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RINGS WITH MANY REGULAR ELEMENTS

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Abstract. In this paper we introduce rings that satisfy regular 1-stable range. These rings are left-right symmetric and are generalizations of unit 1-stable range. We investigate characterizations of these kind of rings and show that these rings are closed under matrix rings and Morita Context rings.

1. Introduction

Let R be an associative ring with an identity. We say that R has stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists some $y \in R$ such that $a + by \in U(R)$, where $U(R)$ denotes the set of all units in R. One of the most important features of stable range one is the cancellation of related modules from direct sums. Evans [15, Theorem 2] proved that if A, B, C are R-modules such that $A \oplus B \cong A \oplus C$, and $\mathrm{End}_R(A)$ has stable range one, then $B \cong C$. Stable range conditions have been studied in [1], [8], [9], [11], [14], [19] and [21]. Goodearl and Mental [16] defined the concept of unit 1-stable range: we say that R satisfies unit 1-stable range provided that for any $a, b \in R$, $aR + bR = R$ implies there exists a $y \in U(R)$ such that $a + by \in U(R)$. Many authors have studied this class of rings such as [7], [12], [13] and [16]. Here we generalize this concept as bellow.

Definition 1.1. A ring R is said to satisfy regular 1-stable range provided that for any $a, b \in R$, $aR + bR = R$ implies there exists a regular (von Neumann) element $r \in R$ such that $a + br \in U(R)$.

Obviously, if R satisfies unit 1-stable range, then it satisfies regular 1-stable range. But the converse is not true in generally. For example, $\mathbb{Z}/2\mathbb{Z}$ (the ring of integers modulo 2) satisfies regular 1-stable range, while it does not satisfies unit 1-stable range.

In this paper, we will prove that a ring satisfies regular 1-stable range is left-right symmetric. In other words, a ring R satisfies regular 1-stable range

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if and only if whenever $Ra + Rb = R$, there exists a regular $r \in R$ such that $a + rb \in U(R)$.

A Morita context (A, B, M, N, ψ, ϕ) consists of two rings A and B, two bimodules ${}_AN_B$ and ${}_BM_A$, and a pair of bimodule homomorphisms $\psi : N \otimes_B M \to A$ and $\phi : M \otimes_A N \to B$ which satisfy the following associativity:

$\psi(n \otimes m)n' = n\phi(m \otimes n')$ and $\phi(m \otimes n)m' = m\psi(n \otimes m')$ for any $m, m' \in M$, $n, n' \in N$. We can form $C = \{\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right) \mid a \in A, b \in B, n \in N, m \in M\}$, and define a multiplication on C as follows:

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes m') & an' + nb' \\ ma' + bm' & \phi(m \otimes n') + bb' \end{pmatrix}.$$

With this multiplication and entry-wise addition, C becomes an associative ring. We call C a Morita Context ring. Obviously, the class of the rings of Morita Contexts includes all 2×2 matrix rings and all formal triangular matrix rings. Many authors studied Morita Contexts such as [5], [10] and [17].

We characterize rings that satisfies regular 1-stable range and show that these kind of rings are closed under matrix rings and Morita Context rings. Finally, we prove that a ring R satisfies regular 1-stable range if and only if so does the ring of all $n \times n$ lower (resp., upper) triangular matrices over R.

Throughout this paper, R denotes an associative ring with unity, $U(R)$ the group of units, $Id(R)$ the set of idempotents, $J(R)$ the Jacobson radical of R and $M_n(R)$ the ring of all $n \times n$ matrices over R. Further $Reg(R) = \{a \in R \mid$ a is regular (von Neumann)$\}$.

2. Main results

In this section we first give some properties of rings satisfy regular 1-stable range.

Proposition 2.1. *The following statements are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *Whenever $a, b \in R$ satisfy $aR + bR = R$, there exists $r \in Reg(R)$ such that $a + br$ is left invertible.*
(3) *Whenever $a, b \in R$ satisfy $aR + bR = R$, there exists $r \in Reg(R)$ such that $a + br$ is right invertible.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1). Given $aR + bR = R$, then there exists $r \in Reg(R)$ such that $a + br = u$ is left invertible. Say $vu = 1$ for some $v \in R$. Since $vR + 0R = R$, so there exists $r_1 \in Reg(R)$ such that $v + 0.r_1 = v$ is left invertible. Therefore $a + br \in U(R)$.

(3) $\Rightarrow$ (1). Given $aR + bR = R$, then there exists $r \in Reg(R)$ such that $a + br = u$ is right invertible. Say $uv = 1$ for some $v \in R$. Since $vR + (1 - vu)R = R$, so there exists $r_1 \in Reg(R)$ such that $v + (1 - vu)r_1 = r_2$ is right invertible. Hence $ur_2 = u(v + (1 - vu)r_1) = 1$. Thus $r_2 \in U(R)$. Therefore $a + br \in U(R)$. $\square$

Lemma 2.1. *If $r \in Reg(R)$, then $ru \in Reg(R)$ and $ur \in Reg(R)$ for any $u \in U(R)$.*

Proof. Since $r \in Reg(R)$, so there exists $y \in R$ such that $ryr = r$. Hence $ruu^{-1}yr = r$ for any $u \in U(R)$. Thus $ruu^{-1}yru = ru$. Therefore $ru \in Reg(R)$. Also $uryu^{-1}ur = ur$. So $ur \in Reg(R)$. □

The proofs of the following two lemmas are analogous to [22, Lemma 4.4 and Theorem 4.5].

Lemma 2.2. *The following statements are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *Whenever $a, b \in R$ satisfy $ax + b = 1$, there exists $r \in Reg(R)$ such that $a + br \in U(R)$.*
(3) *Whenever $a, b \in R$ satisfy $ax + b = 1$, there exists $y \in R$ such that $a + by \in U(R)$ and $1 - xy \in Reg(R)$.*

Lemma 2.3. *The following statements are equivalent for any ring R:*

(1) *Whenever $a, b \in R$ satisfy $ax + b = 1$, there exists $r \in Reg(R)$ such that $a + br \in U(R)$.*
(2) *Whenever $a, b \in R$ satisfy $ax + b = 1$, there exists $r \in Reg(R)$ such that $x + rb \in U(R)$.*

The opposite ring R^{op} consists of formal elements $\{a^{op} : a \in R\}$ with addition and multiplication given by

$$a^{op} + b^{op} = (a + b)^{op}, a^{op}.b^{op} = (ba)^{op}.$$

From Lemma 2.3, we see that R satisfies regular 1-stable range if and only if so does R^{op}. Hence a ring satisfies regular 1-stable range is left-right symmetric. Vaserstein [20] showed that a ring R has stable range one if and only if so does $R/J(R)$. Now, we consider the similar case for rings satisfying regular 1-stable range.

Lemma 2.4. *Let I be an ideal of R with $I \subseteq J(R)$. If R satisfies regular 1-stable range, then so does R/I.*

Proof. Assume that $\overline{ax} + \overline{b} = \overline{1}$ in $\overline{R} = R/I$. Then $ax + b = 1 + k \in U(R)$ for some $k \in J(R)$. Hence $ax(1+k)^{-1} + b(1+k)^{-1} = 1$. So there exists $r \in Reg(R)$ such that $a + b(1 + k)^{-1}r \in U(R)$. But $(1 + k)^{-1}r \in Reg(R)$ by Lemma 2.1. Hence $\overline{(1 + k)^{-1}r} \in Reg(\overline{R})$ and since $\overline{a} + \overline{b(1 + k)^{-1}r} \in U(\overline{R})$, so $\overline{R}$ satisfies regular 1-stable range. □

Corollary 2.1. *Let R be an abelian ring (all its idempotents are central) and idempotents can be lifted modulo $J(R)$. If I is any ideal of R with $I \subseteq J(R)$, then R satisfies regular 1-stable range if and only if so does R/I.*

Proof. One direction is trivial by Lemma 2.4. Conversely, suppose that R/I satisfies regular 1-stable range. Let $ax+b = 1$ in R. Then $\overline{ax}+\overline{b} = \overline{1} \in \overline{R} = R/I$.

Thus there exists $\overline{r} \in Reg(\overline{R})$ such that $\overline{a+br} = \overline{v} \in U(\overline{R})$. But since $\overline{R}$ is abelian, so $\overline{r} = \overline{eu}$ for some $\overline{e} \in Id(\overline{R})$ and $\overline{u} \in U(\overline{R})$. Now, as units and idempotents can be lifted modulo $J(R)$, so we assume that $e \in Id(R)$ and $u \in U(R)$. Hence $a + br = v + k \in U(R)$ with $v \in U(R)$, $k \in J(R)$ and $r = eu \in Reg(R)$, as required. □

Theorem 2.1. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *Whenever $aR + bR = dR$ with $a, b, d \in R$, there exist $u \in U(R)$ and $r \in Reg(R)$ such that $au + br = d$.*
(3) *Whenever $a_1R + \cdots + a_nR = dR$ with $n \geqslant 2$, $a_1, \ldots, a_n, d \in R$, there exist $u_1 \in U(R)$ and $r_2, \ldots, r_n \in Reg(R)$ such that $a_1u_1 + a_2r_2 + \cdots + a_nr_n = d$.*

Proof. Both (2) ⇒ (1) and (3) ⇒ (2) are obvious.

(1) ⇒ (2). Since R satisfies regular 1-stable range, then R has stable range one. Given $aR + bR = dR$ with $a, b, d \in R$, the sets $\{a, b\}$ and $\{d, 0\}$ generate the same R-submodule of R^2. Therefore there exists $U = (u_{ij}) \in GL_2(R)$ such that $(a, b) = (d, 0)U$ by [8, Lemma 2.1]. Obviously, $u_{11}R + u_{12}R = R$. Thus there exists $r \in Reg(R)$ such that $u_{11} + u_{12}r = v \in U(R)$. Hence $a + br = dv$. Therefore $av^{-1} + brv^{-1} = d$, where $v^{-1} \in U(R)$ and $rv^{-1} \in Reg(R)$ by Lemma 2.1.

(2) ⇒ (3). Given $a_1R + \cdots + a_nR = dR$ with $n \geqslant 2$, $a_1, \ldots, a_n, d \in R$. If $n = 2$, then the result follows from (2). Assume that the result holds for $n \leqslant k$ ($k \geqslant 2$). Let $n = k + 1$. Then there exist $x_1, \ldots, x_{k+1} \in R$ such that $a_1x_1 + \cdots + a_{k+1}x_{k+1} = d$. Thus $a_1R + \cdots + a_{k-1}R + (a_kx_k + a_{k+1}x_{k+1})R = dR$. Hence $a_1u_1 + a_2r_2 + \cdots + (a_kx_k + a_{k+1}x_{k+1})r_k = d$ for some $u_1 \in U(R), r_2, \ldots, r_k \in Reg(R)$. Therefore $(a_1u_1 + a_2r_2)R + \cdots + a_kR + a_{k+1}R = dR$. Hence $(a_1u_1 + a_2r_2)v_1 + \cdots + a_kv_{k-1} + a_{k+1}v_k = a_1u_1v_1 + a_2r_2v_1 + \cdots + a_kv_{k-1} + a_{k+1}v_k = d$ for some $v_1 \in U(R), v_2, \ldots, v_k \in Reg(R)$. Note that $u_1v_1 \in U(R)$ and $r_2v_1, v_2, \ldots, v_k \in Reg(R)$, thus we complete the proof. □

Corollary 2.2. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *Whenever $Ra + Rb = Rd$ with $a, b, d \in R$, there exist $u \in U(R)$ and $r \in Reg(R)$ such that $ua + rb = d$.*
(3) *Whenever $Ra_1 + \cdots + Ra_n = dR$ with $n \geqslant 2$, $a_1, \ldots, a_n, d \in R$, there exist $u_1 \in U(R)$ and $r_2, \ldots, r_n \in Reg(R)$ such that $u_1a_1 + r_2a_2 + \cdots + r_na_n = d$.*

Let

$$B_{12}(*) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ and } B_{21}(*) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

We use $[u, v]$ to denote the diagonal matrix $diag(u, v)$ with $u, v \in U(R)$.

Theorem 2.2. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *For any $A \in GL_2(R)$, there exists $r \in Reg(R)$ such that $A = [*,*]B_{21}(*)B_{12}(*)B_{21}(r)$.*
(3) *For any $A \in GL_2(R)$, there exists $r \in Reg(R)$ such that $A = [*,*]B_{21}(r)B_{12}(*)B_{21}(*)$.*

Proof. (1) $\Rightarrow$ (2). Given $A = (a_{ij}) \in GL_2(R)$, then $a_{11}R + a_{12}R = R$ we have $u_1 \in U(R)$ and $v_1 \in Reg(R)$ such that $a_{11}u_1 + a_{12}v_1 = 1$ by Theorem 2.1. Hence $a_{11} + a_{12}v_1u_1^{-1} = u_1^{-1}$. Thus we have

$$AB_{21}(v_1u_1^{-1}) = \begin{pmatrix} u_1^{-1} & a_{12} \\ a_{21} + a_{22}v_1u_1^{-1} & a_{22} \end{pmatrix}.$$

Set $u = u_1^{-1}$, $v = a_{22} - (a_{21} + a_{22}v_1u_1^{-1})u_1^{-1}a_{12}$ and $r = -v_1u_1^{-1}$. Then $A = [u,v]B_{21}(*)B_{12}(*)B_{21}(r)$, where $r \in Reg(R)$ by Lemma 2.1.

(2) $\Rightarrow$ (3). Let $A = (a_{ij}) \in GL_2(R)$. Then $A^{-1} \in GL_2(R)$. Hence there are $u, v \in U(R)$ and $r \in Reg(R)$ such that $A^{-1} = [u,v]B_{21}(*)B_{12}(*)B_{21}(r)$. Thus we have

$$A = B_{21}(-r)B_{12}(*)B_{21}(*)[u^{-1}, v^{-1}] = [u^{-1}, v^{-1}]B_{21}(-vru^{-1})B_{12}(*)B_{21}(*),$$

where $-vru^{-1} \in Reg(R)$ by Lemma 2.1.

(3) $\Rightarrow$ (1). Let $ax + b = 1$ in R. Then $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in GL_2(R)$. So there exists $u, v \in U(R)$ and $r \in Reg(R)$ such that

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = [u,v]B_{21}(r)B_{12}(*)B_{21}(*).$$

Hence $B_{21}(-r)[u^{-1}, v^{-1}]\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = B_{12}(*)B_{21}(*)$. Thus $x - vru^{-1}b = v \in U(R)$, as required. □

Corollary 2.3. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *For any $A \in GL_2(R)$, there exists $r \in Reg(R)$ such that $A = [*,*]B_{12}(*)B_{21}(*)B_{12}(r)$.*
(3) *For any $A \in GL_2(R)$, there exists $r \in Reg(R)$ such that $A = [*,*]B_{12}(r)B_{21}(*)B_{12}(*)$.*

Proof. (1) $\Rightarrow$ (2). Given any $A \in GL_2(R)$, then $(A^T)^o \in GL_2(R^{op})$. But by Theorem 2.2, there are $u, v \in U(R)$ and $r \in Reg(R)$ such that $(A^T)^o = [u^o, v^o]B_{21}(r^o)B_{12}(*^o)B_{21}(*^o)$. Therefore $A = [*,*]B_{12}(*)B_{21}(*)B_{12}(r)$ as required.

(2) $\Rightarrow$ (3). Given any $A = (a_{ij}) \in GL_2(R)$. Then $A^{-1} \in GL_2(R)$. Hence, there are $u, v \in U(R)$ and $r \in Reg(R)$ such that $A^{-1} = [u,v]B_{12}(*)B_{21}(*)$ $B_{12}(r)$. Therefore

$$A = B_{12}(-r)B_{21}(*)B_{12}(*)[u^{-1}, v^{-1}] = [u^{-1}, v^{-1}]B_{12}(-urv^{-1})B_{21}(*)B_{12}(*).$$

Thus we complete proof by Lemma 2.1.

(3) $\Rightarrow$ (1). Given $ax + b = 1$ in R, then $\left(\begin{smallmatrix} x & -1 \\ b & a \end{smallmatrix}\right) \in GL_2(R)$. Thus there are $u, v \in U(R)$ and $r \in Reg(R)$ such that

$$\begin{pmatrix} x & -1 \\ b & a \end{pmatrix} = [u, v]B_{12}(*)B_{21}(*)B_{12}(r).$$

Therefore $a - br = v \in U(R)$, as required. □

Theorem 2.3. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *Whenever $a, b \in R$ generate a principal right ideal of R, there exists some $r \in Reg(R)$ such that $aR + bR = (a + br)R$.*
(3) *Whenever $a, b \in R$ generate a principal left ideal of R, there exists some $r \in Reg(R)$ such that $Ra + Rb = R(a + rb)$.*

Proof. (1) $\Rightarrow$ (2) is clear by Theorem 2.1.

(2) $\Rightarrow$ (1). Let $aR + bR = R$ with $a, b \in R$. Then there exists $r \in Reg(R)$ such that $R = aR + bR = (a + br)R$. Let $a + br = w$. Then there exists $v \in R$ such that $wv = 1$. Now, since $vR + (1 - vw)R = R$, so $R = vR + (1 - vw)R = (v + (1 - vw)t)R$ for some $t \in Reg(R)$. Thus $(v + (1 - vw)t)s = 1$ for some $s \in R$. Therefore $w = w.1 = w(v + (1 - vw)t)s = s$. Thus $a + br \in U(R)$.

(1) $\Leftrightarrow$ (3). Sufficient that applying (1) $\Leftrightarrow$ (2) to R^{op}. □

A ring R is called clean if every element of R can be written as the sum of a unit and an idempotent in R. For a positive integer n, a ring R is called n-clean if every element of R can be written as the sum of n units and an idempotent in R. By [23, Theorem 6], If R is an abelian clean ring, then R has stable range one.

Theorem 2.4. *Let R be an abelian ring and R satisfy regular 1-stable range. Then R is 2-clean.*

Proof. Let $a \in R$. Then $aR + (-1)R = R$. Thus there exists $r \in Reg(R)$ such that $a + (-1)r = u \in U(R)$. So $a = r + u$. Now, since R is abelian, so $r = ev$ for some $e \in Id(R)$ and $v \in U(R)$. Set $f = 1 - e$. Then $r = f + (ev - f)$, where $w := ev - f \in U(R)$ and $f \in Id(R)$. Hence $a = f + w + u$ is 2-clean. □

We say that R satisfies unit regular 1-stable range one provided that for any $a, b \in R$, $aR + bR = R$ implies there exists a unit regular element $y \in R$ such that $a + by \in U(R)$. Obviously, if R satisfies unit regular 1-stable range, then satisfies regular 1-stable range. Camillo and Yu [4, Theorem 3], proved that an exchange rings R has stable range one if and only if every regular element of R is unit-regular in R. Hence if R is an exchange ring that satisfies regular 1-stable range, then satisfies unit regular 1-stable range. Therefore we have following result:

Proposition 2.2. *Let R be an exchange ring. Then R satisfies regular 1-stable range if and only if R satisfies unit regular 1-stable range.*

Lemma 2.5. *Let $e \in Id(R)$. If $w_1 \in Reg(eRe)$ and $w_2 \in Reg((1-e)R(1-e))$, then $diag(w_1, w_2) \in Reg(R)$.*

Proof. Set $T = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. Clearly, we have a ring isomorphism $\varphi : R \cong T$ given by $\varphi(r) = \begin{pmatrix} ere & er(1-e) \\ (1-e)re & (1-e)r(1-e) \end{pmatrix}$ for every $r \in R$. But there exists $y_1 \in eRe$ and $y_2 \in (1-e)R(1-e)$ such that $w_1y_1w_1 = w_1$ and $w_2y_2w_2 = w_2$. Hence

$$diag(w_1, w_2)diag(y_1, y_2)diag(w_1, w_2) = diag(w_1, w_2). \qquad \square$$

Theorem 2.5. *Let $e \in Id(R)$. If eRe and $(1-e)R(1-e)$ are satisfies regular 1-stable range, then so is R.*

Proof. By using a technique similar to the proof of [22, Theorem 5.8] and Lemma 2.5 the result follows. $\square$

Corollary 2.4. *Let R be a ring and $e_1, \ldots, e_n \in Id(R)$. If $e_1Re_1, \ldots, e_nRe_n$ satisfies regular 1-stable range, then so does the following ring*

$$\begin{pmatrix} e_1Re_1 & . & . & . & e_1Re_n \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ e_nRe_1 & . & . & . & e_nRe_n \end{pmatrix}.$$

Proof. By Theorem 2.5 and induction. $\square$

Corollary 2.5. *The following are equivalent for any ring R:*

(1) *R satisfies regular 1-stable range.*
(2) *There exists a complete orthogonal set of idempotents, $\{e_1, \ldots, e_n\}$, such that all e_iRe_i satisfies regular 1-stable range.*

Corollary 2.6. *Let $M_1, \ldots, M_n$ be right R-modules. If $\mathrm{End}_R(M_1), \ldots, \mathrm{End}_R(M_n)$ satisfies regular 1-stable range, then so does $\mathrm{End}_R(M_1 \oplus \cdots \oplus M_n)$.*

Corollary 2.7. (1) *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If A and B satisfy regular 1-stable range, then so does T.*

(2) *If R satisfies regular 1-stable range, then so does $M_n(R)$ for every $n \geqslant 1$.*

Proof. For the proof of (1), Set $e = diag(1_A, 0)$. Since $eTe \cong A$ and $(1_T - e)T(1_T - e) \cong B$, the result follows from Theorem 2.5. The assertion in (2) follows from (1). $\square$

Theorem 2.6. *Let R satisfies regular 1-stable range. Then every $n \times n$ matrix over R is the sum of an invertible matrix and a regular matrix.*

Proof. Let $A \in M_n(R)$. Since R satisfies regular 1-stable range, so does $M_n(R)$. But $AM_n(R) + I_nM_n(R) = M_n(R)$. Thus there exists $W \in Reg(M_n(R))$ such that $A + I_nW = U \in GL_n(R)$. Therefore $A = -W + U$ is the sum of an invertible matrix and a regular matrix. $\square$

Theorem 2.7. *Let R be an abelian ring and $e \in Id(R)$. If R satisfies regular 1-stable range, then so does eRe.*

Proof. Let $a, x, b \in eRe$ with $ax + b = e$. Since $a(1-e) = x(1-e) = 0$, so $(a+1-e)(x+1-e)+b = 1$. Hence there exists $r \in Reg(R)$ such that $(a+1-e)+br \in U(R)$. Thus $((a+1-e)+br)v = v((a+1-e)+br) = 1$ for some $v \in R$. Now, since e is central, so $(a+b(ere))(eve) = ((a+1-e)+br)ve = e$ and $(eve)(a+b(ere)) = ev((a+1-e)+br) = e$. Let $y = ere \in eRe$. Then $a + by \in U(R)$ and it is easy to check that $y \in Reg(eRe)$, as required. □

Theorem 2.8. *The following are equivalent for any ring R:*

(1) *A_1, A_2 and A_3 satisfies regular 1-stable range.*

(2) *The formal triangular matrix ring $T = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ satisfies regular 1-stable range.*

Proof. (1) $\Rightarrow$ (2). Set $B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}$ and $M = \begin{pmatrix} M_{21} \\ M_{32} \end{pmatrix}$. Since A_2 and A_3 satisfies regular 1-stable range, so is the ring B by Corollary 2.7. Therefore $\begin{pmatrix} A_1 & 0 \\ M & B \end{pmatrix} = T$ satisfies regular 1-stable range by Corollary 2.7 again.

(2) $\Rightarrow$ (1). Given $ax+b = 1$ in A_1, then $diag(a,0,0)diag(x,0,0)+diag(b,1,1) = 1_T$. Thus there exists $\begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix} \in Reg(T)$ such that

$$diag(a,0,0) + diag(b,1,1)\begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix} \in U(T).$$

Now, since $\begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix} \in Reg(T)$, so

$$\begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix}\begin{pmatrix} y_1 & 0 & 0 \\ * & y_2 & 0 \\ * & * & y_3 \end{pmatrix}\begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix} = \begin{pmatrix} w_1 & 0 & 0 \\ * & w_2 & 0 \\ * & * & w_3 \end{pmatrix}$$

for some $\begin{pmatrix} y_1 & 0 & 0 \\ * & y_2 & 0 \\ * & * & y_3 \end{pmatrix} \in T$. Hence $w_1y_1w_1 = w_1$. So $w_1 \in Reg(A_1)$. Clearly, $U(T) = \begin{pmatrix} U(A_1) & 0 & 0 \\ * & U(A_2) & 0 \\ * & * & U(A_3) \end{pmatrix}$. Thus $a + bw_1 = u_1 \in U(A_1)$. Therefore A_1 satisfies regular 1-stable range. Likewise, A_2 and A_3 satisfies regular 1-stable range. □

Corollary 2.8. *A ring R satisfies regular 1-stable range if and only if so does the ring of all $n \times n$ lower (resp., upper) triangular matrices over R.*

An element $a \in R$ is said to be r-clean if $a = e + r$, where e is an idempotent and r is a regular (von Neumann) element in R. If every element of R is r-clean, then R is called an r-clean ring. We introduced r-clean rings and gave some properties of this kind of rings in [2] and [3].

Proposition 2.3. *Every abelian r-clean ring R satisfies regular 1-stable range.*

Proof. As R is exchange the result is clear by [6, Theorem 12]. $\square$

Now, we give a non abelian r-clean ring, while it indeed satisfies regular 1-stable range.

Example 2.1. Let $R = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. Since $\mathbb{Z}/2\mathbb{Z}$ is an r-clean ring, so R is an r-clean ring by [3, Theorem 2.14]. It is clear that R is not abelian. Also as $\mathbb{Z}/2\mathbb{Z}$ satisfies regular 1-stable range, so R satisfies regular 1-stable range by Corollary 2.8.

Let M be an R-R-bimodule. The trivial extension of R by M is the ring $T(R, M)$ of pairs (r, m), where $r \in R$ and $m \in M$, and with the usual addition and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$.

Theorem 2.9. *Let R be a ring and M be an R-R-bimodule. Then $T(R, M)$ satisfies regular 1-stable range if and only if so does R.*

Proof. Assume that $T(R, M)$ satisfies regular 1-stable range. Given $ax + b = 1$ in R, then $(a, 0)(x, 0) + (b, 0) = (1, 0)$ in $T(R, M)$. So there exists $(r_1, r_2) \in Reg(T(R, M))$ such that $(a, 0) + (b, 0)(r_1, r_2) = (a + br_1, br_2) \in U(T(R, M))$. But as $(r_1, r_2) \in Reg(T(R, M))$, so $(r_1, r_2)(y_1, y_2)(r_1, r_2) = (r_1, r_2)$ for some $(y_1, y_2) \in T(R, M)$. Hence $r_1 \in Reg(R)$. Now, since $a + br_1 \in U(R)$, so R satisfies regular 1-stable range.

Conversely, let R satisfy regular 1-stable range. Given $(a, m)(x, n) + (b, p) = (1, 0)$ in $T(R, M)$, then $ax + b = 1$ in R. So there exists $r \in Reg(R)$ such that $a + br = u \in U(R)$. Hence $(a, m) + (b, p)(r, 0) = (u, m + pr)$. Now, as $r_1 \in Reg(R)$, so there exists $y \in R$ such that $ryr = r$. Thus $(r, 0)(y, 0)(r, 0) = (r, 0)$. Hence $(r, 0) \in Reg(T(R, M))$. But $uv = vu = 1$ for some $v \in R$. Therefore $(u, m + pr)(v, -v(m + pr)v) = (v, -v(m + pr)v)(u, m + pr) = (1, 0)$. Thus $(a, m) + (b, p)(r, 0) \in U(T(R, M))$, where $(r, 0) \in Reg(T(R, M))$, as required. $\square$

References

[1] P. Ara, *Strongly Π-regular rings have stable range one*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3293–3298.

[2] N. Ashrafi and E. Nasibi, *r-clean rings*, Math. Rep. (Bucur.) **15** (65) (2013), no. 2, 125–132.

[3] ———, *Rings in which elements are the sum of an idempotent and a regular element*, Bull. Iranian Math. Soc. **39** (2013), no. 3, 579–588.

[4] V. P. Camillo and H. P. Yu, *Stable range one for rings with many idempotents*, Trans. Amer. Math. Soc. **347** (1995), no. 8, 3141–3147.

[5] M. C. Campany, M. G. Lozano, and M. S. Molina, *Exchange Morita rings*, Comm. Algebra **29** (2001), no. 2, 907–925.

[6] H. Chen, *Rings with many idempotents*, Int. J. Math. Math. Sci. **22** (1999), no. 3, 547–558.

[7] ———, *Exchange rings satisfying unit 1-stable range*, Kyushu J. Math. **54** (2000), no. 1, 1–6.

[8] ———, *On stable range conditions*, Comm. Algebra **28** (2000), no. 8, 3913–3924.

[9] ———, *Units, idempotents, and stable range conditions*, Comm. Algebra **29** (2001), no. 2, 703–717.
[10] ———, *Morita contexts with many units*, Comm. Algebra **30** (2002), no. 3, 1499–1512.
[11] ———, *Exchange rings having stable range one*, Chinese J. Contemp. Math. **29** (2008), no. 1, 95–102.
[12] H. Chen and M. Chen, *On unit 1-stable range*, J. Appl. Algebra Discrete Struct. **1** (2003), no. 3, 189–196.
[13] ———, *Unit 1-stable range for ideals*, Int. J. Math. Math. Sci. **2004** (2004), no. 45-48, 2477–2482.
[14] H. Chen and F. Li, *Rings with many unit-regular elements*, Chinese J. Contemp. Math. **21** (2000), no. 1, 33–38.
[15] E. G. Evans, *Krull-Schmidt and cancellation over local rings*, Pacifica J. Math. **46** (1973), 115–121.
[16] K. R. Goodearl and P. Mental, *Stable range one for rings with many units*, J. Pure Appl. Algebra **54** (1988), no. 2-3, 261–287.
[17] A. Haghany, *Hopfcity and co-Hopfcity for Morita contexts*, Comm. Algebra **27** (1999), no. 1, 477–492.
[18] W. K. Nicholson, *Lifting idempotent and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
[19] ———, *Strongly clean rings and fittings lemma*, Comm. Algebra **27** (1999), no. 8, 3583–3592.
[20] L. N. Vaserstein, *Bass's first stable range condition*, J. Pure Appl. Algebra **34** (1984), no. 2-3, 319–330.
[21] R. B. Warfield, Jr., *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91** (1980), no. 2, 457–485.
[22] G. Xiao and W. Tong, *n-clean rings and weakly unit stable range rings*, Comm. Algebra **33** (2005), no. 5, 1501–1517.
[23] H. P. Yu, *Stable range one for exchange rings*, J. Pure Appl. Algebra **98** (1995), no. 1, 105–109.

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