square$

The following corollary may be stated.

Corollary 3.4. *Let T be a t-norm. If there is no $p > 0,\ p \neq 1$, such that $T_p = T$, then $\mathcal{P}(T)$ is infinite.*

Proof. With our assumption, the map f (previously defined) is one-to-one and by the first theorem of isomorphism we can deduce that $(0, \infty)$ and $\mathcal{P}(T)$ are isomorphic. We then conclude that $\mathcal{P}(T)$ is infinite. $\square$

4. On a binary relation among t-norms

Let T be a t-norm. Throughout this section, we set $g_T(x) = T(x,x)$ for every $x \in [0,1]$. We then say that g_T is the associated function of T. We ask the question: is it true that $g_T = g_S$ implies $T = S$? Put another way, we ask if T is characterized by its associated function g_T. In the general case we have no affirmative answer. For the lower and upper t-norms W and M, however the two following results may be stated.

Proposition 4.1. *Let T be a t-norm such that $g_T = g_M$, i.e., $T(x,x) = x$ for each $x \in [0,1]$. Then $T = M$.*

Proof. Assume that $T(x,x) = x$ for all $x \in [0,1]$. We would like to show that $T(x,y) = M(x,y) = \min(x,y)$ for all $x,y \in [0,1]$. It is always true that $T(x,y) \leq M(x,y)$ for all $x,y \in [0,1]$ and so it is necessary to prove only that $T(x,y) \geq M(x,y)$ for all $x,y \in [0,1]$. By the commutativity axiom (i) we can assume, without loss the generality, that $x \leq y$. In this case, the monotonicity axiom (ii) of T yields, for all $x,y \in [0,1]$,

$$T(x,y) \geq T(x,x) = x = \min(x,y) = M(x,y).$$

The proof of the proposition is so completed. $\square$

Proposition 4.2. *Let T be a t-norm such that $g_T = g_W$. Then $T = W$.*

Proof. First, it is easy to see that

$$g_W(x) := W(x,x) = 1 \text{ if } x = 1, \;\; g_W(x) = 0 \text{ if } x \neq 1.$$

The inequality $W \leq T$ is always true and we have to prove that $T(x,y) \leq W(x,y)$ for all $x,y \in [0,1]$. Since T is a t-norm then

$$T(x,1) = x = W(x,1) \text{ and } T(1,y) = y = W(1,y).$$

We can then assume that $x,y \in [0,1)$, further with $x \leq y$ by virtue of the commutativity axiom (i). Then, by the monotonicity axiom (ii), we have

$$T(x,y) \leq T(y,y) = g_T(y) = g_W(y) = 0 = W(x,y),$$

which completes the proof. $\square$

We can interpret the previous results in another point of view. In fact, let $\mathcal{T}$ be the set of all t-norms. For $T, S \in \mathcal{T}$, we define the following binary relation

$$T \equiv S \text{ if and only if } g_T = g_S,$$

which is obviously an equivalence relation. The coset of T is given by

$$c(T) := \{S \in \mathcal{T}, \;\; g_S = g_T\}.$$

The above propositions mean that $c(M) = \{M\}$ and $c(W) = \{W\}$, respectively. Some questions, posed here as open problems, arise from the above study.

Problem 1: *Find $c(T)$ for $T \in \{\Pi, L, N, H\}$.*

Problem 2: *Let T and S be two t-norms such that $g_T = g_S$. Is it true that $T = S$?*

It is not hard to verify that the three functions g_M, g_Π and g_H belong to $C^+\big([0,1]\big)$ while the three ones g_L, g_W and g_N do not. We then put the following open question.

Problem 3: *Let T be a given t-norm. Under what (necessary and sufficient) condition on T we have $g_T \in C^+\big([0,1]\big)$?*

In Example 2.1, (ii) we have seen that $H_\Phi = H$ with $\Phi(x) = \frac{2x}{x+1}$. We then deduce that $H_{\Phi^{-1}} = H$ with

$$\forall x \in [0,1] \qquad \Phi^{-1}(x) = \frac{x}{2-x} = g_H(x).$$

We can then write $H_{g_H} = H$. This allows us to put the following.

Problem 4: *How can t-norms T such that $T_{g_T} = T$ be described?*

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