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An improvement of estimators for the multinormal mean vector with the known norm†

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Abstract

Consider the problem of estimating a $p \times 1$ mean vector θ $(p \geq 3)$ under the quadratic loss from multi-variate normal population. We find a James-Stein type estimator which shrinks towards the projection vectors when the underlying distribution is that of a variance mixture of normals. In this case, the norm $\|\boldsymbol{\theta} - K\boldsymbol{\theta}\|$ is known where K is a projection vector with $rank(K) = q$. The class of this type estimator is quite general to include the class of the estimators proposed by Merchand and Giri (1993). We can derive the class and obtain the optimal type estimator. Also, this research can be applied to the simple and multiple regression model in the case of $rank(K) \geq 2$.

Keywords: James-Stein type estimator, optimal estimator, projection vector, quadratic loss.

1. Introduction

There has been considerable interesting in the problem of estimating a $p \times 1$ mean vector $\boldsymbol{\theta}$ $(p \geq 3)$ of a compound multinormal distribution, under the quadratic loss function when the norm $\|\boldsymbol{\theta} - K\boldsymbol{\theta}\|$ is known, where K is an idempotent and projection matrix with $rank(K) = q$. In these assumptions, we find a James-Stein type estimator which shrinks towards a projection vector. Such a class was introduced by James-Stein (1961) and Lindley (1962) in order to prove that some of its members dominate the natural estimator in the multinormal case. A similar result for the more general case was also derived by Strawderman (1974).

The problem of estimation of a mean under constraint has focussed in the context of curved model in the works of Kariya (1989), Perron and Giri (1989), Merchand and Giri (1993), and Baek and Lee (2005) among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger (1975).

George (1990) suggested that it might be possible to use the improved variance estimator to improve the James-Stein type estimator with some shrinkage points. Kim *et al.* (2002)

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produced a class of estimators dominating the James-Stein type estimator with some shrinkage points and Park and Baek (2014) considered the generalized Bayes estimator dominating same type estimators.

This paper improves the James-Stein type estimator which shrinks towards a projection vector when the underlying distribution is that of a variance mixture of normals. In section 2, we present the general setting of our problem and develop necessary notations. In section 3, we examine the estimation problem based on a James-Stein estimator shrinkage toward some projection vectors when the norm $\|\boldsymbol{\theta} - K\boldsymbol{\theta}\|$ is known. In this case, we give to the class of James-Stein estimators shrinkage toward vector which dominate the usual estimator. The result of Merchand and Giri (1993) and Baek (2000) are special cases of this paper when $rank(K) = 0$ and 1, respectively. Also, we can apply this result to the regression model in case of $rank(K) \geq 2$ in concluding remarks.

2. Preliminaries

Let $\boldsymbol{Y} = (Y_1, \cdots, Y_p)'$, $p - q \geq 3$, be a single sample from a compound multinormal distribution with unknown location parameter $\boldsymbol{\theta}(p \times 1)$ and mixture parameter $H(\bullet)$, where $H(\bullet)$ represents a known c.d.f defined on the positive real number. We can represent

$$\pounds(\boldsymbol{Y}|S = s) = N_p(\boldsymbol{\theta}, s\boldsymbol{I}_p), \ \forall s > 0, \tag{2.1}$$

where S is the positive random variable with c.d.f. $H(\bullet)$.

We estimate the location parameter $\boldsymbol{\theta}$ with loss function.

$$\boldsymbol{L}(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{Y})) = (\boldsymbol{\delta}(\boldsymbol{Y}) - \boldsymbol{\theta})'(\boldsymbol{\delta}(\boldsymbol{Y}) - \boldsymbol{\theta}),$$

with $\boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda = \{\boldsymbol{\theta} \in R^p | \ \|\boldsymbol{\theta} - K\boldsymbol{\theta}\| = \lambda, 0 \leq \lambda < \infty\}$, where K is an idempotent and projection matrix with $rank(K) = q$. Consider the estimator

$$\boldsymbol{\delta}(\boldsymbol{Y}) = K\boldsymbol{Y} + \left(1 - \frac{c}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - K\boldsymbol{Y})}\right)(\boldsymbol{Y} - K\boldsymbol{Y}), \ c \in \boldsymbol{R}.$$

Restated in terms of the family of probability density functions of $\boldsymbol{Y}$, the distributional assumption give by expression (2.1) and the restriction on the location parameter $\boldsymbol{\theta}$ indicate that the p.d.f. of $\boldsymbol{Y}$ is

$$P_\theta(\boldsymbol{y}) = \int_{(0,\infty)} (2\pi s)^{-p/2} \exp\left(\frac{\|\boldsymbol{y} - \boldsymbol{\theta}\|^2}{2s}\right) dH(s), \tag{2.2}$$

$\boldsymbol{y} \in \boldsymbol{R}^p$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda$. It will be also assumed that $E(S) < \infty$, the covariance matrix $\boldsymbol{\Sigma} = Cov(\boldsymbol{Y}) = E(S)\boldsymbol{I}_p$, and the mean vector $E(\boldsymbol{Y}) = \boldsymbol{\theta}$. The risk function of the estimator $\boldsymbol{\delta}$ is

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}) = E_{\boldsymbol{\theta}}[L(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{Y}))] = E_{\boldsymbol{\theta}}[(\boldsymbol{\delta}(\boldsymbol{Y}) - \boldsymbol{\theta})'(\boldsymbol{\delta}(\boldsymbol{Y}) - \boldsymbol{\theta})], \ \boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda.$$

3. The improved class of James-Stein type estimators towards projection vector

In this section, the best estimator is derived within

$$D_K = \left\{ \boldsymbol{\delta}^c : \boldsymbol{R}^p \to \boldsymbol{R}^p | \boldsymbol{\delta^c}(\boldsymbol{Y}) = \boldsymbol{Y} + \left(1 - \frac{c}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - \boldsymbol{KY})}(\boldsymbol{Y} - \boldsymbol{KY})\right), c \in \boldsymbol{R} \right\}$$

where the parameter space is of the form

$$\boldsymbol{\Theta}_\lambda = \{\boldsymbol{\theta} \in \boldsymbol{R}^p | \ \|\boldsymbol{\theta} - K\boldsymbol{\theta}\| = \lambda\}, \ \lambda \geq 0.$$

The following lemmas will prove useful in the evaluation of the risk function of the estimator $\boldsymbol{\delta}^c, \ c \in R$.

Lemma 3.1 Let $\boldsymbol{Y}$ be a random multinormal vector $N_p(\boldsymbol{\theta}, \mathrm{I_p})$, $p \geq q + 3$ and $\boldsymbol{\theta} \in R^p$. Then

$$\text{(i)} \ \ E_{\boldsymbol{\theta}} \left(\frac{1}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - \boldsymbol{KY})} \right) = E^L \left(\frac{1}{p - q + 2L - 2} \right)$$

and

$$\text{(ii)} \ \ E_{\boldsymbol{\theta}} \left(\frac{(\boldsymbol{Y} - \boldsymbol{\theta})'(\boldsymbol{Y} - \boldsymbol{KY})}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - \boldsymbol{KY})} \right) = E^L \left(\frac{p - q - 2}{p - q + 2L - 2} \right)$$

where L is a Poisson random variable with mean $(\boldsymbol{\theta} - K\boldsymbol{\theta})'(\boldsymbol{\theta} - K\boldsymbol{\theta})/2$

Proof: See James and Stein(1961) and use Stein's Identity □

Lemma 3.2 Let $\boldsymbol{Y}$ be a compound multinormal vector with location parameter $\boldsymbol{\theta}$; $p \geq q+3$ and $\boldsymbol{\theta} \in R^p$; and known mixture parameter H(•) with p.d.f. of the form given in (2.2). Then, with $\lambda = \|\boldsymbol{\theta} - K\boldsymbol{\theta}\|$

$$\text{(i)} \ \ E_{\boldsymbol{\theta}} \left(\frac{1}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - K\boldsymbol{Y})} \right) = \int_{(0,\infty)} f_{p-q}(\lambda, s) \frac{dH(s)}{s},$$

$$\text{(ii)} \ \ E_{\boldsymbol{\theta}} \left(\frac{(\boldsymbol{Y} - \boldsymbol{\theta})'(\boldsymbol{Y} - K\boldsymbol{Y})}{(\boldsymbol{Y} - K\boldsymbol{Y})'(\boldsymbol{Y} - K\boldsymbol{Y})} \right) = (p - q - 2) \int_{(0,\infty)} f_{p-q}(\lambda, s) dH(s),$$

where the function $f_{p-q}(\bullet,\bullet) : [0, \infty) \to (0, \infty)$, is defined by the relation

$$f_{p-q}(\lambda, s) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2s}} \left(\frac{\lambda^2}{2s}\right)^j}{j!(p - q - 2j - 2)}$$

Proof: (i) Using both the representation given in (2.1) and part(i) of Lemma 3.1, we obtain

$$
\begin{aligned}
E_{\boldsymbol{\theta}}\left(\frac{1}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right) =& E^S\left\{S^{-1}E^{\boldsymbol{X}|S}\left[\frac{S}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right]\right\} \\
=& \int_{(0,\infty)} s^{-1}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^j}{j!(p-q-2j-2)}dH(s) \\
=& \int_{(0,\infty)} f_{p-q}(\lambda,s)\frac{dH(s)}{s}
\end{aligned}
$$

(ii) Again, combining the representation given in (2.1) and part(ii) of Lemma 3.1, we obtain

$$
\begin{aligned}
E_{\boldsymbol{\theta}}\left(\frac{(\boldsymbol{Y}-\boldsymbol{\theta})'(\boldsymbol{Y}-K\boldsymbol{Y})}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right) =& E^S\left\{E_{\boldsymbol{\theta}}^{\boldsymbol{Y}|S}\left[\frac{\left(\frac{\boldsymbol{Y}-\boldsymbol{\theta}}{\sqrt{S}}\right)'\left(\frac{\boldsymbol{Y}-K\boldsymbol{Y}}{\sqrt{S}}\right)}{\left(\frac{\boldsymbol{Y}-K\boldsymbol{Y}}{\sqrt{S}}\right)'\left(\frac{\boldsymbol{Y}-K\boldsymbol{Y}}{\sqrt{S}}\right)}\right]\right\} \\
=& \int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^j}{j!}\frac{p-q-2}{p-q+2j-2}dH(s) \\
=& (p-q-2)\int_{(0,\infty)} f_{p-q}(\lambda,s)dH(s)
\end{aligned}
$$

□

The main result of this section now follows.

Theorem 3.1 Let $\boldsymbol{Y}$ be a single sample from a p-dimensional location parameter with p.d.f. of the form given by (2.2). Under the assumptions $\boldsymbol{\theta} \in \Theta_\lambda$, $p \geq q+3$ and E[S] $< \infty$, the unique best estimator within the class D_M is given by $\delta^{c^*(\lambda)}$ where

$$
c^*(\lambda) = (p-q-2)\frac{\int_{(0,\infty)} f_{p-q}(\lambda,s)dH(s)}{\int_{(0,\infty)} f_{p-q}(\lambda,z)\frac{dH(s)}{s}} \tag{3.1}
$$

Proof: Under the assumptions above, we can easily derive the result $E_{\boldsymbol{\theta}}(\boldsymbol{Y}'\boldsymbol{Y}) = \boldsymbol{\theta}'\boldsymbol{\theta} + pE(S)$. Combining this with Lemma 3.2, we have

$$
\begin{aligned}
R(\boldsymbol{\theta},\delta^c) =& E_{\boldsymbol{\theta}}[(\delta^c(\boldsymbol{Y})-\boldsymbol{\theta})'(\delta^c(\boldsymbol{Y})-\boldsymbol{\theta})] \\
=& pE(S) + \left[c^2E_{\boldsymbol{\theta}}\left\{\frac{1}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right\} - 2cE_{\boldsymbol{\theta}}\left\{\frac{(\boldsymbol{Y}-\boldsymbol{\theta})'(\boldsymbol{Y}-K\boldsymbol{Y})}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right\}\right] \\
=& pE(S) + \left[c^2\int_{(0,\infty)} f_{p-q}(\lambda,s)\frac{dH(s)}{s} - 2c(p-q-2)\int_{(0,\infty)} f_{p-q}(\lambda,s)dH(s)\right] \\
=& pE(S) + \left[\int_{(0,\infty)}\left\{\frac{c^2}{s} - 2c(p-q-2)\right\} f_{p-q}(\lambda,s)dH(s)\right]
\end{aligned} \tag{3.2}
$$

□

From this last equality, we obtain easily that

$$f_{c\in R} R(\boldsymbol{\theta}, \delta^c) = R(\boldsymbol{\theta}, \delta^{c^*(\lambda)})$$

with $c^*(\lambda)$ given by expression (3.1).

Using expression (3.2), the minimum risk attained by the best James-Stein type estimator is equal to

$$R(\boldsymbol{\theta}, \delta^{c^*(\lambda)}) = pE(S) - (p-q-2)^2 \frac{\left[\int_{(0,\infty)} f_{p-q}(\lambda, s) dH(s)\right]^2}{\int_{(0,\infty)} f_{p-q}(\lambda, s) \frac{dH(s)}{s}}, \ \boldsymbol{\theta} \in \Theta_\lambda.$$

When $\|\boldsymbol{\theta} - K\boldsymbol{\theta}\| = \lambda$, the use of other estimators of the James-Stein class other than $\delta^{c^*(\lambda)}$ will incur risk which is a strictly increasing function of distance $|c - c^*(\lambda)|$. To see this, we can define $t(\lambda)$ such that $c = t(\lambda)c^*(\lambda)$ and, using expression (3.2), express $R(\boldsymbol{\theta}, \delta^c)$ as

$$pE(S) + (p-q-2)^2[t^2(\lambda) - 2t(\lambda)] \frac{\left[\int_{(0,\infty)} f_{p-q}(\lambda, s) dH(s)\right]^2}{\int_{(0,\infty)} f_{p-q}(\lambda, s) \frac{dH(s)}{s}} \tag{3.3}$$

From this we can write

$$R(\boldsymbol{\theta}, \delta^c) - R(\boldsymbol{\theta}, \delta^{c^*(\lambda)}) = |c - c^*(\lambda)|^2 \int_{(0,\infty)} f_{p-q}(\lambda, s) \frac{dH(s)}{s} \tag{3.4}$$

The natural estimator $\delta^o(\boldsymbol{X}) = \boldsymbol{X}$ is a member of the James-Stein class and has a constant risk function equal to $pE(S)$. We can also characterize the estimators of the James-Stein type that dominate the natural estimator δ^o.

Corollary 3.1 Under the conditions of Theorem 3.1, the estimator δ^c will dominate the natural estimator δ^o if and only if $0 < c < 2c^*(\lambda)$.

Proof: Using expression (3.3), one easily sees that, for $\boldsymbol{\theta} \in \Theta_\lambda$,

$$\begin{aligned} R(\boldsymbol{\theta}, \delta^c) &< R(\boldsymbol{\theta}, \delta^o) = pE(S) \\ &\Leftrightarrow t^2(\lambda) - 2t(\lambda) < 0 \\ &\Leftrightarrow 0 < t(\lambda) < 2 \\ &\Leftrightarrow 0 < c < 2c^*(\lambda). \end{aligned}$$

□

4. Examples

The class of compound multinormal distributions is quite large and, in this section, we present some examples of the evaluation of the best James-Stein type estimator for different choices of the underlying distribution of $\boldsymbol{Y}$ or, equivalently, of the mixture parameter H(•)

Example 4.1 For $\boldsymbol{Y} \sim N_p(\boldsymbol{\theta}, \sigma^2 I_p), p \geq q+3$, (i.e., $H(s) = 1_{(\sigma^2,\infty)}(s)$ with $1_A(\bullet)$ being the indicator function of the set A); we deduce from Theorem 3.3 that

$$c^*(\lambda) = (p-q-2)\frac{f_p(\lambda, \delta^2)}{f_p(\lambda, \sigma^2)/\sigma^2} = (p-q-2)\sigma^2,$$

and that the best estimator within the James-Stein class D_M is equal to

$$\delta^{(p-q-2)\sigma^2}(\boldsymbol{Y}) = K\boldsymbol{Y} + \left(1 - \frac{(p-q-2)\sigma^2}{(\boldsymbol{Y}-K\boldsymbol{Y})'(\boldsymbol{Y}-K\boldsymbol{Y})}\right)(\boldsymbol{Y}-K\boldsymbol{Y}),$$

regardless of the value of the norm $\lambda = \|\boldsymbol{\theta} - K\boldsymbol{\theta}\|$

For non-normal cases, the following explicit formula for the quantity $f^*_{p-q}(\gamma) = E^L[(p-q+2L-2)^{-1}], L \sim$ Poisson (γ), given by Egerton and Laycock (1982) prove useful for the evaluation of the function $c^*(\lambda), \lambda \geq 0$.

Lemma 4.1 Let L be a Poisson random variable with mean $\gamma > 0$, and

$$\text{(i)} \quad f^*_{p-q}(\gamma) = e^{-\gamma}\int_{(0,1)} t^{p-q-3}e^{\gamma t^2}dt,$$

and

$$\text{(ii)} \quad f^*_{p-q+2}(\gamma) = (2\gamma)^{-1}[1-(p-q-2)f^*_{p-q}(\gamma)] \tag{4.1}$$

For odd values of the dimension $p-q$, the recurrence formula given by expression (4.1) permits the expression of the function f^*_{p-q} $(\bullet)$ as a function of f^*_3 $(\bullet)$. From part(i) of the preceding lemma,

$$\begin{aligned} f^*_3(\gamma) =& e^{-\gamma}\int_{(0,1)} e^{\gamma t^2}dt \\ =& \gamma^{-\frac{1}{2}}D(\gamma^{\frac{1}{2}}), \end{aligned}$$

where $D(y) = e^{-y^2}\int_{(0,y)} e^{t^2}dt, y > 0$, is known as Dawson's integral which is tabulated in Abramowitz and Stegun (1965). For even values of the dimension $p-q$, the recurrence formula given by expression (4.1) permits the expression of the function f^*_{p-q} ($\bullet$ as a function of f^*_4 $(\bullet)$. From part(i) of Lemma 4.1,

$$\begin{aligned} f^*_4(\gamma) =& e^{-\gamma}\int_{(0,1)} te^{\gamma t^2}dt \\ =& (2\gamma)^{-1}(1-e^{-\gamma}) \end{aligned} \tag{4.2}$$

We now proceed with the evaluation of the best James-Stein estimator in the contaminated multinormal case.

Example 4.2 Setting $H(s) = \sum_{j=1}^n \epsilon_j 1_{[\sigma_j^2,\infty)}(s)$ in expression (2.2), where $0 < \epsilon_j < 1, \sigma_j^2 > 0$ for $j \in (1, \cdots, n)$ and $\sum_{j=1}^n \epsilon_j = 1$, we obtain the family of contaminated multinormal distributions with mean parameter $\boldsymbol{\theta}$ and known dispersion parameters $(\sigma_1^2, \epsilon_1), \cdots, (\sigma_n^2, \epsilon_n)$. The function $c^*(\lambda), \lambda \geq 0$, defined by (3.1) becomes

$$c^*(\lambda) = (p-q-2)\frac{\sum_{j=1}^{n} \epsilon_j f_p(\lambda, \sigma_j^2)}{\sum_{j=1}^{n} \frac{\epsilon_j}{\sigma_j^2} f_p(\lambda, \sigma_j^2)},$$

and the decision rule $\delta^{c^*(\lambda)}$ represents, by Theorem 3.3, the best James-Stein type estimator when $\boldsymbol{\theta} \in \Theta_\lambda$. The quantities $f_{p-q}(\lambda, \sigma_j^2)$ can be evaluated by using the results of Lemma 4.1. In particular, for $p-q=6$, using expressions (4.1) and (4.2), we obtain

$$\begin{aligned} f_6(\lambda, s) =& f_6^*\left(\frac{\lambda^2}{2s}\right) \\ =& \lambda^{-4} s(\lambda^2 - 2s + 2se^{-\lambda^2/2s}), \lambda > 0, s > 0, \end{aligned}$$

and

$$c^*(\lambda) = 4\frac{\sum_{j=1}^{n} \epsilon_j \sigma_j^2 (\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}{\sum_{j=1}^{n} \epsilon_j (\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}$$

Example 4.3 Setting $\mathcal{L}(s^{-1}) = \text{Gamma}(a, b)$, $a > 1$ and $b > 0$, in the representation given by expression (2.1), we obtain the family of multivariate student distributions with mean parameter θ (the condition $a > 1$ guaranteeing the existence of a covariance matrix) and known dispersion parameter (a, b). Here, we extend the usual class of multivariate student location families with n degrees of freedom, where $n = 2a = 2b$ and $n \in (1, 2, \cdots)$, to include other values of the dispersion parameter (a, b). For the particular case where $p - q = 4$, we obtain by expressions (3.1) and (4.2),

$$f_4(\lambda, s) = f_4^*\left(\frac{\lambda^2}{2s}\right) = \lambda^{-2} s(1 - e^{-\lambda^2/2s}), \lambda > 0, s > 0,$$

and

$$\begin{aligned} c^*(\lambda) =& 2\frac{\int_{(0,\infty)} s(1 - e^{-\lambda^2/2s}) dH(s)}{\int_{(0,\infty)} (1 - e^{-\lambda^2/2s}) dH(s)} \\ =& 2\frac{\int_{(0,\infty)} (v^{-1} - v^{-1} e^{-\lambda^2 v/2}) v^{a-1} e^{-bv} dv}{\int_{(0,\infty)} (1 - e^{-\lambda^2 v/2}) v^{a-1} e^{-bv} dv} \\ =& \frac{2b}{a-1} \frac{[1 - \left(\frac{2b}{2b+\lambda^2}\right)^{a-1}]}{[1 - \left(\frac{2b}{2b+\lambda^2}\right)^{a}]}. \end{aligned}$$

5. Concluding remarks

K has several special cases as follows :
Let the $O_{p\times p}$ and J be the $p\times p$ matrices all entries are $0's$ and $1's$, respectively. The estimators in Marchand and Giri (1993) and Baek (2000) are the case of $K = O_{p\times p}$ and $K = (1/p)J$. Another case is $K = T(T'T)^{-1}T'$ when $T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_p \end{pmatrix}$ and $\theta_i = \alpha + \beta t_i$ for known t_i and unknown α and β (Lehmann and Casella, 1999), this is the case of $rank(K) = 2$. More general case would be represented as follows. When

$$T = \begin{pmatrix} 1 & 1 & \cdots 1 \\ t_{11} & t_{12} & \cdots t_{1p} \\ \vdots & & \vdots \\ t_{h1} & t_{h2} & \cdots t_{hp} \end{pmatrix}$$

and $\theta_i = \alpha + \beta_1 t_{1i} + \beta_2 t_{2i} + \cdots + \beta_h t_{hi}$ for known t_{1i}, t_{2i}, $\cdots$, t_{hi} and unknown α, and β_1, β_2, $\cdots$, β_h, such projection matrices $K = T(T'T)^{-1}T'$ are symmetric and idempotent of rank $h+1$. It is left to further research to execute the simulation of these results.

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