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Comparison of different estimators of P(Y<X) for two parameter Lindley distribution

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Abstract: Stress-strength reliability problems arise frequently in applied statistics and related fields. In the context of reliability, the stress–strength model describes the life of a component, which has a random strength X and is subjected to random stress Y. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever X > Y. The problem of estimation the reliability parameter in a stress-strength model $R = P[Y < X]$, when X and Y are two independent two-parameter Lindley random variables is considered in this paper. The maximum likelihood estimator (MLE) and Bayes estimator of R are obtained. Also, different confidence intervals of R are obtained. Simulation study is performed to compare the different proposed estimation methods. Example in real data is used as practical application of the proposed procedure.

Key Words: *Bayes estimator, Boot-P method, Boot-t method,* $LD(\theta,\alpha)$ *, maximum likelihood method, simulation, stress-strength model, two-parameter Lindley distribution (two parameter)*

1. INTRODUCTION

Lindley distribution originally developed by Lindley (1958, 1965) in the context of Bayesian statistics, as a counter example of fiducial statistics and some classical statistic properties are investigated by Ghitany et al. (2008). The one-parameter Lindley distribution, known as Lindley distribution has the probability density function as

$$f(x;\theta) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x} \quad , x > 0 \, , \, \theta > 0 \ . \tag{1}$$

Shanker et al. (2013) introduced a two- parameter Lindley distribution with parameters θ and α which has the probability function as

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$$f(x;\theta,\alpha)=\frac{\theta^2}{(\theta+\alpha)}(1+\alpha x)e^{-\theta x}, x>0,\ \theta>0\ ,\theta>-\alpha \tag{2}$$

It can seen that the two-parameter Lindley distribution is a mixture of exponential distribution with parameter θ and gamma distribution with parameters 2 and θ as follows:

$$f(x;\theta,\alpha)=pf_1(x)+(1-P)f_2(x)$$

where, $p=\dfrac{\theta}{\theta+\alpha}$, $f_1(x)=\theta e^{-\theta x}$ and $f_2(x)=\theta^2 x e^{-\theta x}$.

Also if $\alpha=1$, the two-parameter Lindley distribution reduces to the one- parameter Lindley distribution and if $\alpha=0$ it reduces to exponential distribution.

The cumulative distribution of the two parameter Lindley distribution is given by

$$F(x)=P[X\le x]=1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\quad x>0,\ \theta>0\ ,\theta>-\alpha$$

and survival function

$$S(x)=P[X\ge x]=\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\quad x>0,\ \theta>0\ ,\theta>-\alpha\quad .$$

We denoted to an absolutely continuous random variables having Two-parameter Lindley distribution with parameters θ and α by two parameter $LD(\theta,\alpha)$.

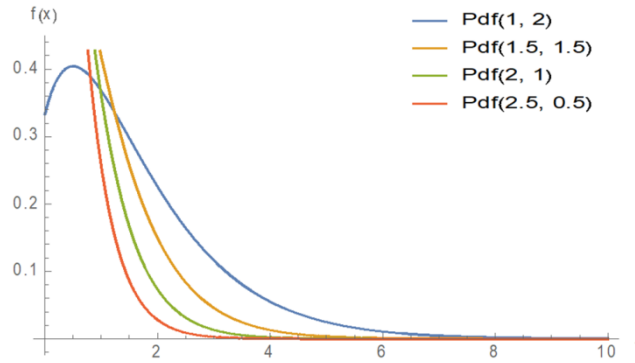


Figure 1. Shows the graph two parameter $LD(\theta,\alpha)$ for different values of parameters θ and α

It is clear that from Figure 1 two parameter $LD(\theta,\alpha)$ is positively skewed distribution.

In this paper, we are interested in estimating the parameter of the stress-strength model which is used in many applications of physics and engineering such that failure and system collapse. The stress-strength model plays an important in the reliability analysis. For examples if X is the strength of system which is subjected to stress Y, then R measures the system performance and it provides the probability of a system failure if the system fails whenever the applied stress is greater than strength. Many authors have studied the estimation of the stress-strength parameter R such that Ghitany et al (2015)

considered the estimation of R for power Lindley distribution; Hussian (2013) considered the estimation of R for generalized inverted exponential distribution; Wong (2012) considered the estimation of R for the generalized Pareto, Al-Mutairi et al (2013) considered the estimation of R for the Lindley distribution; Kundu and Gupta (2005) and Raqab et al. (2008) considered the estimation of R for the two-parameter and three-parameter generalized exponential distributions, respectively; Kundu and Raqab (2009) considered the estimation of R when X and Y are independent and both having three-parameter Weibull distributions with common shape and location parameters but different scale parameters; Gupta and Peng (2009) have studied the estimation of R in the context of proportional odds ratio models and Shahsanaei and Daneshkhah (2013) considered the estimator of R when stress and strength variates are linear failure rate distribution, Hassan (2016, 2017) considered the estimator of R when stress and strength variates are generalized linear failure rate distribution and Lindly distribution respectively . For more literature see Kotz et al (2003).

The main objective in this paper, is the estimation of $R = P[Y < X]$ when X and Y are two parameter $LD(\theta,\alpha)$ and two parameter $LD(\eta,\alpha)$ respectively. Our paper is organized as follows; the system R is derived in section 2. the maximum likelihood estimator (MLE) and asymptotic confidence interval of R in section 3. The bootstrap confidence intervals of R by using different algorithms are presented in section 4. The Bayes estimator of R by using Lindley approximation in section 5. Simulation study is made in order to give an assessment of the performance of different estimation methods in section 6. Example in real data is used as practical application of the proposed procedure in section 7. Finally, conclusion of our paper is presented in section 8.

2. SYSTEM RELIABILITY

Let X and Y are two independent two parameter LD random variables with parameters (θ,α) and (η,α) respectively. The stress-strength model is defined as

$$\begin{aligned} R &= P[Y < X] \\ &= \int_0^\infty \int_0^x \left(\frac{\theta^2}{\theta+\alpha}\right)(1+\alpha x)e^{-\theta x}\left(\frac{\eta^2}{\eta+\alpha}\right)(1+\alpha y)e^{-\eta y}\,dy\,dx \\ &= \frac{\theta^2(\eta(\eta+\theta)^2+\alpha^2(3\eta+\theta)+\alpha(\eta+\theta)(3\eta+\theta))}{(\alpha+\eta)(\alpha+\theta)(\eta+\theta)^3} \end{aligned} \tag{3}$$

3. MAXIMUM LIKELIHOOD ESTIMATION OF R

Suppose that $X_1,...,X_n$ and $Y_1,...,Y_m$ are two random samples of sizes n and m taken from two parameter $LD(\theta,\alpha)$ and two parameter $LD(\eta,\alpha)$ respectively. The likelihood function of the observed samples is

$$L(\Theta)=(\frac{\theta^2}{\theta+\alpha})^n\prod_{i=1}^{n}(1+\alpha x_i)e^{-n\theta\bar{x}}(\frac{\eta^2}{\eta+\alpha})^m\prod_{j=1}^{m}(1+\alpha y_j)e^{-m\eta\bar{y}} \tag{4}$$

Where, $\Theta=(\theta,\alpha,\eta)$. The maximum estimators of $\Theta=(\theta,\alpha,\eta)$ denoted by $\hat{\Theta}=(\hat{\theta},\hat{\alpha},\hat{\eta})$ can be obtain as follows:

(i) Find the log-likelihood function by taking the natural algorithm to equation(4) we get

$$Ln[L(\Theta)]=n(Ln[\theta^2]-Ln[\theta+\alpha])+\sum_{i=1}^{n}Ln[1+\alpha x_i]-n\theta\bar{x}$$

$$+m(Ln[\eta^2]-Ln[\eta+\alpha])+\sum_{j=1}^{m}Ln[1+\alpha y_j]-m\eta\bar{y}$$

(ii) Maximize the log-likelihood function to $\Theta=(\theta,\eta,\alpha)$ (i.e. $\frac{\partial Ln[L(\Theta)]}{\partial\Theta}=0$)

$$\frac{\partial Ln[L(\Theta)]}{\partial\theta}=0=\frac{2n}{\theta}-\frac{n}{\theta+\alpha}-n\bar{x} \tag{5}$$

$$\frac{\partial Ln[L(\Theta)]}{\partial\eta}=0=\frac{2m}{\eta}-\frac{m}{\eta+\alpha}-m\bar{y} \tag{6}$$

$$\frac{\partial Ln[L(\Theta)]}{\partial\alpha}=0=-\frac{n}{\theta+\alpha}-\frac{m}{\eta+\alpha}+\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i}+\sum_{j=1}^{m}\frac{y_j}{1+\alpha y_j} \tag{7}$$

From equation (5) and (6) we get

$$\hat{\theta}=\frac{(1-\hat{\alpha}\bar{x})+\sqrt{(\hat{\alpha}\bar{x}-1)^2+8\hat{\alpha}\bar{x}}}{2\bar{x}}$$

and

$$\hat{\eta}=\frac{(1-\hat{\alpha}\bar{y})+\sqrt{(\hat{\alpha}\bar{y}-1)^2+8\hat{\alpha}\bar{y}}}{2\bar{y}}$$

Where $\hat{\alpha}$ is the solution of nonlinear equation (7).

Using the invariance property of maximum likelihood estimation method to get the maximum estimator of $R=R(\theta,\eta,\alpha)$ denotes by $\hat{R}=R(\hat{\theta},\hat{\eta},\hat{\alpha})$ is given by

$$\hat{R}=\frac{\hat{\eta}^2(\hat{\theta}(\hat{\eta}+\hat{\theta})^2+\hat{\alpha}^2(\hat{\eta}+3\hat{\theta})+\hat{\alpha}(\hat{\eta}+\hat{\theta})(\hat{\eta}+3\hat{\theta}))}{(\hat{\alpha}+\hat{\eta})(\hat{\alpha}+\hat{\theta})(\hat{\eta}+\hat{\theta})^3}. \tag{8}$$

Since the exact distribution of $\hat{R}$ is difficult to obtain, then we find the asymptotic distribution and confidence interval of R.

To derive the asymptotic properties of $\hat{R}$, using the asymptotic properties and the general conditions of maximum likelihood estimators see Lehmann and Casella (1998), we obtain the asymptotic distribution of $\hat{\Theta}$ satisfies

$$\sqrt{n+m}(\hat{\Theta}-\Theta)\xrightarrow{d}N_3(0,I^{-1}(\Theta))$$

Where, $\xrightarrow{d}$ denotes convergence in distribution, $N_3(.,.)$ denotes the trinomial distribution and $I^{-1}(\Theta)$ is the inverse of matrix $I(\Theta)=\underset{n,m\to\infty}{Lim}\,E[(\frac{-1}{n+m}H(\Theta))]$ is the asymptotic expected Fisher information matrix when $H(\Theta)=[\frac{\partial^2 Ln[L(\Theta)]}{\partial\Theta^T\partial\Theta}]$ is the Hessian matrix of $Ln[L(\Theta)]$.

Now to calculate,

$$I(\Theta)=\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

Where,

$$I_{11}=\underset{n,m\to\infty}{Lim}\,E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\theta^2}]=P_1(\frac{2}{\theta^2}-\frac{1}{(\theta+\alpha)^2}),$$

$$I_{22}=\underset{n,m\to\infty}{Lim}\,E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\eta^2}]=P_2(\frac{2}{\eta^2}-\frac{1}{(\eta+\alpha)^2}),$$

$$I_{33}=\underset{n,m\to\infty}{Lim}\,E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\alpha^2}]=-\frac{P_1}{(\theta+\alpha)^2}-\frac{P_2}{(\eta+\alpha)^2}-$$

$$P_1(\frac{\alpha(\alpha-\theta)+e^{\frac{\theta}{\alpha}}\theta^2((\Gamma(0,\frac{\theta}{\alpha})+Ln[\alpha]-Ln[\theta]+Ln[\frac{\theta}{\alpha}])}{\alpha^3(\alpha+\theta)})-$$

$$P_2(\frac{\alpha(\alpha-\eta)+e^{\frac{\eta}{\alpha}}\theta^2((\Gamma(0,\frac{\eta}{\alpha})+Ln[\alpha]-Ln[\eta]+Ln[\frac{\eta}{\alpha}])}{\alpha^3(\alpha+\eta)}),$$

$$I_{12}=I_{21}=\underset{n,m\to\infty}{Lim}\,E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\theta\partial\eta}]=0$$

$$I_{13}=I_{31}=\underset{n,m\to\infty}{Lim}\,E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\theta\partial\alpha}]=-\frac{P_1}{(\theta+\alpha)^2}$$

$$I_{23} = I_{32} = \lim_{n,m\to\infty} E[\frac{-1}{n+m}\frac{\partial^2 Ln[L(\Theta)]}{\partial\eta\partial\alpha}] = -\frac{P_2}{(\eta+\alpha)^2}$$

$$P_1 = \lim_{n,m\to\infty}[\frac{-n}{n+m}] \text{ and } P_2 = \lim_{n,m\to\infty}[\frac{-m}{n+m}].$$

The asymptotic variance-covariance matrix of $\hat{\Theta}$ is given by

$$\frac{1}{n+m}I^{-1}(\Theta) = \frac{1}{n+m}\begin{pmatrix} Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\eta}) & Cov(\hat{\theta},\hat{\alpha}) \\ Cov(\hat{\eta},\hat{\theta}) & Var(\hat{\eta}) & Cov(\hat{\eta},\hat{\alpha}) \\ Cov(\hat{\alpha},\hat{\theta}) & Cor(\hat{\alpha},\hat{\eta}) & Var(\hat{\alpha}) \end{pmatrix}.$$

Now, to derive the asymptotic normality of R

Let, $C(\Theta) = (\frac{\partial R}{\partial\theta},\frac{\partial R}{\partial\eta},\frac{\partial R}{\partial\alpha})^T = (C_1, C_2, C_3)^T$, where T is the transpose operation and

$$C_1 = \frac{\partial R}{\partial\theta} = \frac{\eta^2\theta(6\alpha^3 + \theta(\eta+\theta)^2 + 6\alpha^2(\eta+2\theta) + 2\alpha(\eta^2 + 4\eta\theta + 3\theta^2))}{(\alpha+\eta)(\alpha+\theta)^2(\eta+\theta)^4}$$

$$C_2 = \frac{\partial R}{\partial\eta} = -\frac{\eta\theta^2(6\alpha^3 + \eta(\eta+\theta)^2 + 6\alpha^2(2\eta+\theta) + 2\alpha(3\eta^2 + 4\eta\theta + \theta^2))}{(\alpha+\eta)(\alpha+\theta)^2(\eta+\theta)^4}$$

and

$$C_3 = \frac{\partial R}{\partial\alpha} = -\frac{\eta^2(\eta-\theta)\theta^2(2\alpha+\eta+\theta)}{(\alpha+\eta)(\alpha+\theta)^2(\eta+\theta)^4}$$

Using the delta method, the asymptotic distribution of the maximum estimator of R satisfies $\sqrt{n+m}(\hat{R}-R) \xrightarrow{d} N(0, C^T(\Theta)I^{-1}(\Theta)C(\Theta))$. The asymptotic variance of $\hat{R}$ is given by

$$Var(\hat{R}) = \frac{1}{n+m}C^T\Theta)I^{-1}(\Theta)C(\Theta)$$

$$= C_1^2 Var(\hat{\theta}) + C_2^2 Var(\hat{\eta}) + C_3^2 Var(\hat{\alpha}) + 2C_1C_2Cov(\hat{\theta},\hat{\eta}) + 2C_1C_3Cov(\hat{\theta},\hat{\alpha}) + 2C_2C_3Cov(\hat{\eta},\hat{\alpha})$$

Hence, an asymptotic $100(1-\alpha)\%$ confidence interval for R can obtain as $\hat{R} \mp Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{R})}$ where $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$-quantile of standard normal distribution. The asymptotic confidence interval results do not perform very well for small sample size. For this, we propose two confidence intervals based on nonparametric bootstrap in the next section.

4. BOOTSTRAP CONFIDENCE INTERVAL OF R

Eforn (1982) and Hall (1988) suggested two confidence intervals based on nonparametric bootstrap methods Boot-p and Boot-t respectively. We use Boot-p and Boot-t to evaluate the confidence interval of R as follows:

Algorithm (1): (Boot-P method)

1. For given values of θ , η and α simulate two independent random samples $X_1,...,X_n$ and $Y_1,...,Y_m$ from two parameter $LD(\theta,\alpha)$ and two parameter $LD(\eta,\alpha)$ respectively. To compute $\hat{R} = R(\hat{\theta},\hat{\eta},\hat{\alpha})$ by maximum likelihood method.
2. Generate independent bootstrap samples $X^*{}_1,...,X^*{}_n$ and $Y^*{}_1,...,Y^*{}_m$ using $\hat{\theta},\hat{\eta}$ and $\hat{\alpha}$. To compute $\hat{R}^* = R(\hat{\theta}^*,\hat{\eta}^*,\hat{\alpha}^*)$.
3. Repeat Step 2 B-times.
4. Compute the approximate $(1-\alpha)\%$ Boot-p confidence interval of R as $(\hat{R}_{Boot-P}(\frac{\alpha}{2}),\hat{R}_{Boot-p}(1-\frac{\alpha}{2}))$ where $\alpha = 0.05$ and $\hat{R}_{Boot-P}$ is the cumulative distribution of $\hat{R}^*$.

Algorithm (2) (Boot-t method)

1. For given values of θ , η and α simulate two independent random samples $x_1,...,x_n$ and $y_1,...,y_m$ two parameter $LD(\theta,\alpha)$ and two parameter $LD(\eta,\alpha)$ respectively. To compute $\hat{R} = R(\hat{\theta},\hat{\eta},\hat{\alpha})$ by maximum likelihood method.
2. Generate independent bootstrap samples $x^*{}_1,...,x^*{}_n$ and $y^*{}_1,...,y^*{}_m$ using $\hat{\theta},\hat{\eta}$ and $\hat{\alpha}$. To compute $\hat{R}^* = R(\hat{\theta}^*,\hat{\eta}^*,\hat{\alpha}^*)$.
3. Compute the following statistic $T^* = \dfrac{\hat{R}^* - \hat{R}}{SD^*(\sqrt{\hat{R}^*})}$
4. Repeat Step 2 and 3 NBOOT times.
5. Compute the approximate $(1-\alpha)\%$ Boot-t confidence interval of R as

$$(\hat{R}_{Boot-t}(\frac{\alpha}{2}),\hat{R}_{Boot-t}(1-\frac{\alpha}{2}))$$

Where $SD^*(\hat{R}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{R}_i^* - \overline{\hat{R}}_i^*)^2}$, $\hat{R}_{Boot-t} = \hat{R} + SD^*(\hat{R})H^{-1}(x)$ and $H(x)$ the cumulative distribution function of T^* .

5. BAYES ESTIMATION OF R

In this section, we derived the Bayesian estimator of the unknown parameters $\hat{\Theta} = (\hat{\theta}, \hat{\alpha}, \hat{\eta})$ and stress strength parameter R. We use Lindely approximation technique to get Bayes estimate of R which denotes by R^B. Many authors have used this technique to obtain the Bayes estimates of parameters of various lifetime distributions.
Lindley (1980) consider the posterior expectation of the function $f(\Theta \mid data)$ as the ratio of two integrals as

$$f^*(\Theta \mid data) = \frac{\int_{\Theta} f(\Theta) e^{LogL(\Theta)+\rho(\Theta)} d\Theta}{\int_{\Theta} f(\Theta) e^{LogL(\Theta)+\rho(\Theta)} d\Theta}$$

where,

- $\Theta = (\theta_1, ..., \theta_p)$
- $f(\Theta)$ is a parametric function
- $LogL(\Theta)$ is a log likelihood function, and
- $\rho(\Theta)$ is the log of the joint ratio prior Θ.

For large sample and under some conditions Lindley (1980) approximate (8) as

$$f^*(\Theta \mid data) = f(\Theta) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(f_{ij} + 2f_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}L_{ijk}f_l\sigma_{ij}\sigma_{kl} \qquad (9)$$

where

- $f_{ij} = \dfrac{\partial^2 f}{\partial\theta_i \partial\theta_j}$, $i, j = 1, ..., p$.
- $\dfrac{\partial^3 LogL(\Theta)}{\partial\theta_i \partial\theta_j \partial\theta_k}$, $i, j = 1, ..., p$.
- $\rho_j = \dfrac{\partial\rho}{\partial\theta_j}$, $j = 1, ..., p$.
- $\sigma_{ij} = [I_{ij}]^{-1}$, , $i, j = 1, ..., p$.

then the Bayes estimator of stress strength reliability parameter R is given by replace f by R in (9).

6. SIMULATION STUDY

Simulation study is constructed to investigate the performance of maximum likelihood and bootstrap estimation methods of R. Generate 1000 samples with sizes $(n_1, n_2) = (20,20), (30,30), (40,40), (50,50), (100,100), (200,200)$ with B=1000

replicated from each independent $TL(\theta,\alpha)$ and $TL(\eta,\alpha)$ distributions where (θ,η,α) =(1,2.5,2), (1,1,2) and (3,1,2) and correspond values of $R = 0.21$, 0.5,0.83 respectively. In Tables 3-5 we calculate the bias and mean square error (MSE) as follows

Table 1. $(\theta,\eta,\alpha) = (1, 2.5, 2)$, $R = 0.21$

(n_1,n_2)	Maximum likelihood method		Parametric Bootstrap		Nonparametric bootstrap	
	Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
(20,20)	-0.215	0.0652	0.2476	0.0624	-0.0573	0.0044
(30,30)	0.2468	0.0623	0.1146	0.0139	-0.1903	0.0369
(40,40)	-0.1511	0.0239	-0.0637	0.0046	-0.3686	0.1364
(50,50)	0.2613	0.0691	-0.1619	0.0266	-0.4668	0.2188
(100,100)	-0.0610	0.0041	-0.1043	0.0111	-0.4092	0.1676
(200,200)	-0.0450	0.0022	-.0325	0.001	-0.3374	0.1139

Table 2. $(\theta,\eta,\alpha) = (1, 1, 2)$, $R = 0.5$

(n_1,n_2)	Maximum likelihood method		Parametric Bootstrap		Nonparametric bootstrap	
	Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
(20,20)	0.1261	0.0166	-0.0990	0.0105	-0.1289	0.0173
(30,30)	-0.0634	0.0045	-0.0084	0.0005	-0.0383	0.0019
(40,40)	0.1742	0.0307	0.1777	0.0319	0.1478	0.0221
(50,50)	-0.1042	0.0111	-0.1598	0.0258	-0.1897	0.0362
(100,100)	-0.1291	0.0168	0.0396	0.0017	0.0097	0.0002
(200,200)	-0.0037	0.0001	0.0498	0.0025	0.0199	0.0004

Table 3. $(\theta,\eta,\alpha) = (3, 1, 2)$, $R = 0.83$

(n_1,n_2)	Maximum likelihood method		Parametric Bootstrap		Nonparametric bootstrap	
	Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
(20,20)	-0.0314	0.0034	-0.2884	0.0875	0.0209	0.0209
(30,30)	-0.0795	0.0079	0.2469	0.0638	0.5562	0.3122
(40,40)	-0.0857	0.0085	0.3376	0.1161	0.6469	0.4206
(50,50)	-0.1196	0.0153	0.2144	0.0477	0.5247	0.2760
(100,100)	0.4466	0.1994	-0.2966	0.0888	0.127	0.0010
(200,200)	-0.1305	0.0172	-0.2141	0.0462	0.0952	0.0095

From Table 1 we get the mean square error of the parametric bootstrap method is smaller than MSE which obtained by maximum likelihood method and nonparametric bootstrap, in Table 2 no difference between three methods, but in Table 3 the maximum likelihood method has smaller mean square error.

7. EXAMPLE

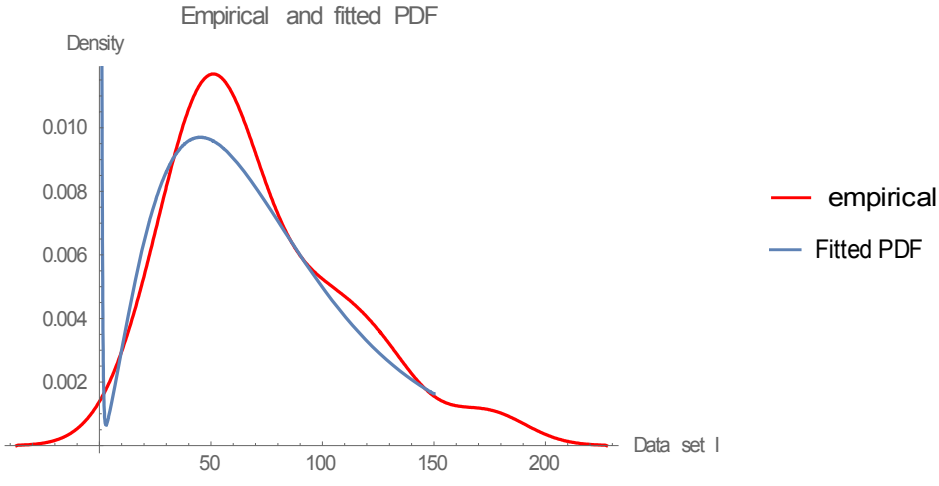
We use two data sets reported by Lawless (1982) and Proschan (1963). The first data set is obtained from Lawless (1982) and it represents the number of revolution before failure of each of 23 ball bearings in the life tests and they are as follows:

Data Set I: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

The second data set is obtained from Proschan (1963) and represents times between successive failures of 15 air conditioning (AC) equipment in a Boeing 720 airplane and they are as follows:

Data Set II: 12, 21, 26, 27, 29, 29, 48, 57, 59, 70, 74, 153, 326, 386, 502

Suppose that, the given data sets I and II are two independent random samples are drawn from $TL(\theta,\alpha)$ and $TL(\eta,\alpha)$ respectively. Figures 2 and 3 show diagnostic plots for data sets I and II respectively as follows



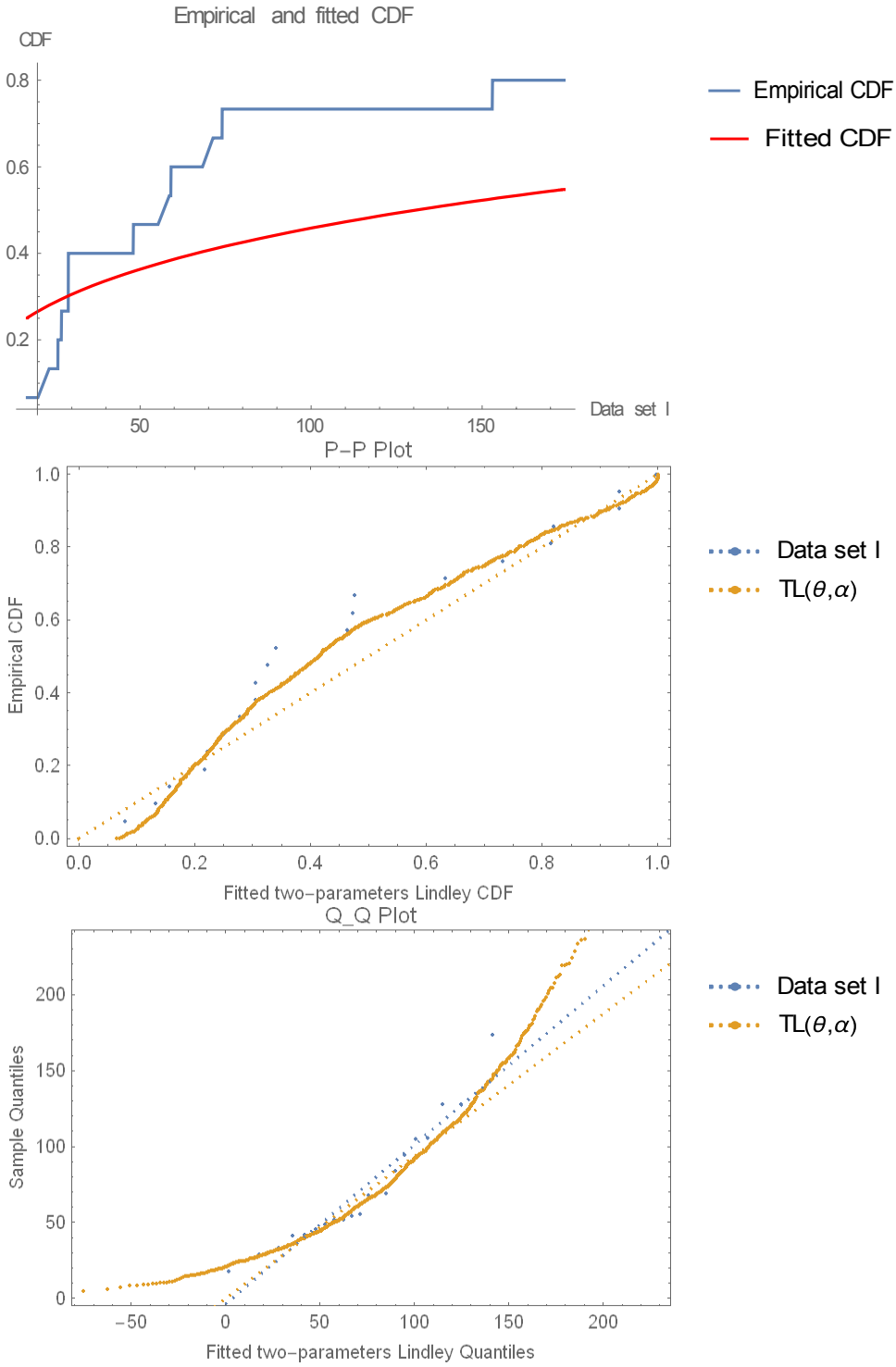
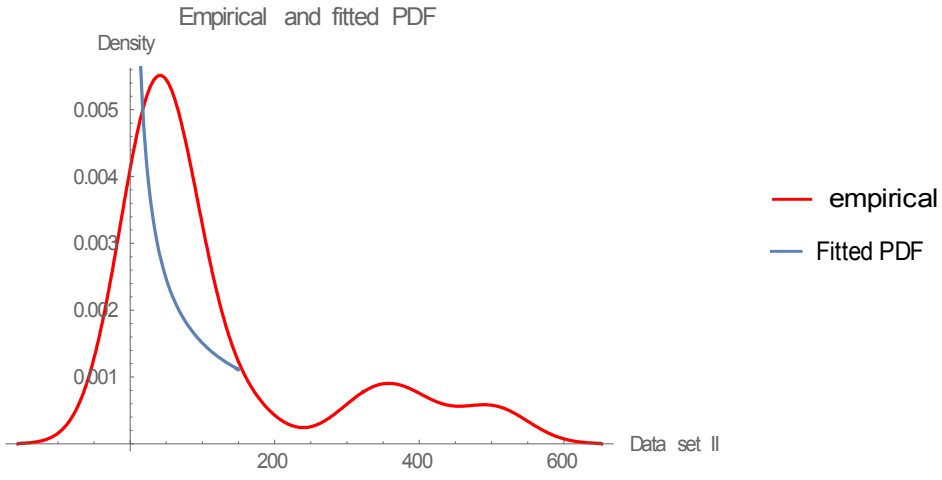
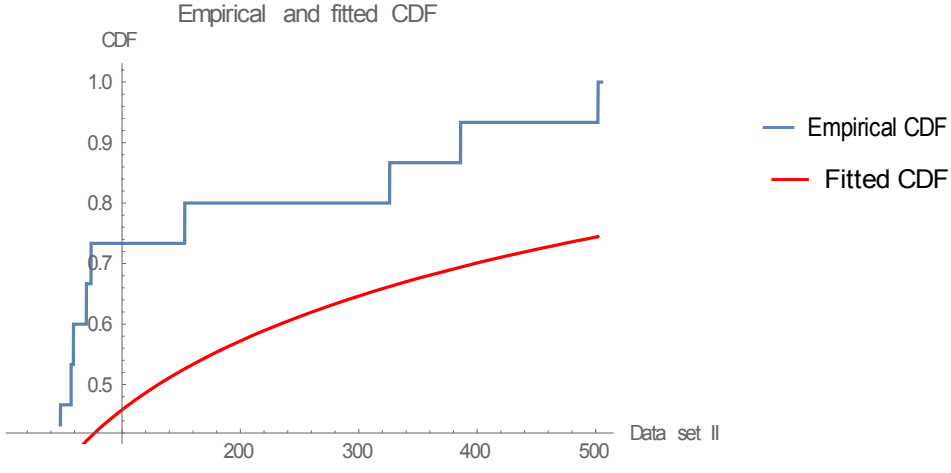


Figure 2. Diagnostic plots for the fitted Two-parameter Lindley model for Data set I


Empirical and fitted PDF
Density
0.005
0.004
0.003
0.002
0.001
200
400
600
Data set II
empirical
Fitted PDF


Empirical and fitted CDF
CDF
1.0
0.9
0.8
0.7
0.6
0.5
200
300
400
500
Data set II
Empirical CDF
Fitted CDF

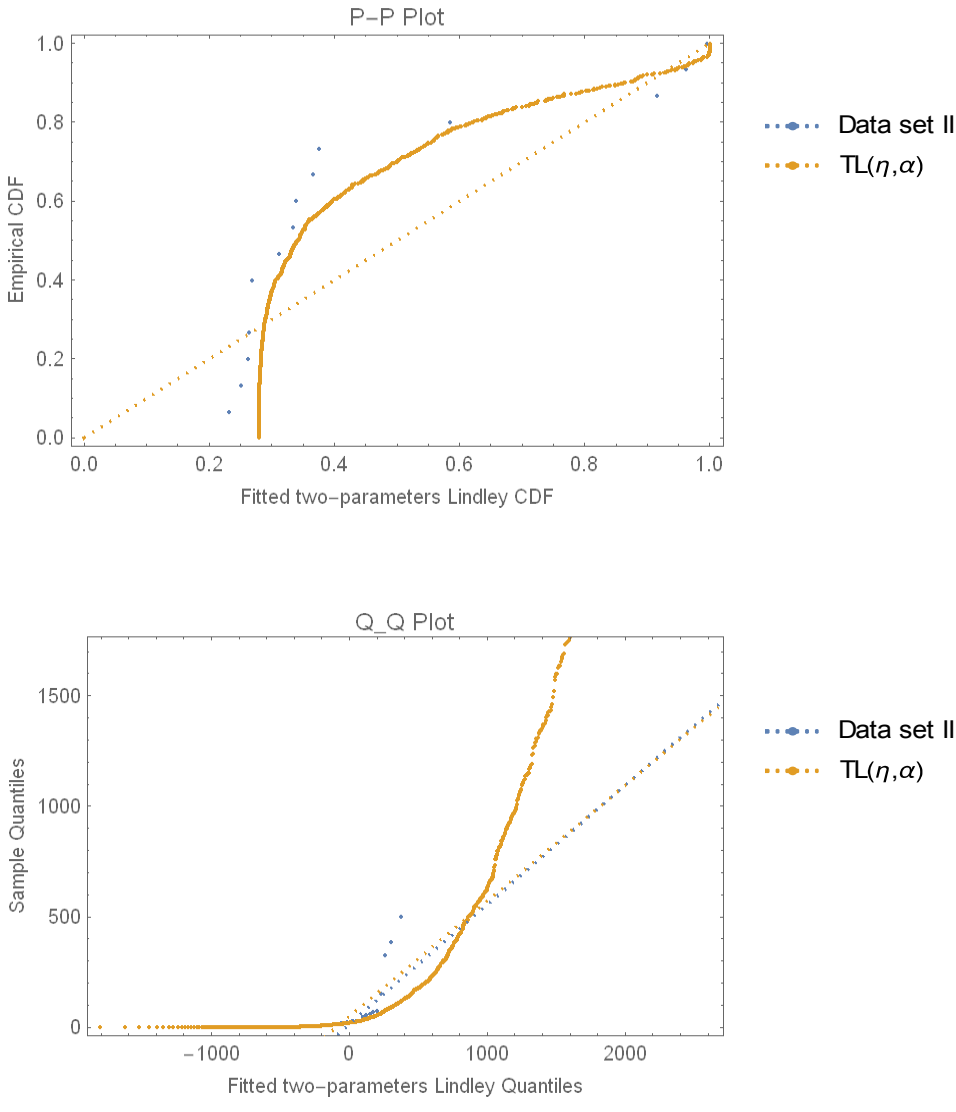


Figure 3. Diagnostic plots for the fitted Two-parameter Lindley model for Data set II

Table 4 shows the Kolmogorov-Smirnov (K-S) tests for data sets I and II with corresponding P-value. Therefore, it is clear that two parameters Lindley distribution is a good fit to both data sets I and II as follows

Table 4. Maximum likelihood estimator and K-S goodness of fit tests

Data	Maximum likelihood estimator	K-S Statistic value	P-value
Data set I	$\hat{\theta} = 0.3523$ $\hat{\alpha} = 28.29$	**0.1913**	**0.0614**
Data set II	$\hat{\eta} = 2.6$ $\hat{\alpha} = 28.29$	**0.357**	**0.00429**

Table 5 shows the different estimators of R and different confidence intervals as follows:

Table 5. Point and interval estimator of R

Estimation Method	Maximum likelihood method	Parametric Bootstrap	Nonparametric Bootstrap
$\hat{R}$	**0.042**	**0.5**	**0.508**
95% confidence interval	**(-1.26,1.34)**	**(-0.40,1.40)**	**(0.502,0.505)**

8. CONCLUSION

In this paper, the problem of estimating $R = P[Y < X]$ for two- parameters Lindley distribution is considered. From Figure (1) it is clear that the two- parameters Lindley distribution is positively skewed distribution. The maximum likelihood estimator (MLE) and Bayes estimator of R are obtained. Also, different confidence intervals of R are obtained. Simulation study is performed to compare the different proposed estimation methods. The results of simulation study is obtained as follows: from equation (3) and Tables 1 and 2, we get $R = P[Y < X]$ depends on three parameters (θ, η, α), also when, value of $R = 0.5$ and the value of R increases when θ increases and η decreases. From Table 3 we get the parametric bootstrap method is the best estimation method for R because MSE is smaller than MSE which obtained by maximum likelihood method and nonparametric bootstrap, in Table 4 no difference between three methods, but in Table 5 the maximum likelihood method is the best. In future studies, we hope to find the estimation of R using the ranked set sampling (RSS).

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