

SOME INTEGRAL REPRESENTATIONS AND TRANSFORMS FOR EXTENDED GENERALIZED APPELL'S AND LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper, we generalize the extended Appell's and Lauricella's hypergeometric functions which have recently been introduced by Liu [9] and Khan [7]. Also, we aim at establishing some (presumably) new integral representations and transforms for the extended generalized Appell's and Lauricella's hypergeometric functions.

1. Introduction and preliminaries

For nonnegative integers p and q , the generalized hypergeometric function in a variable (argument) z with p numeratorial and q denominatorial parameters is defined by (see, *e.g.*, [2, 13, 15, 16])

$$(1.1) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = {}_pF_q [\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z] \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

whenever this series converges and elsewhere by analytic continuation. Here Γ is the familiar Gamma function and $(\cdot)_m$ stands for the Pochhammer(or shifted factorial) symbol defined for any complex number α and nonnegative integers m by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1)$. The series defining ${}_pF_q$ converges for all values of z when $p \leq q$. If $p = q + 1$, then the series (1.1) converges when $|z| < 1$, it is absolutely convergent on the unit circle if $\Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) > 0$ and it is convergent on the circle $|z| = 1$ except at $z = 1$ if $-1 < \Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) \leq 0$.

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A generalization of Euler's beta function is defined by [12]

$$(1.2) \quad B_{\gamma}^{(\alpha, \beta)}(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1 \left[\alpha; \beta; \frac{-\gamma}{t(1-t)} \right] dt \\ (\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(x) > 0, \Re(y) > 0).$$

For $\gamma = 0$, it reduces to the well-known Euler's beta function [13, p. 18].

Very recently, Luo *et al.* [10, 14] introduced the following extended generalized hypergeometric type function and investigated its various properties. The extended generalized hypergeometric function is defined, for $z \in \mathbb{C}$, by

$$(1.3) \quad {}_pF_q^{(\alpha, \beta; \kappa, \mu)} \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z; \gamma \right] := \sum_{n=0}^{\infty} \Theta(n|p, q) \frac{z^n}{n!} \\ (\min\{\Re(\kappa), \Re(\mu)\} \geq 0, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0)$$

whose coefficient is given by

$$\Theta(n|p, q) = \begin{cases} (a_1)_n \prod_{j=1}^q \frac{B_{\gamma}^{(\alpha, \beta; \kappa, \mu)}(a_{j+1} + n, b_j - a_{j+1})}{B(a_{j+1}, b_j - a_{j+1})} \\ \quad (p = q + 1; \Re(b_j) > \Re(a_{j+1}) > 0; |z| < 1), \\ \prod_{j=1}^q \frac{B_{\gamma}^{(\alpha, \beta; \kappa, \mu)}(a_j + n, b_j - a_j)}{B(a_j, b_j - a_j)} \\ \quad (p = q; \Re(b_j) > \Re(a_j) > 0), \\ \prod_{i=1}^r \frac{1}{(b_i)_n} \prod_{j=1}^p \frac{B_{\gamma}^{(\alpha, \beta; \kappa, \mu)}(a_j + n, b_r + j - a_j)}{B(a_j, b_r + j - a_j)} \\ \quad (r = q - p > 0; \Re(b_{r+j}) > \Re(a_j) > 0). \end{cases}$$

Here the generalized Beta function $B_{\gamma}^{(\alpha, \beta; k, \mu)}(x, y)$ is defined by

$$(1.4) \quad B_{\gamma}^{(\alpha, \beta; \kappa, \mu)}(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^{\kappa}(1-t)^{\mu}} \right) dt \\ (\min\{\Re(x), \Re(y), \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\kappa), \Re(\mu)\} > 0).$$

For $\kappa = \mu = 1$, it obviously reduces to the usual generalized beta function (1.2).

Some special cases of hypergeometric functions of two variables, $F_1[a, b_1, b_2; c; x, y]$ and $F_2[a, b_1, b_2; c_1, c_2; x, y]$, as well as Lauricella's hypergeometric function of three variables $F_D^{(3)}[a, b_1, b_2; b_3; c; x, y, z]$ are the following Appell functions (see [16, 17]):

$$(1.5) \quad F_1[a, b_1, b_2; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (|x| < 1, |y| < 1);$$

(1.6)

$$F_2 [a, b_1, b_2; c_1, c_2; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (|x| + |y| < 1);$$

(1.7)

$$F_D^{(3)} [a, b_1, b_2, b_3; c; x, y, z] = \sum_{m,n,r=0}^{\infty} \frac{(a)_{m+n+r}(b_1)_m(b_2)_n(b_3)_r}{(c)_{m+n+r}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \\ (\max\{|x|, |y|, |z|\} < 1).$$

In 2014, using $B_\gamma^{(\alpha, \beta)}(x, y)$, Liu [9, p. 115] defined the extended Appell's hypergeometric functions of two variables,

$$F_1^{(\alpha, \beta)} [a, b, c; d; x, y; \gamma] \text{ and } F_2^{(\alpha, \beta, \alpha', \beta')} [a, b, c; d, e; x, y; \gamma],$$

as well as Lauricella's hypergeometric function of three variables

$$F_{D,\gamma}^{(3;\alpha, \beta)} [a, b, c, d; e; x, y, z].$$

We define the new generalized extended Appell's hypergeometric functions, by using (1.2),

$$F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma], \quad F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)} [a, b, c, d, e; x, y; \gamma]$$

and the new generalized extended Lauricella's hypergeometric function

$$F_{D,\gamma}^{(3;\alpha, \beta; \kappa, \mu)} [a, b, c, d; e; x, y, z; \gamma] \quad (\text{see, e.g. [1, p. 406]}),$$

by

$$(1.8) \quad F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma] \\ = \sum_{m,n=0}^{\infty} \frac{B_\gamma^{(\alpha, \beta; \kappa, \mu)}(a+m+n, d-a)(b)_m(c)_n}{B(a, d-a)} \frac{x^m}{m!} \frac{y^n}{n!} \\ (|x| < 1, |y| < 1);$$

$$(1.9) \quad F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma] \\ = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} B_\gamma^{(\alpha, \beta; \kappa, \mu)}(b+n, d-b)}{B(a, d-a)} \frac{B_\gamma^{(\alpha', \beta'; \nu, \eta)}(c+m, e-c)}{B(c, e-c)} \frac{x^m}{m!} \frac{y^n}{n!} \\ (|x| + |y| < 1);$$

$$(1.10) \quad F_{D,\gamma}^{(3;\alpha, \beta; \kappa, \mu)} [a, b, c, d; e; x, y, z; \gamma] \\ = \sum_{m,n,r=0}^{\infty} \frac{B_\gamma^{(\alpha, \beta; \kappa, \mu)}(a+m+n+r, e-a)}{B(a, e-a)} \\ (b)_m(c)_n(d)_r \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \quad (\max\{|x|, |y|, |z|\} < 1).$$

The Mellin transform of $f(x)$ is defined in the following way (see, e.g., [9, p. 1993])

$$(1.11) \quad \mathcal{M}\{f(x); s\} = \int_0^\infty f(x)x^{s-1} dx,$$

provided the integral converges. Its inverse transform is

$$(1.12) \quad f(x) = \mathcal{M}^{-1}\{\varphi(s)\} := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s} \varphi(s) ds,$$

where the integration path in the imaginary axis starting from $-i\infty$ and ending at the point $i\infty$.

In this paper, we generalize the extended Appell's and Lauricella's hypergeometric functions which has recently been introduced by Liu [9], Khan [7], and Agarwal et al. [1]. Also, we aim at establishing some (presumably) new integral representations and transforms for the extended generalized Appell's and Lauricella's hypergeometric functions.

2. Integral representations

In this section, we present three integral representations of the generalized extended Appell's hypergeometric functions and the generalized extended Lauricella's hypergeometric function.

Theorem 1. *For the generalized extended Appell's hypergeometric function $F_1^{(\alpha, \beta; \kappa, \mu)}[a, b, c; d; x, y; \gamma]$, we have*

$$(2.1) \quad \begin{aligned} & F_1^{(\alpha, \beta; \kappa, \mu)}[a, b, c; d; x, y; \gamma] \\ &= \frac{1}{B(a, d-a)} \int_0^1 t^{a-1} (1-t)^{d-a-1} (1-xt)^{-b} (1-yt)^{-c} \\ & \quad {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) dt. \end{aligned}$$

Proof. Interchanging the order of summation and integration, using the generalized beta function (1.4) and binomial theorem, we have

$$(2.2) \quad \begin{aligned} & \int_0^1 t^{a-1} (1-t)^{d-a-1} (1-xt)^{-b} (1-yt)^{-c} {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) dt \\ &= \sum_{m,n=0}^{\infty} (b)_m (c)_n \frac{x^m}{m!} \frac{y^n}{n!} \left\{ \int_0^1 t^{a+m+n-1} (1-t)^{d-a-1} {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) dt \right\} \\ &= \sum_{m,n=0}^{\infty} B_\gamma^{(\alpha, \beta; \kappa, \mu)}(a+m+n, d-a) (b)_m (c)_n \frac{x^m}{m!} \frac{y^n}{n!} \\ &= B(a, d-a) F_1^{(\alpha, \beta; \kappa, \mu)}[a, b, c; d; x, y; \gamma]. \end{aligned}$$

Therefore, we obtain the desired result. \square

Theorem 2. *For the generalized extended Appell's hypergeometric function $F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma]$, we have*

$$(2.3) \quad \begin{aligned} & F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma] \\ &= \frac{1}{B(a, d-b)B(c, e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \\ & \quad \times {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) {}_1F_1\left(\alpha'; \beta'; -\frac{\gamma}{s^\nu(1-s)^\eta}\right) dt ds. \end{aligned}$$

Proof.

$$(2.4) \quad \begin{aligned} & \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \\ & \quad \times {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) {}_1F_1\left(\alpha'; \beta'; -\frac{\gamma}{s^\nu(1-s)^\eta}\right) dt ds. \end{aligned}$$

Taking into account the summation formula

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{m,n=0}^{\infty} f(m+n) \frac{x^m y^n}{m! n!},$$

interchanging the order of summation and integration, we have

$$(2.5) \quad \begin{aligned} & \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \\ & \quad \times {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) {}_1F_1\left(\alpha'; \beta'; -\frac{\gamma}{s^\nu(1-s)^\eta}\right) dt ds \\ &= \sum_{m,n=0}^{\infty} (a)_{m+n} \frac{x^m y^n}{m! n!} \int_0^1 t^{b+m-1}(1-t)^{d-b-1} {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) dt \\ & \quad \times \int_0^1 s^{c+n-1}(1-s)^{e-c-1} {}_1F_1\left(\alpha'; \beta'; -\frac{\gamma}{s^\nu(1-s)^\eta}\right) ds. \end{aligned}$$

By the generalized beta function (1.4), and Theorem 1,

$$(2.6) \quad \begin{aligned} & \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \\ & \quad \times {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa(1-t)^\mu}\right) {}_1F_1\left(\alpha'; \beta'; -\frac{\gamma}{s^\nu(1-s)^\eta}\right) dt ds \\ &= \sum_{m,n=0}^{\infty} (a)_{m+n} \frac{x^m y^n}{m! n!} B_\gamma^{(\alpha, \beta; \kappa, \mu)}(b+m, d-b) B_\gamma^{(\alpha', \beta'; \nu, \eta)}(c+n, e-c) \\ &= B(b, d-b) B(c, e-c) F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma]. \quad \square \end{aligned}$$

Theorem 3. *For the new generalized extended Lauricella's hypergeometric function*

$$F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)} [a, b, c, d; e; x, y, z; \gamma],$$

we have

$$(2.7) \quad \begin{aligned} & F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)} [a, b, c, d; e; x, y, z; \gamma] \\ &= \frac{1}{B(a, e-a)} \int_0^1 t^{a-1} (1-t)^{e-a-1} (1-xt)^{-b} (1-yt)^{-c} (1-zt)^{-d} \\ & \quad \times {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) dt. \end{aligned}$$

Proof. By changing the order of integration and summation which may be verified under the conditions, and using the generalized Beta function (1.4) and binomial theorem, we have

$$(2.8) \quad \begin{aligned} & \int_0^1 t^{a-1} (1-t)^{e-a-1} (1-xt)^{-b} (1-yt)^{-c} (1-zt)^{-d} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) dt \\ &= \sum_{m,n,r=0}^{\infty} \left\{ \int_0^1 t^{a+m+n+r-1} (1-t)^{e-a-1} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) dt \right\} \\ & \quad \times (b)_m (c)_n (d)_r \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \\ &= \sum_{m,n,r=0}^{\infty} B_{\gamma}^{(\alpha,\beta;\kappa,\mu)} (a+m+n+r, e-a) (b)_m (c)_n (d)_r \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \\ &= B(a, e-a) F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)} [a, b, c, d; e; x, y, z; \gamma]. \end{aligned}$$

Hence, we obtain the desired result. \square

3. Mellin transforms

In this section, we will give the Mellin transforms of

$$\begin{aligned} & F_1^{(\alpha,\beta;\kappa,\mu)} [a, b, c; d; x, y; \gamma], F_2^{(\alpha,\beta,\alpha',\beta';\kappa,\mu,\nu,\eta)} [a, b, c; d, e; x, y; \gamma] \text{ and} \\ & F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)} [a, b, c, d; e; x, y, z; \gamma]. \end{aligned}$$

Recall that the Mellin transform of $f(x)$ is defined in the following (see, *e.g.*, [9])

$$(3.1) \quad \mathcal{M}\{f(x); s\} = \int_0^\infty f(x) x^{s-1} dx,$$

provided the integral converges, and its inverse transform is

$$(3.2) \quad f(x) = \mathcal{M}^{-1}\{\varphi(s)\} := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s} \varphi(s) ds,$$

where the integration path in the imaginary axis starting from $-i\infty$ and ending at the point $i\infty$.

Theorem 4. *For the generalized extended Appell's hypergeometric function $F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma]$, the following Mellin transform formula holds true:*

$$(3.3) \quad \begin{aligned} & \mathcal{M}\{F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma] ; s\} \\ &= \frac{\Gamma(s)B(\alpha - s, s)B(d - a + \mu s, a + \kappa s)}{B(a, d - a)B(\beta - s, s)} F_1 [a + \kappa s, b, c; d + \kappa s + \mu s; x, y]. \end{aligned}$$

Proof. In order to prove the assertion (3.3), by taking the Mellin transform of (2.1), we obtain

$$(3.4) \quad \begin{aligned} & \mathcal{M}\{F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma] ; s\} \\ &= \frac{1}{B(a, d - a)} \\ & \quad \times \int_0^\infty \gamma^{s-1} \left\{ \int_0^1 t^{a-1} (1-t)^{d-a-1} (1-xt)^{-b} (1-yt)^{-c} \right. \\ & \quad \left. {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) dt \right\} d\gamma \\ &= \frac{1}{B(a, d - a)} \int_0^1 t^{a-1} (1-t)^{d-a-1} (1-xt)^{-b} (1-yt)^{-c} \\ & \quad \times \left\{ \int_0^\infty \gamma^{s-1} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) d\gamma \right\} dt. \end{aligned}$$

According to [18],

$$(3.5) \quad \int_0^\infty \gamma^{s-1} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) d\gamma = t^{\kappa s} (1-t)^{\mu s} \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-s)}.$$

By changing the order of integration and summation which may be verified under the conditions, and using the generalized beta function (1.4) and binomial theorem, we have

$$(3.6) \quad \begin{aligned} & \mathcal{M}\{F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma] ; s\} \\ &= \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{B(a, d - a)\Gamma(\alpha)\Gamma(\beta-s)} \sum_{m,n=0}^{\infty} (b)_m (c)_n \frac{x^m}{m!} \frac{y^n}{n!} \\ & \quad \times \int_0^1 t^{a+\kappa s+m+n-1} (1-t)^{d-a+\mu s-1} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(s)B(\alpha-s, s)B(d-a+\mu s, a+\kappa s)}{B(a, d-a)B(\beta-s, s)} \sum_{m,n=0}^{\infty} \frac{(a+\kappa s)_{m+n}(b)_m(c)_n}{(d+\kappa s+\mu s)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \frac{\Gamma(s)B(\alpha-s, s)B(d-a+\mu s, a+\kappa s)}{B(a, d-a)B(\beta-s, s)} F_1 [a+\kappa s, b, c; d+\kappa s+\mu s; x, y],
\end{aligned}$$

which is the desired result. \square

Corollary 5. *By Mellin inverse formula, we have the following contour integral representation for the function $F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma]$:*

$$\begin{aligned}
&F_1^{(\alpha, \beta; \kappa, \mu)} [a, b, c; d; x, y; \gamma] \\
(3.7) \quad &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)B(\alpha-s, s)B(d-a+\mu s, a+\kappa s)}{B(a, d-a)B(\beta-s, s)} \\
&\times F_1 [a+\kappa s, b, c; d+\kappa s+\mu s; x, y] \gamma^{-s} ds.
\end{aligned}$$

Theorem 6. *For the generalized extended Appell's hypergeometric function $F_2^{(\alpha, \beta, r, r; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma]$, we have*

$$\begin{aligned}
(3.8) \quad &\mathcal{M}\{F_2^{(\alpha, \beta, r, r; \kappa, \mu, \nu, \eta)} [a, b, c; d, e; x, y; \gamma]; s\} \\
&= \frac{\Gamma(s)}{B(a, d-a)B(c, e-c)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (s)_n B(b-\kappa n, d-b-\mu n+\kappa n) B(c+\nu s+\nu n, \eta s+\eta n-c-\nu s-\nu n)}{n! (-1)^n (\beta)_n} \\
&\times F_2 [a, b-\kappa n, c+\nu s+\nu n; d-\mu n, \eta s+\eta n; x, y].
\end{aligned}$$

Proof. Let \mathcal{L} be the left-hand side of (3.8). By taking the Mellin transform of (2.3), we obtain

$$\begin{aligned}
(3.9) \quad \mathcal{L} &= \frac{1}{B(a, d-a)B(c, e-c)} \\
&\int_0^1 \int_0^1 \frac{t^{b-1} (1-t)^{d-b-1} s^{c-1} (1-s)^{e-c-1}}{(1-xt-ys)^a} \\
&\times \left\{ \int_0^\infty \gamma^{s-1} \exp\left\{ \frac{-\gamma}{s^\nu (1-s)^\eta} \right\} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) d\gamma \right\} dt ds.
\end{aligned}$$

According to [18], there exists the following integral formula for the confluent function ${}_1F_1$:

$$(3.10) \quad \int_0^\infty t^{s-1} \exp\left\{ -ct \right\} {}_1F_1 (\alpha; \beta; -t) dt = c^{-s} \Gamma(s) {}_2F_1 \left(\alpha, s; \beta; -\frac{1}{c} \right).$$

Then, setting $u = \frac{\gamma}{t^\kappa (1-t)^\mu}$, using the above formula, we have

$$\begin{aligned}
(3.11) \quad &\int_0^\infty \gamma^{s-1} \exp\left\{ \frac{-\gamma}{s^\nu (1-s)^\eta} \right\} {}_1F_1 \left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu} \right) d\gamma \\
&= s^{\nu s} (1-s)^{\eta s} \Gamma(s) {}_2F_1 \left(\alpha, s; \beta; -\frac{s^\nu (1-s)^\eta}{t^\kappa (1-t)^\mu} \right).
\end{aligned}$$

Applying (3.11) to (3.9), we get

$$\begin{aligned}
 (3.12) \quad & \mathcal{L} = \frac{\Gamma(s)}{B(a, d-a)B(c, e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c+\nu s-1}(1-s)^{e-c+\eta s-1}}{(1-xt-ys)^a} \\
 & \times {}_2F_1\left(\alpha, s; \beta; -\frac{s^\nu(1-s)^\eta}{t^\kappa(1-t)^\mu}\right) dt ds \\
 & = \frac{\Gamma(s)}{B(a, d-a)B(c, e-c)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(s)_n}{n!(-1)^n(\beta)_n} \\
 & \times \int_0^1 \int_0^1 \frac{t^{b-\kappa n-1}(1-t)^{d-b-\nu n-1}s^{c+\nu s+\nu n-1}(1-s)^{e-c+\eta s+\eta n-1}}{(1-xt-ys)^a} dt ds.
 \end{aligned}$$

By the integral representation of Appell's function F_2 [11], we obtain

$$\begin{aligned}
 (3.13) \quad & \int_0^1 \int_0^1 \frac{t^{b-\kappa n-1}(1-t)^{d-b-\nu n-1}s^{c+\nu s+\nu n-1}(1-s)^{e-c+\eta s+\eta n-1}}{(1-xt-ys)^a} dt ds \\
 & = B(b-\kappa n, d-\mu n-b+\kappa n) B(c+\nu s+\nu n, \eta s+\eta n-c-\nu s-\nu n) \\
 & \times F_2[a, b-\kappa n, c+\nu s+\nu n; d-\mu n, \eta s+\eta n; x, y].
 \end{aligned}$$

Then, by applying (3.13) to (3.12), we have

$$\begin{aligned}
 (3.14) \quad & \mathcal{L} = \frac{\Gamma(s)}{B(a, d-a)B(c, e-c)} \\
 & \sum_{n=0}^{\infty} \frac{(\alpha)_n(s)_n B(b-\kappa n, d-\mu n-b+\kappa n) B(c+\nu s+\nu n, \eta s+\eta n-c-\nu s-\nu n)}{n!(-1)^n(\beta)_n} \\
 & \times F_2[a, b-\kappa n, c+\nu s+\nu n; d-\mu n, \eta s+\eta n; x, y]
 \end{aligned}$$

which leads to the right-hand side of (3.8). \square

Considering the parameter b and d which are symmetric to c and e , respectively, in $F_2^{(\alpha, \beta, r, r; \kappa, \mu, \nu, \eta)}[a, b, c; d, e; x, y; \gamma]$, we get:

Corollary 7. *For the generalized extended Appell's hypergeometric function $F_2^{(\alpha, \beta, \alpha', \beta'; \kappa, \mu, \nu, \eta)}[a, b, c; d, e; x, y; \gamma]$, we have*

$$\begin{aligned}
 (3.15) \quad & \mathcal{M}\{F_2^{r, r, \alpha, \beta; \kappa, \mu, \nu, \eta}[a, b, c; d, e; x, y; \gamma]; s\} \\
 & = \frac{\Gamma(s)}{B(b, d-b)B(c, e-c)} \\
 & \sum_{n=0}^{\infty} \frac{(\alpha)_n(s)_n B(c-\kappa n, e-c-\mu n+\kappa n) B(b+\nu s+\nu n, \eta s+\eta n-b-\nu s-\nu n)}{n!(-1)^n(\beta)_n} \\
 & \times F_2[a, c-\kappa n, b+\nu s+\nu n; e-\mu n, \eta s+\eta n; x, y].
 \end{aligned}$$

Finally, we have:

Theorem 8. *For the new generalized extended Lauricella's hypergeometric function $F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}[a, b, c, d; e; x, y, z; \gamma]$, the following Mellin transform holds:*

$$(3.16) \quad \begin{aligned} & \mathcal{M}\{F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}[a, b, c, d; e; x, y, z; \gamma]; s\} \\ &= \frac{\Gamma(s) B(\alpha - s, s) B(e - a + \mu s, a + \kappa s)}{B(a, e - a) B(\beta - s, s)} \\ & \quad \times F_D^{(3)}[a + \kappa s, b, c, d; e + \kappa s + \mu s; x, y, z]. \end{aligned}$$

Proof. Let \mathcal{L} be the left-hand side of (3.16). By taking the Mellin transform of (2.7), we obtain

$$(3.17) \quad \begin{aligned} \mathcal{L} &= \frac{1}{B(a, e - a)} \int_0^1 t^{a-1} (1-t)^{e-a-1} (1-xt)^{-b} (1-yt)^{-c} (1-zt)^{-d} \\ & \quad \times \left\{ \int_0^\infty \gamma^{s-1} {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu}\right) d\gamma \right\} dt. \end{aligned}$$

According to [18], there exists the following integral formula for the confluent function ${}_1F_1$:

$$(3.18) \quad \int_0^\infty \gamma^{s-1} {}_1F_1\left(\alpha; \beta; -\frac{\gamma}{t^\kappa (1-t)^\mu}\right) d\gamma = t^{\kappa s} (1-t)^{\nu s} \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-s)}.$$

Then, using the above formula, we have

$$(3.19) \quad \begin{aligned} \mathcal{L} &= \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{B(a, e - a)\Gamma(\alpha)\Gamma(\beta-s)} \\ & \quad \times \int_0^1 t^{a+\kappa s-1} (1-t)^{e-a+\mu s-1} (1-xt)^{-b} (1-yt)^{-c} (1-zt)^{-d} dt. \end{aligned}$$

By the binomial theorem and beta function, we get

$$(3.20) \quad \begin{aligned} \mathcal{L} &= \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{B(a, e - a)\Gamma(\alpha)\Gamma(\beta-s)} \\ & \quad \times \sum_{m,n,r=0}^{\infty} (b)_m (c)_n (d)_r \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \int_0^1 t^{a+\kappa s+m+n+r-1} (1-t)^{e-a+\mu s-1} dt \\ &= \frac{\Gamma(s) B(\alpha - s, s) B(e - a + \mu s, a + \kappa s)}{B(a, e - a) B(\beta - s, s)} \\ & \quad \times \sum_{m,n,r=0}^{\infty} \frac{(a + \kappa s)_{m+n+r} (b)_m (c)_n (d)_r}{(e + \kappa s + \mu s)_{m+n+r}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \\ &= \frac{\Gamma(s) B(\alpha - s, s) B(e - a + \mu s, a + \kappa s)}{B(a, e - a) B(\beta - s, s)} \\ & \quad \times F_D^{(3)}[a + \kappa s, b, c, d; e + \kappa s + \mu s; x, y, z], \end{aligned}$$

which, upon using (1.10), leads to the right-hand side (3.16). \square

In case $\kappa = \mu = 1$, the formula (3.16) reduces to the result in [9, Theorem 3.3].

Corollary 9. *We have the following contour integral representation for the function $F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}[a, b, c, d; e; x, y, z; \gamma]$:*

$$(3.21) \quad \begin{aligned} & F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}[a, b, c, d; e; x, y, z; \gamma] \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)B(\alpha-s, s)B(e-a+\mu s, a+\kappa s)}{B(a, e-a)B(\beta-s, s)} \\ & \quad \times F_D^{(3)}[a+\kappa s, b, c, d; e+\kappa s+\mu s; x, y, z] \gamma^{-s} ds. \end{aligned}$$

Upon setting $\kappa = \mu$ from (2.1) to (3.21), we arrive at the known results in [1].

4. Concluding remarks

The integral representations and Mellin transforms used in this paper can provide several (presumably) new transformation formulas involving multiple variables hypergeometric functions. These results may be useful in number theory, the theory of asymptotic expansion and computer science.

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