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A NOTE ON JORDAN DERIVATIONS OF TRIVIAL GENERALIZED MATRIX ALGEBRAS

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Abstract. H. R. Ebrahimi Vishki et al. conjectured in [1], that if every Jordan higher derivation on a trivial generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ is a higher derivation, then either $M = 0$ or $N = 0$. In this note, we will give a class of counter examples.

1. Introduction

Generalized matrix algebras were introduced by Morita in [12] to study Morita duality theory in 1958. They are natural generalizations of triangular algebras. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. When both the pairings are zero, a generalized matrix algebra is called trivial. A class of trivial generalized matrix algebras is the so-called generalized one-point extension algebras introduced in [7] by Li and Wei. In [2], Haghany studied hopfcity and co-hopfcity for trivial generalized matrix algebras.

Recent years, it has been an active research area to study various mappings on generalized matrix algebras, such as commuting mappings, Lie derivations, Jordan derivations, higher derivations and so on. We refer the reader to [5, 6, 7, 8, 9, 10, 11, 13, 15] for details. Note that Jordan derivations are related to the problem of the decomposition of Jordan homomorphisms [4]. The standard problem for Jordan derivations is to find out whether a Jordan derivation degenerate to a derivation. In [3], Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Zhang and Yu proved in [18] that all Jordan derivations of a triangular algebra with the faithful condition are derivations. More examples include incidence algebras [14] and dual extension algebras [8]. It is helpful to point out that the

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problem has the following higher version: to find out whether a Jordan higher derivation degenerate to a higher derivation. We refer the reader to [16, 17] for some results on it.

H. R. Ebrahimi Vishki et al. conjectured in [1] that if every Jordan higher derivation on a trivial generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ is a higher derivation, then either $M = 0$ or $N = 0$. In this note, we will give a class of counter examples.

2. Preliminaries

We recall some definitions in this section. Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ a unital algebra over $\mathcal{R}$. We denote the Jordan product by $a \circ b = ab + ba$ for all $a, b \in \mathcal{A}$. Recall that an $\mathcal{R}$-linear mapping Θ from $\mathcal{A}$ into itself is called a derivation if

$$\Theta(ab) = \Theta(a)b + a\Theta(b)$$

for all $a, b \in \mathcal{A}$, an anti-derivation if

$$\Theta(ab) = \Theta(b)a + b\Theta(a)$$

for all $a, b \in \mathcal{A}$, and a Jordan derivation if

$$\Theta(a \circ b) = \Theta(a) \circ b + a \circ \Theta(b).$$

Every derivation is obviously a Jordan derivation. The converse statement is in general not true. Those Jordan derivations which are not derivations are said to be proper.

Let $\mathbb{N}$ be the set of all non-negative integers. Let $D := \{d_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{R}$-linear mappings on $\mathcal{A}$ with $d_0 = id_{\mathcal{A}}$ (the identity mapping of $\mathcal{A}$). If

$$d_n(xy) = \sum_{i=0}^{n} d_i(x)d_{n-i}(y) \qquad \forall\, x, y \in \mathcal{A} \quad \forall\, n \in \mathbb{N},$$

then $\{d_n\}_{n\in\mathbb{N}}$ is called a higher derivation of $\mathcal{A}$. A sequence D is called a Jordan higher derivation of $\mathcal{A}$ if

$$d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x) \qquad \forall\, x \in \mathcal{A},\ n \in \mathbb{N}.$$

The definition of generalized matrix algebras is given by a Morita context. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow$

A and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ satisfying the following commutative diagrams:

$$\begin{tikzcd} M \underset{B}{\otimes} N \underset{A}{\otimes} M \arrow[r, "\Phi_{MN}\otimes I_M"] \arrow[d, "I_M\otimes\Psi_{NM}"] & A \underset{A}{\otimes} M \arrow[d, "\cong"] \\ M \underset{B}{\otimes} B \arrow[r, "\cong"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N \underset{A}{\otimes} M \underset{B}{\otimes} N \arrow[r, "\Psi_{NM}\otimes I_N"] \arrow[d, "I_N\otimes\Phi_{MN}"] & B \underset{B}{\otimes} N \arrow[d, "\cong"] \\ N \underset{A}{\otimes} A \arrow[r, "\cong"] & N\,. \end{tikzcd}$$

Let us write this Morita context as $(A, B,_A M_B,_B N_A, \Phi_{MN}, \Psi_{NM})$. If $(A, B,_A M_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\left[\begin{array}{cc} A & M \\ N & B \end{array}\right] = \left\{ \left[\begin{array}{cc} a & m \\ n & b \end{array}\right] \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

forms an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication. There is no constraint condition concerning the bimodules M and N. Of course, they are possibly equal to zeros. Such an $\mathcal{R}$-algebra is called a generalized matrix algebra of order 2 and is usually denoted by $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ or $\mathcal{G} = (A, M, N, B)$ in brief. When the pairings are both zero, we call $\mathcal{G}$ a trivial generalized matrix algebra.

3. Main results

In order to give the main result of this note, we need the following lemmas about derivations and Jordan derivations of generalized matrix algebras.

Lemma 3.1 ([5, Proposition 4.2]). *An additive map Θ from $\mathcal{G}$ into itself is a derivation if and only if it has the form*

$$(\bigstar 1) \qquad \Theta\left(\left[\begin{array}{cc} a & m \\ n & b \end{array}\right]\right) = \left[\begin{array}{cc} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + \nu_3(n) & n_0 m + nm_0 + \mu_4(b) \end{array}\right], \quad \forall \left[\begin{array}{cc} a & m \\ n & b \end{array}\right] \in \mathcal{G},$$

where $m_0 \in M$, $n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \nu_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *δ_1 is a derivation of A with $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;*
(2) *μ_4 is a derivation of B with $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;*
(3) *$\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;*
(4) *$\nu_3(na) = \nu_3(n)a + n\delta_1(a)$ and $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$.*

Lemma 3.2 ([9, Proposition 4.2])**.** *An additive map Θ from $\mathcal{G}$ into itself is a Jordan derivation if and only if it is of the form*

$$\begin{aligned}(\bigstar 2)\qquad &\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\\ &= \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix},\\ &\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},\end{aligned}$$

where $m_0 \in M, n_0 \in N$ and

$$\begin{aligned} &\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \tau_3 : N \longrightarrow M, \\ &\nu_2 : M \longrightarrow N, \quad \nu_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B \end{aligned}$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *δ_1 is a Jordan derivation on A and $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;*
(2) *μ_4 is a Jordan derivation on B and $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;*
(3) *$\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;*
(4) *$\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$ and $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$;*
(5) *$\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n) = 0$, $\tau_3(n)n = 0$;*
(6) *$\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$, $m\nu_2(m) = 0$, $\nu_2(m)m = 0$.*

Clearly, the mappings τ_3 and ν_2 in ($\bigstar$2) play an important role for a Jordan derivation. Let us study them now.

Lemma 3.3. *Let K be a field and A a finite dimensional K-algebra. Let M be a simple left A-module and N a simple right A-module with $\dim_K M \neq \dim_K N$. Then we have*:

(1) *Let ν be a K-linear map from M to N. If $\nu(am) = \nu(m)a$ for all $a \in A$, $m \in M$, then $\nu = 0$.*
(2) *Let τ be a K-linear map from N to M. If $\tau(na) = a\tau(n)$ for all $a \in A$, $n \in N$, then $\tau = 0$.*

Proof. Since $\dim_K M \neq \dim_K N$, without loss of generality, suppose that $\dim_K M < \dim_K N$.

(1) If $\nu \neq 0$, then there exists some $m \in M$ such that $\nu(m) \neq 0$. It follows from $\nu(am) = \nu(m)a$ that the cyclic module $\langle \nu(m) \rangle$ generated by $\nu(m)$ is contained in $\operatorname{Im}\nu$, the image of ν. Note that N is a simple as a right A-module. This implies that $\langle \nu(m) \rangle = N$ and consequently, $\operatorname{Im}\nu = N$. However, this is impossible for $\dim_K M < \dim_K N$.

(2) Since $\dim M < \dim N$, the kernel of τ is not zero. Let $0 \neq n \in \mathrm{Ker}\tau$. Then the condition $\tau(na) = a\tau(n)$ gives that $0 \neq \langle n \rangle \subseteq \mathrm{Ker}\tau$ and hence $\mathrm{Ker}\tau = N$ for N is simple as a right A-module, that is, $\tau = 0$. $\square$

Now we are in a place to give the main result of this note.

Theorem 3.4. *Let A and B be two finite dimensional K-algebras and let $\mathcal{G} = (A, {}_AM_B, {}_BN_A, B)$ be a trivial generalized matrix algebra. Suppose that $\mathcal{G}$ satisfies the following conditions*:

(1) *All Jordan derivations of A and B are derivations.*
(2) *M is simple as a left A-module and N is simple as a right A-module*;
(3) $\dim_K M \neq \dim_K N$.

Then each Jordan derivation of $\mathcal{G}$ is a derivation.

Proof. Since all Jordan derivations of A and B are derivations, comparing the form ($\bigstar$1) of Lemma 3.1 with that ($\bigstar$2) of Lemma 3.2 yields that we only need to prove $\nu_2 = 0$ and $\tau_3 = 0$. By Lemma 3.3 this is clear and we complete the proof now. $\square$

In order to give a counter example for [1, Conjecture 3.2], let us recall a result of Xiao and Wei on Jordan higher derivations.

Lemma 3.5 ([17, Proposition 3.1])**.** *Let A be an associative algebra over a field of characteristic zero. If every Jordan derivation on A is a derivation, then every Jordan higher derivation on A is a higher derivation.*

According to Theorem 3.4 and Lemma 3.5, we can construct an example as follows.

Example 3.6. Let K be a field of characteristic zero. Denote by $M_r(K)$ the algebra of all matrices of degree r. Let $A = M_2(K) \bigoplus M_3(K)$ and let $B = K$. Clearly A is a semi-simple algebra. Then the semisimplicty of A implies that there are only two non-isomorphic simple left(right) A-modules, which are of dimensions 2 and 3, respectively. Take M to be the simple left A-module of dimension 2 and take N to be the simple right A-module of dimension 3. Clearly M can be viewed as an A-K-bimodule and N can be viewed as a K-A-bimodule. Let $\mathcal{G}$ be the corresponding trivial generalized matrix algebra $(A, {}_AM_K, {}_KN_A, K)$. Then it is easy to check that $\mathcal{G}$ satisfies the conditions of Theorem 3.4. Thus all of the Jordan derivations of $\mathcal{G}$ are derivations. Then it follows from Lemma 3.5 that all Jordan higher derivations of $\mathcal{G}$ are higher derivations. However, neither M nor N is zero. Then we have construct a counter example for [1, Conjecture 3.2].

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