

INCOMPLETE EXTENDED HURWITZ-LERCH ZETA FUNCTIONS AND ASSOCIATED PROPERTIES

RAKESH K. PARMAR AND RAM K. SAXENA

ABSTRACT. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [*Integral Transforms Spec. Funct.* 23 (2012), 659–683] by means of the incomplete Pochhammer symbols $(\lambda; \kappa)_\nu$ and $[\lambda; \kappa]_v$, we first introduce incomplete Fox-Wright function. We then define the families of incomplete extended Hurwitz-Lerch Zeta function. We then systematically investigate several interesting properties of these incomplete extended Hurwitz-Lerch Zeta function which include various integral representations, summation formula, fractional derivative formula. We also consider an application to probability distributions and some special cases of our main results.

1. Introduction, definitions and preliminaries

The familiar *incomplete Gamma functions* $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$(1.1) \quad \gamma(s, x) := \int_0^x t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0)$$

and

$$(1.2) \quad \Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (x \geq 0; \Re(s) > 0 \text{ when } x = 0),$$

respectively, satisfy the following decomposition formula:

$$(1.3) \quad \gamma(s, x) + \Gamma(s, x) := \Gamma(s) \quad (\Re(s) > 0).$$

Throughout this paper, \mathbb{N} , \mathbb{Z}^- and \mathbb{C} denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

Received November 30, 2015.

2010 *Mathematics Subject Classification.* Primary 11M06, 11M35, 33B15, 33C60; Secondary 11B68, 33C65, 33C90.

Key words and phrases. gamma functions, incomplete gamma functions, Pochhammer symbol, incomplete Pochhammer symbols, incomplete generalized hypergeometric functions, incomplete Fox-Wright function, generalized Hurwitz-Lerch zeta function, incomplete Hurwitz-Lerch zeta function, fractional derivative operator, probability density function.

Moreover, the parameter $x \geq 0$ used above in (1.1) and (1.2) and elsewhere in this paper is independent of $\Re(z)$ of the complex number $z \in \mathbb{C}$.

Recently, Srivastava *et al.* [22] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$$(1.4) \quad {}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

and

$$(1.5) \quad {}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where, in terms of the incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by (1.1) and (1.2), respectively, the *incomplete* Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ ($\lambda, \nu \in \mathbb{C}; x \geq 0$) are defined as follows:

$$(1.6) \quad (\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0)$$

and

$$(1.7) \quad [\lambda; x]_\nu := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0),$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ satisfy the following decomposition relation:

$$(1.8) \quad (\lambda; x)_\nu + [\lambda; x]_\nu := (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; x \geq 0).$$

Here, and in what follows, $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(1.9) \quad (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ -quotient exists (see, for details, [26, p. 21 *et seq.*]).

It is observed that the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$(1.10) \quad {}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] + {}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \\ = {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right]$$

for the familiar generalized hypergeometric function ${}_pF_q$.

The *Hurwitz-Lerch Zeta function* $\Phi(z, s, a)$ is defined by (see, e.g., [9, p. 27, Eq. 1.11(1)]; see also [23, p. 121] and [24, p. 194]):

$$(1.11) \quad \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

The Hurwitz-Lerch Zeta function has the well-known integral representation (see, e.g., [9, p. 27, Eq. 1.11(3)]; see also [24, p. 194, Eq. 2.5(4)]):

$$(1.12) \quad \Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t - z} dt$$

$$(\Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1).$$

Various generalizations of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ have been investigated by several authors (see, e.g., [1, 4, 5, 9, 12, 21, 25, 27, 29]). In particular, Goyal and Laddha [11, p. 100, Eq. (1.5)], Lin and Srivastava [14, p. 727, Eq. (8)] and Garg *et al.* [10, p. 313, Eq. (1.7)] studied certain functions which are, respectively, defined by

$$(1.13) \quad \Phi_\mu^*(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}$$

$$(\mu \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s-\mu) > 1 \text{ when } |z| = 1),$$

$$(1.14) \quad \Phi_{\mu;\nu}^{(\rho,\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

$$(\mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma \in \mathbb{R}^+; \rho < \sigma \text{ when } s, z \in \mathbb{C}; \rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < \delta := \rho^{-\rho} \sigma^{-\sigma}; \rho = \sigma \text{ and } \Re(s+\nu-\mu) > 1 \text{ when } |z| = \delta)$$

and

$$(1.15) \quad \Phi_{\lambda,\mu;\nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s}$$

($\lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s+\nu-\lambda-\mu) > 1 \text{ when } |z| = 1$), or, equivalently, by means of an integral representations of (1.13), (1.14) and (1.15) which are given, respectively, by

$$(1.16) \quad \Phi_\mu^*(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{(1 - ze^{-t})^\mu} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{(e^t - z)^\mu} dt$$

$$(\Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1),$$

$$(1.17) \quad \Phi_{(\mu;\nu)}^{(\rho,\kappa)}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2\Psi_1^* \left[\begin{matrix} (1, 1), (\mu, \rho); \\ (\nu, \kappa); \end{matrix} ze^{-t} \right] dt$$

$$(\min\{\Re(s), \Re(a)\} > 0, \kappa > \rho > 0 \text{ when } z \in \mathbb{C}; \kappa \geq \rho > 0 \text{ when } |z| < \rho^{-\rho} \kappa^\kappa)$$

and

$$(1.18) \quad \Phi_{\lambda,\mu;\nu}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2F_1(\lambda, \mu; \nu; ze^{-t}) dt$$

$$(\Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1).$$

Motivated essentially by the demonstrated potential for applications of these incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [2, 6, 7, 8, 16, 30, 31, 32, 33, 34] and the references cited therein), here, we aim here at systematically investigating the family of the incomplete extended Hurwitz-Lerch Zeta function. We, first introduce incomplete Fox-Wright Psi function. We then systematically investigate several interesting properties of incomplete extended Hurwitz-Lerch Zeta function which include other families of incomplete generalized Hurwitz-Lerch Zeta function, various integral representations, sum-integral formula, summation formula and fractional derivative formula. We also consider an application to probability distributions and some special cases of our main results.

2. The incomplete Fox-Wright function

In this section, we introduce a family of the incomplete Fox-Wright function ${}_p\psi_q(z)$ and ${}_p\bar{\psi}_q(z)$, which is a further generalization of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ defined by (1.4) and (1.5), with p numerator and q denominator parameters, such that $\alpha_1, \dots, \alpha_p \in \mathbb{C}$, and $\beta_1, \dots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$ are defined by

$$(2.1) \quad {}_p\psi_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\gamma(\alpha_1 + A_1 n, x) \Gamma(\alpha_2 + A_2 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}$$

and

$$(2.2) \quad {}_p\bar{\psi}_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n, x) \Gamma(\alpha_2 + A_2 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}$$

or its normalization ${}_p\psi_q^*$ and ${}_p\bar{\psi}_q^*$, respectively, are defined by

$$\begin{aligned} {}_p\psi_q^* \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_{A_1 n} (\alpha_2)_{A_2 n} \cdots (\alpha_p)_{A_p n}}{(\beta_1)_{B_1 n} \cdots (\beta_q)_{B_q n}} \frac{z^n}{n!} \end{aligned}$$

$$(2.3) \quad = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} {}_p\bar{\psi}_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right]$$

and

$$(2.4) \quad \begin{aligned} & {}_p\bar{\psi}_q^* \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] \\ &= \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_{A_1 n} [\alpha_2]_{A_2 n} \cdots [\alpha_p]_{A_p n}}{(\beta_1)_{B_1 n} \cdots (\beta_q)_{B_q n}} \frac{z^n}{n!} \\ &= \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} {}_p\bar{\psi}_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right]. \end{aligned}$$

Remark 1. Since it is easily seen that

$$|(\lambda; x)_{\rho n}| \leq |(\lambda)_{\rho n}| \quad \text{and} \quad |[\lambda; x]_{\rho n}| \leq |(\lambda)_{\rho n}| \quad (\lambda \in \mathbb{C}; n \in \mathbb{N}_0, \rho > 0; x \geq 0),$$

the precise sufficient conditions under which the infinite series in definitions (2.3) and (2.4) would converge absolutely can be derived from those in the case of the Fox-Wright function ${}_p\Psi_q(z)$ or ${}_p\Psi_q^*(z)$ ($p, q \in \mathbb{N}_0$) (see [15, 26, 28]). Indeed, in their special case when $x = 0$, both ${}_p\psi_q^*(z)$ ($p, q \in \mathbb{N}_0$) and ${}_p\bar{\psi}_q^*(z)$ ($p, q \in \mathbb{N}_0$) would reduce immediately to the Fox-Wright Psi function ${}_p\Psi_q(z)$ or ${}_p\Psi_q^*(z)$ ($p, q \in \mathbb{N}_0$). In view of (1.8), these families of incomplete Fox-Wright generalized hypergeometric function satisfy the following decomposition formula:

$$(2.5) \quad \begin{aligned} & {}_p\psi_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] \\ &+ {}_p\bar{\psi}_q \left[\begin{matrix} (\alpha_1, A_1, x), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] \\ &= {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right], \end{aligned}$$

where ${}_p\Psi_q(z)$ or ${}_p\Psi_q^*(z)$ is the Fox-Wright generalized hypergeometric function [15, 26, 28].

Remark 2. The special case when

$$A_j = B_k = 1 \quad (j = 1, \dots, p; k = 1, \dots, q)$$

in (2.3) and (2.4) immediately reduces to (1.4) and (1.5), respectively.

Remark 3. It is interesting to note that if we set

$$p \mapsto p + 1, \quad \alpha_{p+1} = A_{p+1} = 1, \quad q \mapsto q + 1, \quad \beta_{q+1} = \beta, \quad B_{p+1} = \alpha$$

$$\text{and } A_j = B_k = 1 \quad (j = 1, \dots, p; k = 1, \dots, q)$$

in (2.3) and (2.4), we obtain the following family of incomplete generalized M-series:

$$(2.6) \quad {}_p\gamma_q^{\alpha, \beta} \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

and

$$(2.7) \quad {}_p\Gamma_q^{\alpha,\beta} \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

respectively. So that it satisfy the decomposition formula:

$${}_p\gamma_q^{\alpha,\beta} + {}_p\Gamma_q^{\alpha,\beta} = {}_pM_q,$$

where ${}_pM_q$ is so-called generalized M-series introduced by Sharma and Jain [20] (see also, [18, 19]).

3. The incomplete extended Hurwitz-Lerch zeta function

In this section, we introduce the families of the incomplete extended Hurwitz-Lerch Zeta function $\phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a)$ and $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a)$ as follows:

$$(3.1) \quad \phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda; x)_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda, \mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma, \kappa \in \mathbb{R}^+; \kappa - \rho - \sigma > -1 \text{ when } s, z \in \mathbb{C};$
 $\kappa - \rho - \sigma = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \delta^* := \rho^{-\rho} \sigma^{-\sigma} \kappa^\kappa;$
 $\kappa - \rho - \sigma = -1 \text{ and } \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = \delta^*)$

and

$$(3.2) \quad \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda, \mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma, \kappa \in \mathbb{R}^+; \kappa - \rho - \sigma > -1 \text{ when } s, z \in \mathbb{C};$
 $\kappa - \rho - \sigma = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \delta^* := \rho^{-\rho} \sigma^{-\sigma} \kappa^\kappa;$
 $\kappa - \rho - \sigma = -1 \text{ and } \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = \delta^*).$

In view of (1.8), these families of incomplete extended Hurwitz-Lerch Zeta function satisfy the following decomposition formula:

$$(3.3) \quad \phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a) + \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a) = \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a),$$

where $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a)$ is the extended Hurwitz-Lerch Zeta function [29, p. 491, Eq. (1.20)]:

$$(3.4) \quad \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

$(\lambda, \mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma, \kappa \in \mathbb{R}^+; \kappa - \rho - \sigma > -1 \text{ when } s, z \in \mathbb{C};$
 $\kappa - \rho - \sigma = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \delta^* := \rho^{-\rho} \sigma^{-\sigma} \kappa^\kappa;$
 $\kappa - \rho - \sigma = -1 \text{ and } \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = \delta^*).$

Remark 4. The following special or limit cases of the incomplete extended Hurwitz-Lerch Zeta function $\phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ and $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ defined by (3.1) and (3.2) are useful in present investigation:

Case 1. If we set $\mu = 1 = \sigma$ in (3.1) and (3.2), we obtain another family of incomplete generalized Hurwitz-Lerch Zeta function studied earlier by Lin and Srivastava [14, p. 727, Eq. (8)]:

$$(3.5) \quad \phi_{\lambda;\nu}^{(\rho,\kappa)}(z, s, a) := \phi_{\lambda,1;\nu}^{(\rho,1,\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda;x)_{\rho n}}{(\nu)_{\kappa n}} \frac{z^n}{(n+a)^s}$$

and

$$(3.6) \quad \varphi_{\lambda;\nu}^{(\rho,\kappa)}(z, s, a) := \varphi_{\lambda,1;\nu}^{(\rho,1,\kappa)}(z, s, a) := \sum_{n=0}^{\infty} \frac{[\lambda;x]_{\rho n}}{(\nu)_{\kappa n}} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s+\nu-\mu) > 1 \text{ when } |z| = 1).$

Case 2. If we set $\rho = \sigma = \kappa = 1$ in (3.1) and (3.2), we obtain incomplete generalized Hurwitz-Lerch Zeta function introduced by Garg *et al.* [10, p. 313, Eq. (1.7)]:

$$(3.7) \quad \phi_{\lambda,\mu;\nu}^{(1,1,1)}(z, s, a) := \phi_{\lambda,\mu;\nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda;x)_n(\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s}$$

and

$$(3.8) \quad \varphi_{\lambda,\mu;\nu}^{(1,1,1)}(z, s, a) := \varphi_{\lambda,\mu;\nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{[\lambda;x]_n(\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1;$
 $\Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = 1).$

Case 3. Upon setting $\rho = \sigma = \kappa = 1$ and $\mu = \nu$ in (3.1) and (3.2), we obtain incomplete Hurwitz-Lerch Zeta function of Goyal and Laddha [11, p. 100, Eq. (1.5)]:

$$(3.9) \quad \phi_{\lambda,\mu;\mu}^{(1,1,1)}(z, s, a) := \phi_{\lambda}^{*}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda;x)_n}{n!} \frac{z^n}{(n+a)^s}$$

and

$$(3.10) \quad \varphi_{\lambda,\mu;\mu}^{(1,1,1)}(z, s, a) := \varphi_{\lambda}^{*}(z, s, a) := \sum_{n=0}^{\infty} \frac{[\lambda;x]_n}{n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s - \lambda) > 1 \text{ when } |z| = 1).$

Case 4. If we take limit $|\mu| \rightarrow \infty$ and replacing $\frac{z}{\mu}$ in (3.1) and (3.2), we have the limiting case

$$(3.11) \quad \phi_{\lambda;\nu}^{*(\rho,\kappa)}(z, s, a) := \lim_{|\mu| \rightarrow \infty} \phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}\left(\frac{z}{\mu^\sigma}, s, a\right) := \sum_{n=0}^{\infty} \frac{(\lambda;x)_{\rho n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

and

(3.12)

$$\varphi_{\lambda;\nu}^{*(\rho,\kappa)}(z, s, a) := \lim_{|\mu| \rightarrow \infty} \left\{ \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)} \left(\frac{z}{\mu^\sigma}, s, a \right) \right\} := \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s+\nu-\lambda) > 1 \text{ when } |z| = 1).$

Case 5. Another limit cases of (3.1) and (3.2) are given by

(3.13)

$$\phi_{\lambda}^{*(\rho)}(z, s, a) := \lim_{\min |\mu|, |\nu| \rightarrow \infty} \left\{ \phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)} \left(\frac{z\nu^\kappa}{\mu^\sigma}, s, a \right) \right\} := \sum_{n=0}^{\infty} \frac{(\lambda; x)_{\rho n}}{n!} \frac{z^n}{(n+a)^s}$$

and

(3.14)

$$\varphi_{\lambda}^{*(\rho)}(z, s, a) := \lim_{\min |\mu|, |\nu| \rightarrow \infty} \left\{ \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)} \left(\frac{z\nu^\kappa}{\mu^\sigma}, s, a \right) \right\} := \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}}{n!} \frac{z^n}{(n+a)^s}$$

$(x \geq 0; \lambda \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s+\nu-\lambda) > 1 \text{ when } |z| = 1).$

It is noted in passing that, in view of the decomposition formula (3.3), we had better discuss the properties and characteristics of the incomplete extended Hurwitz-Lerch Zeta function $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$.

4. Integral representations and summation formula

In this section, we present various integral representations of the incomplete extended Hurwitz-Lerch Zeta function $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ defined by (3.2) including their certain special cases. A summation formula is also evaluated.

Theorem 1. *The following integral representation for $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ in (3.2) holds:*

$$(4.1) \quad \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} {}_2\overline{\psi}_1^* \left[\begin{matrix} (\lambda, \rho, x), (\mu, \sigma); \\ (\nu, \kappa); \end{matrix} ze^{-t} \right] dt$$

$(\Re(x) \geq 0, \min\{\Re(s), \Re(a)\} > 0, \rho, \sigma, \kappa \in \mathbb{R}^+ \text{ and } \kappa - \rho - \sigma \geqq -1$
 $\text{when } |z| < \delta^* := \rho^{-\rho} \sigma^{-\sigma} \kappa^\kappa).$

Proof. Using the following Eulerian integral:

$$(4.2) \quad \frac{1}{(n+a)^s} := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt \quad (\min\{\Re(s), \Re(a)\} > 0; n \in \mathbb{N}_0)$$

in (3.2) and interchanging the order of summation and integration under the condition stated in Theorem 1, we get

$$\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left(\sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n} n!} \frac{(ze^{-t})^n}{n!} \right) dt.$$

Using the definition (2.4), we obtain the desired result. \square

The special cases of the integral representation (4.1) for certain incomplete families of generalized Hurwitz-Lerch Zeta functions in (3.6), (3.8) and (3.10) are asserted by Corollary 1 below. We state here the resulting integral representations without proof.

Corollary 1. *Each of the following integral representations for*

$$\varphi_{(\lambda;\nu)}^{(\rho,\kappa)}(z, s, a), \quad \varphi_{\lambda,\mu;\nu}(z, s, a) \text{ and } \varphi_{\lambda}^*(z, s, a)$$

defined by (3.6), (3.8) and (3.10) holds:

$$(4.3) \quad \varphi_{(\lambda;\nu)}^{(\rho,\kappa)}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2\overline{\psi}_1^* \left[\begin{array}{c} (\lambda, \rho, x), (1, 1); \\ (\nu, \kappa); \end{array} ze^{-t} \right] dt$$

$(\Re(x) \geq 0, \min\{\Re(s), \Re(a)\} > 0, \kappa > \rho > 0 \text{ when } z \in \mathbb{C};$
 $\kappa \geq \rho > 0 \text{ when } |z| < \rho^{-\rho} \kappa^\kappa),$

$$(4.4) \quad \varphi_{\lambda,\mu;\nu}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2\Gamma_1((\lambda, x), \mu; \nu; ze^{-t}) dt$$

$(\Re(x) \geq 0, \Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1)$

and

$$(4.5) \quad \varphi_{\lambda}^*(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_1\Gamma_0 [(\lambda, x); \dots; ze^{-t}] dt$$

$(\Re(x) \geq 0, \Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1).$

Theorem 2. *Each of the following integral representations for $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z, s, a)$ in (3.2) holds:*

$$(4.6) \quad \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu-1}}{(1+y)^\nu} \varphi_{\lambda;\nu-\mu}^{*(\rho,\kappa-\sigma)} \left(\frac{zy^\sigma}{(1+y)^\kappa}, s, a \right) dy$$

$(\Re(x) \geq 0, \Re(\nu) > \Re(\mu) > 0; \kappa \geq \sigma > 0; \sigma > 0)$

and

$$(4.7) \quad \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \times \int_0^\infty \int_0^\infty \frac{t^{s-1} e^{-at} y^{\mu-1}}{(1+y)^\nu} {}_1\overline{\psi}_1^* \left[\begin{array}{c} (\lambda, \rho, x); \\ (\nu - \mu, \kappa - \sigma); \end{array} \frac{zy^\sigma e^{-t}}{(1+y)^\kappa} \right] dt dy$$

$(\Re(x) \geq 0, \Re(\nu) > \Re(\mu) > 0; \kappa \geq \sigma > 0; \sigma > 0, \min\{\Re(s), \Re(a)\} > 0).$

Proof. Setting $\alpha = \mu + \sigma n$ and $\beta = \nu + \kappa n$ in the Eulerian Beta-function formula:

$$(4.8) \quad B(\alpha, \beta - \alpha) = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)} \int_0^\infty \frac{y^{\alpha-1}}{(1+y)^{\beta}} dy \quad (\Re(\beta) > \Re(\alpha) > 0),$$

we find that

$$(4.9) \quad \frac{(\mu)_{\sigma n}}{(\nu)_{\kappa n}} = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \frac{1}{(\nu-\mu)(\kappa-\sigma)n} \int_0^\infty \frac{y^{\mu+\sigma n-1}}{(1+y)^{\nu+\kappa n}} dy$$

$$(\Re(\nu) > \Re(\mu) > 0; \kappa \geqq \sigma n \in \mathbb{N}_0),$$

which by appealing to the definition (3.14) immediately yields the first assertion (4.6) of Theorem 2. Moreover, by (4.1) and (4.9), we also obtain

$$\begin{aligned} & \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) \\ &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \sum_{n=0}^\infty \frac{[\lambda;x]_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{(ze^{-t})^n}{n!} dt \\ &= \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{t^{s-1} e^{-at} y^{\mu-1}}{(1+y)^\nu} \sum_{n=0}^\infty \frac{[\lambda;x]_{\rho n}}{(\nu-\mu)(\kappa-\sigma)n n!} \left(\frac{zy^\sigma e^{-t}}{(1+y)^\nu} \right)^n dt dy, \end{aligned}$$

which in view of (2.4) leads us to the second assertion (4.7) of Theorem 2. \square

The case when $\kappa = \sigma$ in the integral representation (4.6) and (4.7), immediately reduce to following integral representations asserted by Corollary 2 below.

Corollary 2. *Each of the following integral representations holds:*

$$(4.10) \quad \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\sigma)}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu-1}}{(1+y)^\nu} \varphi_\lambda^{*(\rho)} \left(\frac{zy^\sigma}{(1+y)^\sigma}, s, a \right) dy$$

$$(\Re(x) \geqq 0, \Re(\nu) > \Re(\mu) > 0; \rho > 0)$$

and

$$(4.11) \quad \begin{aligned} & \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\sigma)}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \\ & \times \int_0^\infty \int_0^\infty \frac{t^{s-1} e^{-at} y^{\mu-1}}{(1+y)^\nu} {}_1\bar{\psi}_0^* \left[\begin{matrix} (\lambda, \rho, x); & zy^\sigma e^{-t} \\ -; & (1+y)^\sigma \end{matrix} \right] dt dy \\ & (\Re(x) \geqq 0, \Re(\nu) > \Re(\mu) > 0). \end{aligned}$$

Further, the special case of (4.10) and (4.11) when $\rho = 1 = \sigma$ reduces to the integral representations asserted by Corollary 3 below.

Corollary 3. *Each of the following integral representation for $\varphi_{\lambda,\mu;\nu}(z, s, a)$ in (3.8) holds:*

$$(4.12) \quad \varphi_{\lambda,\mu;\nu}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu-1}}{(1+y)^\nu} \varphi_\lambda^* \left(\frac{zy}{1+y}, s, a \right) dy$$

$$(\Re(x) \geqq 0, \Re(\nu) > \Re(\mu) > 0)$$

and

$$(4.13) \quad \begin{aligned} \varphi_{\lambda,\mu;\nu}(z, s, a) &= \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \\ &\times \int_0^\infty \int_0^\infty \frac{t^{s-1}e^{-at}y^{\mu-1}}{(1+y)^\nu} {}_1\Gamma_0\left[(\lambda, x); \frac{zye^{-t}}{1+y}\right] dt dy \\ &(\Re(x) \geq 0, \Re(\nu) > \Re(\mu) > 0, \min\{\Re(s), \Re(a)\} > 0). \end{aligned}$$

Theorem 3. *The following summation formula for $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ in (3.2) holds:*

$$(4.14) \quad \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s+k, a) t^n = \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a-t) \\ (x \geq 0, \lambda \in \mathbb{C} \text{ and } |t| < |a|; s \neq 1).$$

Proof. Using the definition (3.2) in right-hand side of the assertion (4.14), we have

$$(4.15) \quad \begin{aligned} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a-t) &= \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!(n+a-t)^s} \\ &= \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!(n+a)^s} \left(1 - \frac{t}{n+a}\right)^{-s} \\ &= \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!(n+a)^s} \left\{ \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \frac{t^k}{(n+a)^k} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda; x]_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!(n+a)^{s+k}} \right\} t^k. \end{aligned}$$

Using the series over n from (3.1), we obtain the formula (4.14) of Theorem 3. \square

5. Fractional derivative formulas

We begin by recalling the *Riemann-Liouville fractional derivative operator* \mathcal{D}_z^μ defined by (see, e.g. [13] and [17, p. 70 et seq.]):

$$(5.1) \quad \mathcal{D}_z^\mu \{f(z)\} := \begin{cases} \frac{1}{\Gamma(-\mu)} \int_0^z (z-t)^{-\mu-1} f(t) dt & (\Re(\mu) < 0) \\ \frac{d^m}{dz^m} \left\{ \mathcal{D}_z^{\mu-m} \{f(z)\} \right\} & (m-1 \leq \Re(\mu) < m \ (m \in \mathbb{N})). \end{cases}$$

We also recall the following known formula:

$$(5.2) \quad \mathcal{D}_z^\mu \{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu} \quad (\Re(\lambda) > -1).$$

Theorem 4. *The following fractional derivative formula for $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ in (3.2) holds:*

$$(5.3) \quad \mathcal{D}_z^{\mu-\tau} \left\{ z^{\mu-1} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z^\kappa, s, a) \right\} = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \varphi_{\lambda,\mu;\tau}^{(\rho,\sigma,\kappa)}(z^\kappa, s, a) \quad (\Re(\nu) > 0; \kappa > 0).$$

Proof. By virtue of the definition (3.2) of $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$ and the formula (5.2), the assertion (5.3) of Theorem 4 follows easily. \square

Further, upon setting $\rho = \sigma = \kappa = 1$ and $\mu = \sigma = 1$ in (5.3), we obtain the following fractional derivative formulas for incomplete families of Hurwitz-Lerch Zeta functions in (3.6) and (3.8), given in Corollary 4.

Corollary 4. *Each of the following fractional derivative formula for*

$$\varphi_{\lambda,\mu;\nu}(z, s, a) \text{ and } \varphi_{\lambda;\nu}^{(\rho,\kappa)}(z, s, a)$$

in (3.6) and (3.8) holds:

$$(5.4) \quad \mathcal{D}_z^{\mu-\tau} \left\{ z^{\mu-1} \varphi_{\lambda,\mu;\nu}(z, s, a) \right\} = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \varphi_{\lambda,\mu;\tau}(z, s, a) \quad (\Re(\nu) > 0)$$

and

$$(5.5) \quad \mathcal{D}_z^{\mu-\tau} \left\{ z^{\mu-1} \varphi_{\lambda;\nu}^{(\rho,\kappa)}(z^\kappa, s, a) \right\} = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \varphi_{\lambda;\tau}^{(\rho,\kappa)}(z^\kappa, s, a) \quad (\Re(\nu) > 0; \kappa > 0).$$

Theorem 5. *The following fractional derivative formula for $\varphi_{\lambda,\mu;\nu}(z, s, a)$ holds:*

$$(5.6) \quad \varphi_{\lambda,\mu;\nu}(z, s, a) = \frac{\Gamma(\nu)}{\Gamma(\mu)} z^{1-\nu} \mathcal{D}_z^{\mu-\nu} \left\{ z^{\mu-1} \varphi_{\lambda}^*(z, s, a) \right\}.$$

Proof. In view of the definition (3.10) and formula (5.2), we obtain the desired fractional derivative formula (5.6) for the function $\varphi_{\lambda,\mu;\nu}(z, s, a)$ in (3.8). \square

6. Application to probability distributions

In this concluding section, we consider a general probability distribution involving the incomplete extended Hurwitz-Lerch Zeta function (3.2) defined as follow:

Definition 1. A random variable ξ is said to be the incomplete extended Hurwitz distributed if its probability density function is given by

$$(6.1) \quad f_\xi(a) =: \begin{cases} \frac{s \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s+1, a)}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)}, & a \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where it is *tacitly* assumed that the arguments z , s and the parameters λ , μ and ν are fixed and suitably constrained so that the probability density function $f_\xi(a)$ remains nonnegative.

Theorem 6. *Let ξ be a continuous random variable with its probability density function defined by (6.1). Then, the moment generating function $M(t)$ of the random variable ξ is given by*

$$(6.2) \quad M(t) = E_s[e^{\xi t}] = \sum_{n=0}^{\infty} E_s[\xi^n] \frac{t^n}{n!},$$

where the moments $E_s[\xi^n]$ of order n are given by

$$(6.3) \quad E_s[\xi^n] = \sum_{k=0}^n \frac{n!}{(n-k)!} \frac{\Gamma(s-k)}{\Gamma(s)} \frac{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s-k, 1)}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)}.$$

Proof. The assertion in (6.2) can be derived easily by using the series expansion of $e^{\xi t}$. To establish (6.3), we observe that

$$(6.4) \quad \frac{d}{da} \left\{ \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) \right\} = -s \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s+1, a),$$

which follows readily from (3.2), and thus from the definition of $E_s[\xi^n]$, we have

$$\begin{aligned} E_s[\xi^n] &= \int_1^\infty a^n f_\xi(a) da \\ &= \frac{s}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \int_1^\infty a^n \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s+1, a) da \\ &= -\frac{1}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \int_1^\infty a^n \frac{d}{da} \left\{ \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) \right\} da \\ &= \left[-\frac{a^n \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \right]_{a=1}^\infty + \frac{n}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \int_1^\infty a^{n-1} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) da \\ &= 1 - \lim_{a \rightarrow \infty} \left\{ \frac{a^n \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \right\} + \frac{n}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \int_1^\infty a^{n-1} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) da \\ (6.5) \quad &= 1 + \frac{n}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \int_1^\infty a^{n-1} \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) da, \end{aligned}$$

where, in addition to the derivative property (6.4), we have used the following limit formula:

$$\begin{aligned} (6.6) \quad &\lim_{a \rightarrow \infty} \left\{ a^n \varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) \right\} \\ &= \lim_{a \rightarrow \infty} \left\{ \frac{a^n}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2\overline{\psi}_1^* \left[\begin{matrix} (\lambda, \rho, x), (\mu, \sigma); & ze^{-t} \\ (\nu, \kappa); & \end{matrix} \right] dt \right\} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \lim_{a \rightarrow \infty} \left\{ a^n e^{-at} \right\} {}_2\overline{\psi}_1^* \left[\begin{matrix} (\lambda, \rho, x), (\mu, \sigma); & ze^{-t} \\ (\nu, \kappa); & \end{matrix} \right] dt = 0. \end{aligned}$$

Consequently, we have the following reduction formula for $E_s[\xi^n]$:

$$(6.7) \quad E_s[\xi^n] = 1 + \frac{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s-1, 1)}{\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, 1)} \frac{n}{s-1} E_{s-1}[\xi^{n-1}],$$

and by iterating the recurrence (6.5), we arrive at the desired result (6.3). \square

7. Multi-parametric extension of $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$

In this section, we further extend incomplete family of extended Hurwitz-Zeta function (3.2) by introducing p numerator and q denominator parameters in the definition (4.12):

$$(7.1) \quad \phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda_1; x)_{n\rho_1} (\lambda_2)_{n\rho_2} \cdots (\lambda_p)_{n\rho_p}}{(\mu_1)_{n\sigma_1} \cdots (\mu_q)_{n\sigma_q}} \frac{z^n}{n!(n+a)^s}$$

and

$$(7.2) \quad \varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a) := \sum_{n=0}^{\infty} \frac{[\lambda_1; x]_{n\rho_1} (\lambda_2)_{n\rho_2} \cdots (\lambda_p)_{n\rho_p}}{(\mu_1)_{n\sigma_1} \cdots (\mu_q)_{n\sigma_q}} \frac{z^n}{n!(n+a)^s}$$

$(x \geq 0; \lambda_j \in \mathbb{C} (j = 1, \dots, p) \quad a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, q);$
 $\rho_j, \sigma_k \in \mathbb{R}^+ (j = 1, \dots, p; \quad k = 1, \dots, q); \quad \Delta > -1 \text{ when } s, z \in \mathbb{C};$
 $\Delta = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \Delta^*;$
 $\Delta = -1 \text{ and } \Re(\Xi) > \frac{1}{2} \text{ when } |z| = \Delta^*,$

where

$$\Delta^* := \left(\prod_{j=0}^p \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=0}^q \sigma_j^{\sigma_j} \right), \quad \Delta = \sum_{j=0}^q \sigma_j - \sum_{j=0}^p \rho_j$$

and

$$\Xi = s + \sum_{j=0}^q \mu_j - \sum_{j=0}^p \lambda_j + \frac{p-q}{2}.$$

The results involving the incomplete extended Hurwitz-Lerch Zeta function

$$\varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a)$$

can be presented by applying the definition (7.2) in the same manner as for the corresponding results involving the incomplete extended Hurwitz-Zeta function $\varphi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$.

Theorem 7. *The following integral representation for $\varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a)$ in (7.2) holds:*

$$(7.3) \quad \begin{aligned} & \varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a) \\ &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_p\psi_q^* \left[\begin{matrix} (\lambda_1, \rho_1, x), (\lambda_2, \rho_2), \dots, (\lambda_p, \rho_p); & ze^{-t} \\ (\mu_1, \sigma_1), \dots, (\mu_q, \sigma_q); & \end{matrix} \right] dt \end{aligned}$$

$$(\Re(x) \geqq 0, \min\{\Re(s), \Re(a)\}).$$

Theorem 8. *The following fractional derivative formula for*

$$\varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a)$$

in (7.2) holds:

$$(7.4) \quad \begin{aligned} & \mathcal{D}_z^{\mu-\tau} \left\{ z^{\mu-1} \varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z^\kappa, s, a) \right\} \\ &= \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \varphi_{\lambda_1, \dots, \lambda_p, \nu; \mu_1, \dots, \mu_q, \tau}^{\rho_1, \dots, \rho_p, \kappa; \sigma_1, \dots, \sigma_q, \kappa}(z^\kappa, s, a) (z^\kappa, s, a) \quad (\Re(\nu) > 0; \kappa > 0). \end{aligned}$$

Theorem 9. *The following sum-integral representation for*

$$\varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a)$$

in (7.2) holds:

$$(7.5) \quad \begin{aligned} & \varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a) (z, s, a) \\ &:= \frac{1}{\Gamma(s)} \sum_{j=0}^{m-1} \frac{\prod_{l=1}^p (\lambda_l)_{j\rho_l}}{\prod_{l=1}^q (\mu_l)_{j\sigma_l}} \frac{z^j}{j!} \int_0^\infty t^{s-1} e^{-(a+j)t} \\ & \quad {}_{p+1}\overline{\psi}_{q+1}^* \left[\begin{array}{l} (\lambda_1 + j\rho_1, m\rho_1, x), (\lambda_2 + j\rho_2, m\rho_2), \dots, (\lambda_p + j\rho_p, m\rho_p), (1, 1); \\ (\mu_1 + j\sigma_1, m\sigma_1), \dots, (\mu_q + j\sigma_q, m\sigma_q), (j+1, m); \end{array}; z^m e^{-mt} \right] dt \\ & \quad (\Re(x) \geqq 0, \min\{\Re(s), \Re(a)\} > 0, m \in \mathbb{N}). \end{aligned}$$

Proof. Using the following elementary series identity:

$$\sum_{n=0}^{\infty} f(n) = \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} f(mn+j) \quad (m \in \mathbb{N})$$

in (7.2), we obtain following series formula:

$$\begin{aligned} & \varphi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q}(z, s, a) (z, s, a) \\ &= m^{-s} \sum_{j=0}^{m-1} \frac{\prod_{l=1}^p (\lambda_l)_{j\rho_l}}{\prod_{l=1}^q (\mu_l)_{j\sigma_l}} \frac{z^j}{j!} \varphi_{\lambda_1 + j\rho_1, \dots, \lambda_p + j\rho_p, 1; \mu_1 + j\sigma_1, \dots, \mu_q + j\sigma_q, j+1}^{m\rho_1, \dots, m\rho_p, 1; m\sigma_1, \dots, m\sigma_q, m} \left(z^m, s, \frac{a+j}{m} \right) \end{aligned}$$

which, in conjunction with (7.3), yields desired sum-integral representation. \square

Corollary 5. *The following sum-integral representation for*

$$\varphi_{\lambda, \nu}^{(\rho, \kappa)}(z, s, a) \text{ and } \varphi_{\lambda}^{(\rho)}(z, s, a)$$

in (3.6) and (3.10) holds:

$$\varphi_{\lambda; \nu}^{(\rho, \kappa)}(z, s, a) := \frac{1}{\Gamma(s)} \sum_{j=0}^{m-1} \frac{(\lambda)_{\rho j}}{(\nu)_{\kappa j}} z^j \int_0^\infty t^{s-1} e^{-(a+j)t}$$

$$(7.6) \quad {}_2\overline{\psi}_1 \left[\begin{matrix} (\lambda + \rho j, \rho m, x), (1, 1); \\ (\nu + \kappa j, \kappa m); \end{matrix} z^m e^{-mt} \right] dt$$

$(\Re(x) \geq 0, \min\{\Re(s), \Re(a)\} > 0, m \in \mathbb{N}),$

$$(7.7) \quad \varphi_{\lambda}^*(z, s, a) := \frac{1}{\Gamma(s)} \sum_{j=0}^{m-1} \frac{(\lambda)_j}{(1)_j} z^j \int_0^\infty t^{s-1} e^{-(a+j)t} {}_2\overline{\psi}_1 \left[\begin{matrix} (\lambda + j, m, x), (1, 1); \\ (1 + j, m); \end{matrix} z^m e^{-mt} \right] dt$$

$(\Re(x) \geq 0, \min\{\Re(s), \Re(a)\} > 0, m \in \mathbb{N}).$

References

- [1] E. W. Barnes, *The asymptotic expansion of integral functions defined by Taylor series*, Philos. Trans. Roy. Soc. London Ser. A **206** (1906), 249–297.
- [2] A. Çetinkaya, *The incomplete second Appell hypergeometric functions*, Appl. Math. Comput. **219** (2013), no. 15, 8332–8337.
- [3] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math. **55** (1994), no. 1, 99–124.
- [4] ———, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall, (CRC Press Company), Boca Raton, London, New York and Washington, D. C., 2001.
- [5] J. Choi, D. S. Jang, and H. M. Srivastava, *A generalization of the Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. **19** (2008), no. 1-2, 65–79.
- [6] J. Choi and R. K. Parmar, *The incomplete Lauricella and fourth Appell functions*, Far East J. Math. Sci. **96** (2015), 315–328.
- [7] J. Choi, R. K. Parmar, and P. Chopra, *The incomplete Lauricella and first Appell functions and associated properties*, Honam Math. J. **36** (2014), no. 3, 531–542.
- [8] ———, *The incomplete Srivastava’s triple hypergeometric functions γ_B^H and Γ_B^H* , Filomat, In Press 2015.
- [9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.
- [10] M. Garg, K. Jain, and S. L. Kalla, *A further study of general Hurwitz-Lerch zeta function*, Algebras Groups Geom. **25** (2008), 311–319.
- [11] S. P. Goyal and R. K. Laddha, *On the generalized zeta function and the generalized Lambert function*, Ganita Sandesh **11** (1997), 99–108.
- [12] D. Jankov, T. K. Pogány, and R. K. Saxena, *An extended general Hurwitz-Lerch zeta function as a Mathieu (a, λ) -series*, Appl. Math. Lett. **24** (2011), 1473–1476.
- [13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
- [14] S. D. Lin and H. M. Srivastava, *Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput. **154** (2004), no. 3, 725–733.
- [15] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Functions: Theory and Applications*, Springer, New York, 2010.
- [16] R. K. Parmar and R. K. Saxena, *The incomplete generalized τ -hypergeometric and second τ -Appell functions*, J. Korean Math. Soc. **53** (2016), no. 2, 363–379.
- [17] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Translated from the Russian: Integrals and Derivatives of

- Fractional Order and Some of Their Applications (“Nauka i Tekhnika”, Minsk, 1987); Gordon and Breach Science Publishers: Reading, UK, 1993.
- [18] R. K. Saxena, *A remark on a paper on M-series*, Fract. Calc. Appl. Anal. **12** (2009), no. 1, 109–110.
 - [19] M. Sharma, *Fractional integration and fractional differentiation of the M-series*, Fract. Calc. Appl. Anal. **11** (2008), no. 2, 187–191.
 - [20] M. Sharma and R. Jain, *A note on a generalized M-series as a special function of fractional calculus*, Fract. Calc. Appl. Anal. **12** (2009), no. 4, 449–452.
 - [21] H. M. Srivastava, *A new family of the λ -generalized Hurwitz-Lerch zeta functions with applications*, Appl. Math. Inf. Sci. **8** (2014), no. 4, 1485–1500.
 - [22] H. M. Srivastava, M. A. Chaudhry, and R. P. Agarwal, *The incomplete Pochhammer symbols and their applications to hypergeometric and related functions*, Integral Transforms Spec. Funct. **23** (2012), no. 9, 659–683.
 - [23] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer, Academic Publishers, Dordrecht, Boston and London, 2001.
 - [24] ———, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science, Publishers, Amsterdam, London and New York, 2012.
 - [25] H. M. Srivastava, D. Jankov, T. K. Pogány, and R. K. Saxena, *Two-sided inequalities for the extended Hurwitz-Lerch zeta function*, Comput. Math. Appl. **62** (2011), no. 1, 516–522.
 - [26] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
 - [27] H. M. Srivastava, M.-J. Luo, and R. K. Raina, *New results involving a class of generalized Hurwitz-Lerch zeta functions and their applications*, Turkish J. Anal. Number Theory **1** (2013), no. 1, 26–35.
 - [28] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
 - [29] H. M. Srivastava, R. K. Saxena, T. K. Pogány, and R. Saxena, *Integral and computational representations of the extended Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. **22** (2011), no. 7, 487–506.
 - [30] R. Srivastava, *Some properties of a family of incomplete hypergeometric functions*, Russian J. Math. Phys. **20** (2013), no. 1, 121–128.
 - [31] ———, *Some generalizations of Pochhammer’s symbol and their associated families of hypergeometric functions and hypergeometric polynomials*, Appl. Math. Inform. Sci. **7** (2013), no. 6, 2195–2206.
 - [32] ———, *Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions*, Appl. Math. Comput. **243** (2014), 132–137.
 - [33] R. Srivastava and N. E. Cho, *Generating functions for a certain class of incomplete hypergeometric polynomials*, Appl. Math. Comput. **219** (2012), no. 6, 3219–3225.
 - [34] ———, *Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials*, Appl. Math. Comput. **234** (2014), 277–285.

RAKESH K. PARMAR
DEPARTMENT OF MATHEMATICS
GOVERNMENT COLLEGE OF ENGINEERING AND TECHNOLOGY
BIKANER 334004, RAJASTHAN STATE, INDIA
E-mail address: rakeshparmar27@gmail.com

RAM K. SAXENA
DEPARTMENT OF MATHEMATICS AND STATISTICS
JAI NARAIN VYAS UNIVERSITY
JODHPUR-342 004, RAJASTHAN STATE, INDIA
E-mail address: ram.saxena@yahoo.com