Journal of the Korean Data & Information Science Society
2017, **28**(2), 421–433

http://dx.doi.org/10.7465/jkdi.2017.28.2.421
한국데이터정보과학회지

Multivariate CTE for copula distributions

Chong Sun Hong[1] · Jae Young Kim[2]

[12]Department of Statistics, Sungkyunkwan University

Received 27 February 2017, revised 23 March 2017, accepted 24 March 2017

Abstract

The CTE (conditional tail expectation) is a useful risk management measure for a diversified investment portfolio that can be generally estimated by using a transformed univariate distribution. Hong *et al.* (2016) proposed a multivariate CTE based on multivariate quantile vectors, and explored its characteristics for multivariate normal distributions. Since most real financial data is not distributed symmetrically, it is problematic to apply the CTE to normal distributions. In order to obtain a multivariate CTE for various kinds of joint distributions, distribution fitting methods using copula functions are proposed in this work. Among the many copula functions, the Clayton, Frank, and Gumbel functions are considered, and the multivariate CTEs are obtained by using their generator functions and parameters. These CTEs are compared with CTEs obtained using other distribution functions. The characteristics of the multivariate CTEs are discussed, as are the properties of the distribution functions and their corresponding accuracy. Finally, conclusions are derived and presented with illustrative examples.

Keywords: Copula, generator, kurtosis, portfolio, quantile vector, risk, skewness.

1. Introduction

Due to the rapid growth of the international financial market and its growing influence on the financial and foreign exchange crisis, risk management of personal financial assets has become a major and growing topic of interest. For this reason, there has been a great deal of research on risk measures to quantify and compare various risks. J.P. Morgan had proposed the VaR (Value at Risk) metric that can estimate the maximum amount of loss that can occur within a fixed target period and a confidence level for basic market prices (Jorion, 2006). In order to complement the shortcomings of the VaR, such as lack of sub-additivity, new measurements have been suggested, including the CTE (Conditional Tail Expectation), CVaR (Conditional Value at Risk), ES (Expected Shortfall), and others (Artzner *et al.*, 1999; Rockafellar and Uryasev, 2000; Andersson *et al.*, 2001; Acerbi and Tasche, 2002; Rockafellar and Uryasev, 2002; Park and Baek, 2014; Ko and Son, 2015).

Many financial companies currently measure the risk of several portfolios rather than a particular one. Studies on these risk measures, such as VaR, CTE, CVaR, and ES have been

[1] Corresponding author : Professor, Department of Statistics, Sungkyunkwan University, Seoul 03063 Korea. E-mail: cshong@skku.edu
[2] Graduate student, Department of Statistics, Sungkyunkwan University, Seoul 03063 Korea.

extended to a multidimensional distribution. Most early research used a transformed one-dimensional distribution with the normal distribution assumption. Even though this method is intuitive, since the risks followed a multivariate distribution that can be represented by one value, each risk cannot be understood. Additionally, various studies have shown that the assumption of a normal distribution is not appropriate for real financial market data. The distribution of the return on US stocks has been reported to have thicker tails (Zangari, 1996), and the distribution of exchange transactions in the foreign exchange market was found to follow non-normality (Li, 1999).

For the above problems, Cai and Li (2005) derived an equation by applying the CTE to the multivariate phase type distribution. Hong and Kwon (2010) measured the risk by applying alternative distributions, such as the skewed t-distribution, the Laplace distribution, and the mixture distribution to gauge the real loss-rate. In addition, Hong and Lee (2011a, 2011b) used copula functions to estimate the VaR of the bivariate and multivariate return rate data. Cousin and Bernardino (2013) extended the definitions of Embrechts and Puccetti (2006) and suggested a multivariate lower-orthant VaR and upper-orthant VaR by using the cumulative distribution function and constructed a survival function of each variable simultaneously. Then, they explored the function properties and relation equations. Cousin and Bernardino (2014) extended the above studies to the CTE. Hong *et al.* (2016) proposed the multivariate quantile vector for multi-dimensional data. Hong and Kim (2016) defined the multivariate CTE vector for the multivariate normal distribution using the quantile vector. This method proposes that each variable's CTE can be calculated by using a multivariate CTE vector and and this CTE vector is estimated with the less risk value.

In this study, we propose a multivariate CTE vector in multivariate loss-rate data by using copula distributions. Copula distributions are a probabilistic tool that can easily show how several random variables are connected. In other words, these distributions provide an efficient method to explore the structure of multivariate distributions by mathematically modeling a complex dependency structure among several random variables. Therefore, for the risk measure concerning the tail of the distribution, it is reasonable to use copula functions rather than the correlation coefficient (Embrechts *et al.*, 1999; Nelson, 2006).

For the data that does not follow the normal distribution, with large skewness and kurtosis values and with strong correlation in the tails, multivariate CTE vectors are obtained by copula distributions, and then these vectors are compared with estimates offered by normal distributions, allowing for the discussion of their differences.

In Section 2, the characteristics of the Clayton, Gumbel, and Frank copulas used in this study are introduced and the parameter estimation methods for bivariate and multivariate distributions are also summarized. The multivariate quantile vectors of each copula function are obtained, and the CTE vectors are calculated by applying the copula distributions in various situations and their properties are discussed in Section 3. With the bivariate and trivariate illustrative data, CTE vectors using the copula distribution are calculated and analyzed in Section 4. Finally, conclusions are derived and discussed in Section 5.

2. Copula distribution

The copula function $C(\cdot, \dots, \cdot)$ is a joint distribution function of random variables following a uniform distribution.

$$C(u_1, \dots, u_n) = P(U_1 \le u_1, \dots, U_n \le u_n),$$

where each U_i follows a uniform distribution $U(0, 1)$. For any random variables $X_1, \dots, X_n$ with a joint cumulative distribution function $F(x_1, \dots, x_n)$ and a marginal cumulative distribution function, $F_1(x_1), \dots, F_n(x_n)$, there exists a copula function such that

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

If each $F_i(x)$ is continuous, the copula function is unique. Sklar's Theorem (Sklar, 1959) allows us to separate the modeling of the marginal distributions from the dependence structure. In other words, the copula function contains all the information on the dependence structure between the components of variables and links the marginal distributions together.

Among the several copula functions, Elliptical and Archimedean copulas are the most commonly used. The elliptical copula is represented with an elliptical distribution such as Gaussian and Student's t copulas. The Archimedean copula is used to express various dependencies according to its generator functions. The Clayton, Gumbel, and Frank copulas belong to the Archimedean copula. In this study, Clayton, Gumbel and Frank copulas, in Archimedean copula family, will be explored.

To estimate bivariate copula parameters, complex processes are generally required. However, bivariate Archimedean copula parameters can easily be obtained by using the calibration method with the generator function of the Archimedean copula and Kendall's τ. Kendall's τ is nonparametric and allows for dependency decision measures and is also known as an alternative correlation coefficient for a non-normal case. In the bivariate Archimedean copula, the equation $\tau = 1+4\int_0^1 \varphi(t)/\varphi'(t)dt$ expresses the relationship between Kendall's τ and the generation function $\varphi(\cdot)$ (Nelsen, 2006). The copula functions, generator functions, and parameter estimation equations are summarized in Table 2.1.

Table 2.1 Function and kendall's τ for copula distributions

	$C[u, v]$	generator $\varphi(t)$	Kendall's τ & parameter
Clayton	$[max\{u^{-\theta} + v^{-\theta} - 1; 0\}]^{-\frac{1}{\theta}}$ $\theta \in [-1, \infty)\backslash\{0\}$	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\tau = \frac{\theta}{\theta+2}$
Gumbel	$exp[-((-log(u))^{\theta} + (-log(v))^{\theta})^{\frac{1}{\theta}}]$ $\theta \in [1, \infty)$	$(-log(t))^{\theta}$	$\tau = \frac{\theta-1}{\theta}$
Frank	$-\frac{1}{\theta}log[1 + \frac{(exp(-\theta u)-1)(exp(-\theta v)-1)}{exp(-\theta)-1}]$ $\theta \in \mathbb{R}\backslash\{0\}$	$-log(\frac{exp(-\theta t)-1}{exp(-\theta)-1})$	$\tau = 1 + \frac{4(D_1(\theta)^{*}-1)}{\theta}$

$*D_1(\theta) = \frac{1}{\theta}\int_0^{\theta} \frac{t}{exp(t)-1}dt$

Assuming that the two marginal distributions are standard normal, the contour plots of the bivariate Clayton, Gumbel, and Frank copula are shown in Figure 2.1 to elucidate the characteristics of the corresponding copula functions. For a given parameter, the Clayton and Gumbel copulas show a strong relation on the lower-left tail and the upper-right tail, respectively. The Frank copula is used to simultaneously explain the dependence of left and right tail. As values of the copula parameter increase, the dependency of the tail of the Clayton and Gumbel copulas increase. When the absolute value of Frank copula parameter increases, the relative contributions of the left and right tails grow.

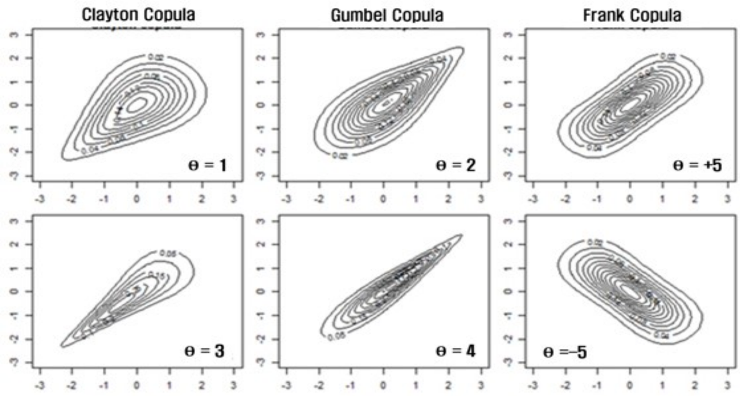

Figure 2.1 Contour plots for copula distributions

For the cases that are more than bivariate Archimedean copula the Exchangeable Archimedean Copula (EAC) can be used. The EAC can represent n-dimensional distributions with one parameter and generation function and is expressed as follows:

$$C(u_1, u_2, \ldots, u_n) = \varphi^{-1}[\varphi(u_1) + \varphi(u_1) + \cdots + \varphi(u_n)].$$

The EAC can represent a dependent structure with one parameter in a multivariate distribution, but it has limitations with regard to expressing various dependent structures as the number of the dimensions increases. Nonetheless, the EAC is widely used because it can be used to simply and easily perform calculations. Since the calibration method cannot be used to estimate more than bivariate Archimedean copula parameters, the Maximum Pseudo-Likelihood (MPL) estimation is utilized in this study. The MPL is estimated by using pseudo data rather than the empirical distribution that is employed in the Canonical Maximum Likelihood (CML) method (Shih and Louis, 1995; Genest *et al.*, 1995).

3. Multivaraite CTE for copula distribution

The Clayton, Gumbel, and Frank Copulas mentioned in Section 2 are applied to obtain the multivariate quantile vectors of each copula functions in this Section. And then the multivariate $\underline{CTE_\alpha}$ are calculated by using the quantile vectors and their properties are explored. In Section 3.1, based on various generating data from the bivariate copula functions, the multivariate quantile vectors are obtained and compared with those of the bivariate normal distribution functions. The bivariate $\underline{CTE_\alpha}$ for the bivariate copula functions are calculated. In Section 3.2, the trivariate quantile vectors and $\underline{CTE_\alpha}$ for the trivariate copula functions are obtained and discussed their properties. Then we make a comparison of the $\underline{CTE_\alpha}$s for both the trivariate copula functions and normal distribution functions.

3.1. Bivariate quantile vector and CTE for the copula distribution

Hong *et al.* (2016) proposed a multivariate quantile vector $\underline{z_\alpha} = (z_{1\alpha}, \ldots, z_{k\alpha})^{\mathrm{T}}$ to estimate the VaR in a multi-dimensional distribution. Taking two marginal distributions are standard normal, the bivariate 0.95 quartile vectors of the three kinds of copula distributions and the standard normal distribution with the correlation coefficient ρ are shown in Figure 3.1.

Regardless of the type of copula function, as the value of the copula parameters increase, it is found that the bivariate 0.95 quantile vectors move to the right and upper space, and these shapes are folded by 45 degrees.

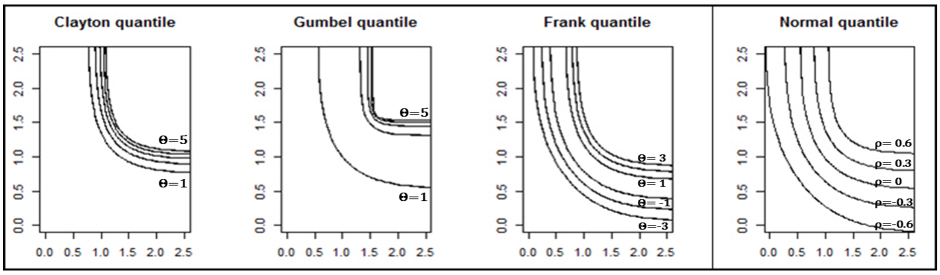

Figure 3.1 0.95 quantile vector for Copula distribution

For various parameter values of the Clayton copula, the shape change of the bivariate 0.95 quantile vectors are the smallest and these quantile vectors are highly similar. With the Gumbel copula, it is shown that the variation of the bivariate 0.95 quantile vector is the most severe, and the shape of the bivariate 0.95 quantile vector changes from a parabola to a right angle as the parameter increases. It is found that the bivariate 0.95 quantile vectors of the Frank copula are located in different regions of parameter space, with various parameter values, and have similar shapes with those of the standard normal distribution.

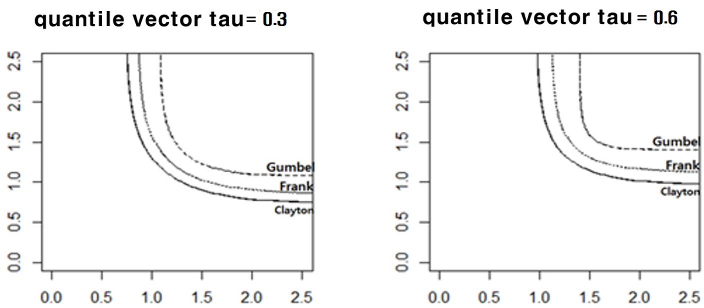

Figure 3.2 0.95 quantile vectors for Copula distributions with Kendall's τ

With fixed Kendall's τ values of 0.3 and 0.6, the bivariate 0.95 quantile vectors corresponding to the three kinds of copula function are represented in Figure 3.2 and their characteristics are discussed. Regardless of the values of Kendall's τ, the bivariate 0.95 quartile vectors appear in the order of Clayton, Frank, and Gumbel copula from the outside, and it can be shown that the distances between the Frank and Gumbel copula are larger than the distances between the Clayton and Frank copula. Additionally, these differences become larger as Kendall's τ value increases. This means that when an inappropriate copula distribution function is applied, even if we know the dependency information of two variables, the actual bivariate 0.95 quantile vector has different results and this difference gets larger as the dependency increases, so that it is important to select an appropriate copula distribution and use the bivariate 0.95 quartile vector.

Hong and Kim (2016) proposed a vector $\underline{CTE_\alpha}$ that estimates the CTE using a multivariate Vector at Risk, $\underline{VaR_\alpha} = \underline{\mu} + \sqrt{diag(\Sigma)}\underline{z_\alpha}$, where $\underline{z_\alpha}$ is a multivariate quantile vector. Because of the difficulty of exactly estimating the correspondence to α for the univariate CTE, it follows that for α^+ and α^- ($\alpha^- < \alpha < \alpha^+$), both $\underline{CTE_\alpha^+} = (CTE_{1,\alpha}^+, \ldots, CTE_{k,\alpha}^+)^T$ and $\underline{CTE_\alpha^-} = (CTE_{1,\alpha}^-, \ldots, CTE_{k,\alpha}^-)^T$ were respectively proposed. Further, the variation of $\underline{CTE_\alpha}$ was explored using various correlation coefficient values in the bivariate standard normal distribution. In this study, assuming that the marginal distribution is a standard normal distribution, the effects of the copula $\underline{CTE_\alpha}$ are investigated with various parameter values of Clayton, Gumbel, and Frank copulas. $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ are calculated and summarized in Table 3.1 when $\alpha = 0.05$ ($\alpha^+ = 0.055, \alpha^- = 0.045$), $\alpha = 0.1$ ($\alpha^+ = 0.105, \alpha^- = 0.095$) and $\alpha = 0.01$ ($\alpha^+ = 0.015, \alpha^- = 0.005$). The $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ of each copula function are shown in Figure 3.3 in order to explore their characteristics only when $\alpha = 0.05$.

Table 3.1 $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$

	Copula	parameter	λ	$\underline{CTE_\alpha^+}$	$\underline{CTE_\alpha^-}$	$E(\underline{VaR_\alpha})$
$\alpha = 0.1$	Clayton	1	0.0954	(1.3301, 1.3301)	(1.3624, 1.3624)	(1.0237, 1.0237)
		2	0.0956	(1.4125, 1.4125)	(1.4451, 1.4451)	(1.1013, 1.1013)
		3	0.0953	(1.4665, 1.4665)	(1.4997, 1.4997)	(1.1517, 1.1517)
		4	0.0952	(1.5053, 1.5053)	(1.5390, 1.5390)	(1.1857, 1.1857)
		5	0.0955	(1.5350, 1.5350)	(1.5693, 1.5693)	(1.2108, 1.2108)
	Gumbel	1	0.0946	(1.1884, 1.1884)	(1.2215, 1.2215)	(0.8715, 0.8715)
		2	0.0951	(1.6428, 1.6428)	(1.6902, 1.6902)	(1.1920, 1.1920)
		3	0.0948	(1.6961, 1.6961)	(1.7435, 1.7435)	(1.2429, 1.2429)
		4	0.0951	(1.7122, 1.7122)	(1.7597, 1.7597)	(1.2603, 1.2603)
		5	0.0945	(1.7192, 1.7192)	(1.7662, 1.7662)	(1.2682, 1.2682)
	Frank	-3	0.0948	(0.8731, 0.8731)	(0.8996, 0.8996)	(0.6210, 0.6210)
		-2	0.0952	(0.9719, 0.9719)	(1.0007, 1.0007)	(0.6981, 0.6981)
		-1	0.0946	(1.0741, 1.0741)	(1.1048, 1.1048)	(0.7800, 0.7800)
		1	0.0952	(1.2611, 1.2611)	(1.2943, 1.2943)	(0.9453, 0.9453)
		2	0.0955	(1.3360, 1.3360)	(1.3700, 1.3700)	(1.0141, 1.0141)
		3	0.0956	(1.3970, 1.3970)	(1.4313, 1.4313)	(1.0726, 1.0726)
$\alpha = 0.05$	Clayton	1	0.1796	(1.5263, 1.5263)	(1.5843, 1.5843)	(1.2735, 1.2735)
		2	0.1868	(1.6115, 1.6115)	(1.6696, 1.6696)	(1.3588, 1.3588)
		3	0.1845	(1.6696, 1.6696)	(1.7244, 1.7244)	(1.4129, 1.4129)
		4	0.1769	(1.7106, 1.7106)	(1.7655, 1.7655)	(1.4548, 1.4548)
		5	0.1758	(1.7414, 1.7414)	(1.7966, 1.7966)	(1.4825, 1.4825)
	Gumbel	1	0.1853	(1.3941, 1.3941)	(1.4552, 1.4552)	(1.1254, 1.1254)
		2	0.1807	(1.9394, 1.9394)	(2.0238, 2.0238)	(1.5561, 1.5561)
		3	0.1873	(1.9882, 1.9882)	(2.0752, 2.0752)	(1.6106, 1.6106)
		4	0.1819	(2.0070, 2.0070)	(2.0909, 2.0909)	(1.6298, 1.6298)
		5	0.1785	(2.0116, 2.0116)	(2.0937, 2.0937)	(1.6337, 1.6337)
	Frank	-3	0.1818	(1.0406, 1.0406)	(1.0909, 1.0909)	(0.8141, 0.8141)
		-2	0.1821	(1.1529, 1.1529)	(1.2067, 1.2067)	(0.9112, 0.9112)
		-1	0.1812	(1.2657, 1.2657)	(1.3216, 1.3216)	(1.0132, 1.0132)
		1	0.1821	(1.4639, 1.4639)	(1.5220, 1.5220)	(1.2031, 1.2031)
		2	0.1818	(1.5412, 1.5412)	(1.5994, 1.5994)	(1.2797, 1.2797)
		3	0.1821	(1.6037, 1.6037)	(1.6619, 1.6619)	(1.3420, 1.3420)
$\alpha = 0.01$	Clayton	1	0.6668	(1.8633, 1.8633)	(2.1075, 2.1075)	(1.7412, 1.7412)
		2	0.6660	(1.9444, 1.9444)	(2.1843, 2.1843)	(1.8241, 1.8241)
		3	0.6659	(1.9997, 1.9997)	(2.2375, 2.2375)	(1.8805, 1.8805)
		4	0.6658	(2.0410, 2.0410)	(2.2774, 2.2774)	(1.9224, 1.9224)
		5	0.6665	(2.0738, 2.0738)	(2.3097, 2.3097)	(1.9557, 1.9557)
	Gumbel	1	0.6665	(1.7585, 1.7585)	(2.0381, 2.0381)	(1.6187, 1.6187)
		2	0.6650	(2.4457, 2.4457)	(2.8112, 2.8112)	(2.2616, 2.2616)
		3	0.6649	(2.4910, 2.4910)	(2.8525, 2.8525)	(2.3088, 2.3088)
		4	0.6624	(2.5048, 2.5048)	(2.8629, 2.8629)	(2.3223, 2.3223)
		5	0.6659	(2.5108, 2.5108)	(2.8716, 2.8716)	(2.3297, 2.3297)
	Frank	-3	0.6668	(1.3557, 1.3557)	(1.6064, 1.6064)	(1.2304, 1.2304)
		-2	0.6661	(1.4845, 1.4845)	(1.7392, 1.7392)	(1.3568, 1.3568)
		-1	0.6669	(1.6064, 1.6064)	(1.8620, 1.8620)	(1.4788, 1.4788)
		1	0.6671	(1.8086, 1.8086)	(2.0585, 2.0585)	(1.6838, 1.6838)
		2	0.6663	(1.8848, 1.8848)	(2.1309, 2.1309)	(1.7616, 1.7616)
		3	0.6666	(1.9463, 1.9463)	(2.1900, 2.1900)	(1.8245, 1.8245)

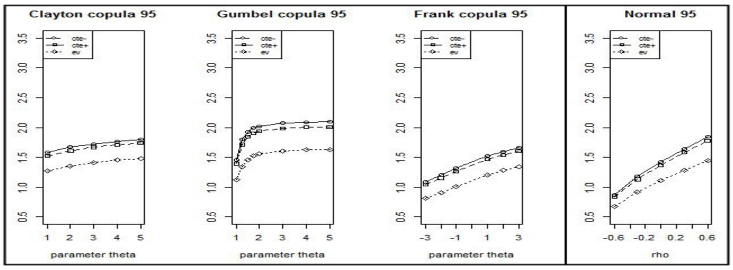

Figure 3.3 $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ when $\alpha = 0.05$

As the parameter increases or α decreases in Table 3.1 and Figure 3.3, it is found that the coordinates of $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ move to the right and upward and that $E(\underline{VaR_\alpha}) < \underline{CTE_\alpha^+} < \underline{CTE_\alpha^-}$, regardless of the type of copula function. As the parameter increases, we can see that the rates of increase for $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ shrink, and these rates are independent of the risk value. Additionally, it can be seen that the averages of the $\underline{CTE_\alpha^+}$ and $\underline{CTE_\alpha^-}$ of the copula function are greater than those of the standard normal distribution.

We can see that as the parameter for the Clayton copula increases, the increase rates of $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ decrease. The $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ for the Gumbel copula increase rapidly when the parameter is between 1 and 2, and increase slowly when the parameter is more than 2. The $\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$, and $E(\underline{VaR_\alpha})$ for the Frank copula increase gradually as the parameter increases, in a manner similar to the Clayton copula, but with a rate of increase slightly greater than that of the Clayton copula.

Next, consider the difference between the $\underline{CTE_\alpha^+}$ and $\underline{CTE_\alpha^-}$. The Clayton and Frank copulas have a constant difference between $\underline{CTE_\alpha^+}$ and $\underline{CTE_\alpha^-}$ (the difference is approximately 0.05 when $\alpha = 0.05$). The differences of Gumbel copula are irregular and increase as α decreases (a risk increase). This behavior can be explained by the relation of Hong and Kim (2016, Theorem 2.1). As $\lambda((\alpha^+ - \alpha^+)/\alpha^+ = 1 - \alpha^-/\alpha^+)$ becomes larger, $\underline{CTE_\alpha^+}$ moves to $E(\underline{VaR_\alpha})$, and as λ becomes smaller, $\underline{CTE_\alpha^+}$ moves to the $\underline{CTE_\alpha^-}$.

To observe the copula $\underline{CTE_\alpha}$ according to the change in the variance, it is assumed that one distribution follows the standard normal and the other follows $N(0, \sigma^2)$. Figure 3.4 shows $\underline{CTE_\alpha^+}$ with various σ and parameter values. When $\sigma = 0.5$ and 1.0, $\underline{CTE_\alpha^+}$ changes with the variance ratio of the marginal distribution as the value of the parameter increases. When $\sigma = 2$, $\underline{CTE_\alpha^+}$ for Clayton and Frank copulas have similar values with those of the Gumbel copula, but the $\underline{CTE_\alpha^+}$ of the Gumbel copula increases more rapidly than the rate of increase of the variance ratio as the value of the parameter increases.

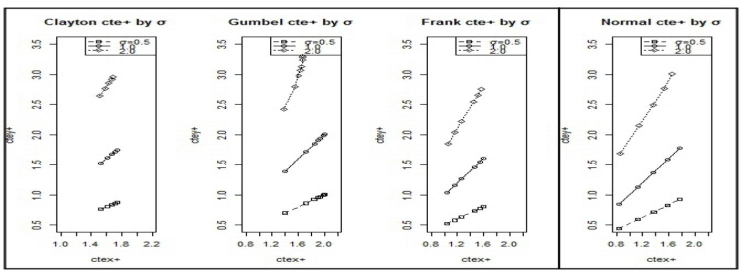

Figure 3.4 $\underline{CTE_\alpha^+}$ with various σ

3.2. Trivariate CTE for the copula distribution

To estimate the CTE of the trivariate copula distribution, each marginal distribution of the three varieties is assumed to be the standard normal. The responses of the copula $\underline{CTE_\alpha}$ to changing the values of the three copula parameter and the risk are obtained and summarized in Table 3.2. For $\alpha = 0.05$, $\underline{CTE^+_{x0.05}}$, $\underline{CTE^-_{x0.05}}$, and $E(\underline{VaR_{x0.05}})$ are drawn in Figure 3.5 and there were compared with the additional $\underline{CTE_\alpha}$ of the trivariate normal distribution to find characteristics of the copulas.

Table 3.2 $\underline{CTE^+_\alpha}$, $\underline{CTE^-_\alpha}$, and $E(\underline{VaR_\alpha})$ with trivariate copula

	parameter	$a = 0.1$			$a = 0.05$			$a = 0.01$		
		$\underline{CTE^+_\alpha}$	$\underline{CTE^-_\alpha}$	$E(\underline{VaR_\alpha})$	$\underline{CTE^+_\alpha}$	$\underline{CTE^-_\alpha}$	$E(\underline{VaR_\alpha})$	$\underline{CTE^+_\alpha}$	$\underline{CTE^-_\alpha}$	$E(\underline{VaR_\alpha})$
Clayton	1	1.7090	1.7255	1.5513	1.8211	1.8545	1.6722	2.0271	2.1858	1.9476
	2	1.8485	1.8318	1.6646	1.9427	1.9758	1.7879	2.1467	2.2992	2.0699
	3	1.9106	1.9303	1.7283	2.0215	2.0566	1.8703	2.2223	2.3711	2.1462
	4	1.9694	1.9850	1.8173	2.0801	2.1125	1.9350	2.2805	2.4301	2.2054
	5	2.0151	2.0321	1.8500	2.1261	2.1576	1.9844	2.3198	2.4657	2.2446
Gumble	1	1.5452	1.5701	1.3124	1.7047	1.7555	1.4732	2.0501	2.3727	1.8834
	2	2.5565	2.5897	2.2391	2.7731	2.8323	2.4897	3.1401	3.4114	3.0064
	3	2.6080	2.6458	2.2821	2.8204	2.8835	2.5449	3.1951	3.4564	3.0601
	4	2.6323	2.6618	2.3207	2.8371	2.8976	2.5522	3.2009	3.4559	3.0593
	5	2.6269	2.6650	2.3162	2.8414	2.9074	2.5623	3.2271	3.4780	3.0852
Frank	1	1.6356	1.6531	1.4632	1.7539	1.7886	1.5946	1.9681	2.1325	1.8870
	2	1.7579	1.7778	1.5779	1.8752	1.9082	1.7222	2.0800	2.2402	2.0009
	3	1.8477	1.8656	1.6754	1.9633	1.9951	1.8101	2.1686	2.3228	2.0908
	4	1.9510	1.9337	1.7380	2.0278	2.0632	1.8732	2.2353	2.3867	2.1574
	5	1.9681	1.9886	1.7869	2.0842	2.1174	1.9358	2.2868	2.4352	2.2090

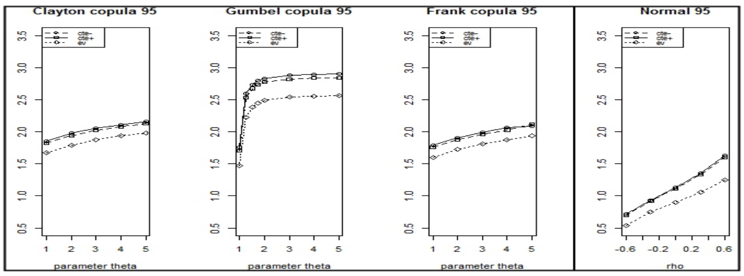

Figure 3.5 $\underline{CTE^+_\alpha}$, $\underline{CTE^-_\alpha}$, $E(\underline{VaR_\alpha})$ with trivariate copula when $\alpha = 0.05$

With the similar results obtained in Section 3.1, $\underline{CTE^+_\alpha}$, $\underline{CTE^-_\alpha}$, and $E(\underline{VaR_\alpha})$ move to the right and upper space as values of the parameter and risk increase. Additionally, the $\underline{CTE^+_\alpha}$ and $\underline{CTE^-_\alpha}$ of the trivariate normal distribution are much less than those of the trivariate copula, and the differences are larger than those of the bivariate case. As number of components of the portfolio increases, the difference between both $\underline{CTE_\alpha}$ values using the normal and copula distributions become larger and the expected values of the risk loss-rates using copula functions are larger than that of the existing method.

The characteristics of each copula function are as follows. The $\underline{CTE^+_\alpha}$, $\underline{CTE^-_\alpha}$, and $E(\underline{VaR_\alpha})$ of the trivariate Clayton copula increase gradually as the value of the parameter increases, as in the bivariate case. Those of the trivariate Gumbel copula increase rapidly when the parameter values are between 1 and 2, and the growth rate is slow in the remaining intervals. However, since the parameter estimates for the trivariate Frank copula are positive, it can be found that the rate of increase is similar to that of the Clayton copula, unlike the case of the bivariate Frank copula.

It can be extended to more than three variate copula distribution functions by using similar methods in Section 3.1 and 3.2. For the bivariate copula distribution functions, the quantile vector appears as a curved line in the two-dimensional plane. And for the trivariate copula distribution functions, the quantile vector is shown as a curved surface in three-dimensional space. However, for more than three variate copula distribution functions, the corresponding quantile vectors cannot be represented with geometrical methods, but more than three dimensional quantile vectors and $\underline{CTE}_\alpha$s could be calculated with ease.

4. Examples

4.1. Bivariate example

Some illustrative examples are considered to compare the $\underline{CTE}_\alpha$ of Hong and Kim (2016) and the copula $\underline{CTE}_\alpha$ proposed in this study for the bivariate and trivariate normal distribution. The first data consists of 246 daily log loss-rates of stocks in Samsung Engineering Co., Ltd. (SE) and Hyundai Engineering & Construction Co., Ltd. (HDEC) from July 13, 2015, to July 12, 2016. The $\underline{CTE}_\alpha^+$ and $\underline{CTE}_\alpha^-$ from this data are calculated when the risk value is 90% ($\alpha^+ = 0.105, \alpha^- = 0.095$), 95% ($\alpha^+ = 0.055, \alpha^- = 0.045$) and 99% ($\alpha^+ = 0.015, \alpha^- = 0.005$).

The basic statistics of the two companies are shown in Table 4.1. The normality test of the joint distribution tells us that the p-values of skewness and kurtosis for the Mardia test are less than 0.0001, and the p-value of Royston test is less than 0.0001, so that it can be concluded that the joint distribution of the two firms does not follow a normal distribution.

Table 4.1 Basic statstics

	SE	*HDEC*
Mean	0.4472	0.0974
Standard deviation	4.5908	2.7623
Skewness	0.6581	0.0687
Kurtosis	5.8105	2.0046

Table 4.2 Log-likelihood, AIC and BIC

	Clayton	Gumbel	Frank
Log-likelihood	25.96	30.48	32.77
AIC	-49.91	-58.95	-63.54
BIC	-46.41	-55.45	-60.04

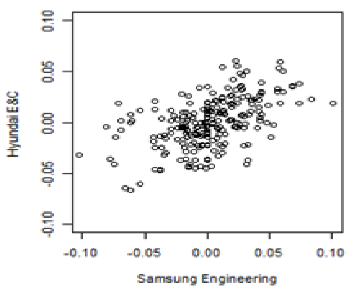

Figure 4.1 Scatter Plot

Based on the scatter plot in Figure 4.1, it is appropriate to choose the Frank copula since the loss- rate of the stocks is not dependent on the specific tail direction. The likelihood, AIC and BIC values of Table 4.2 also show that the Frank copula has a better goodness of fit than those of the Clayton and Gumbel copula. The Frank copula parameter is estimated as 3.363 using the parameter estimation equation, and is then used to compare the copula $\underline{CTE}_\alpha$ and the normal $\underline{CTE}_\alpha$.

The loss-rate range of SE with a 95% risk is the expected value of 3.07% to 3.17%, and that of HDEC ranges from 2.79% to 2.89%, based on the copula $\underline{CTE}_\alpha$. On the other hand, based on the normal $\underline{CTE}_\alpha$, the loss ratios of the SE and HDEC are expected to range from

2.72% to 2.85% and 2.58% to 2.69%, respectively. These results and the additional results with the risk of 90% and 99% are summarized in Table 4.3. Furthermore, the results of Table 4.3 are shown in Figure 4.2, where the rectangle is the visualization of the range of $\underline{CTE_\alpha}$=($\underline{CTE_\alpha^+}$, $\underline{CTE_\alpha^-}$) is obtained and presented in Table 4.3.

Table 4.3 Comparison of loss rate between Normal and Copula

	Normal 89.5%	Copula 89.5%	Normal 90.5%	Copula 90.5%
SE	2.24%	2.68%	2.32%	2.74%
HDEC	2.16%	2.39%	2.23%	2.46%
	Normal 94.5%	Copula 94.5%	Normal 95.5%	Copula 95.5%
SE	2.72%	3.04%	2.85%	3.13%
HDEC	2.58%	2.79%	2.69%	2.89%
	Normal 98.5%	Copula 98.5%	Normal 99.5%	Copula 99.5%
SE	3.32%	3.49%	3.61%	3.70%
HDEC	3.16%	3.31%	3.47%	3.57%

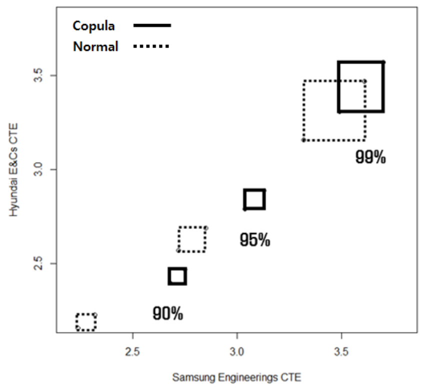

Figure 4.2 Comparison of loss rate between Normal and Copula $\underline{CTE_\alpha}$

Explorations verify that the copula $\underline{CTE_\alpha}$ has a larger value than that of the normal distribution. The $\underline{CTE_\alpha}$ of Hong and Kim (2016) give underestimations because the expectation of the risk depends only on the variance and correlation. Therefore, it can be seen that it is dangerous to aggressively invest with the existing $\underline{CTE_\alpha}$ information. In addition, when the risks are the same, the copula $\underline{CTE_\alpha}$ provides a smaller range of the risk than the normal $\underline{CTE_\alpha}$ (the size of the rectangle is smaller).

With some other illustrative datasets which are not fitted to the multivariate normal distribution, it is found that the copula $\underline{CTE_\alpha}$ is larger than the normal $\underline{CTE_\alpha}$, and the range of the copula $\underline{CTE_\alpha}$ is also small. Therefore, one can conclude that this copula $\underline{CTE_\alpha}$ is not only superior to the normal value, but also accurate.

4.2. Trivariate example

The second dataset consists of 246 daily log loss-rates of the prices of S-Oil Corporation (S-OIL), LG Chem, Ltd. (LGChem) and NH Investment & Securities Co., Ltd (NHIS) from

July 13, 2015, to July 12, 2016. Both the normal and copula $\underline{CTE_\alpha}$ from this dataset are obtained when the risk is 90, 95, and 99%.

As a result of the Mardia test (skewness p-value=0.0843, kurtosis p-value<0.0001), Henze-Zirkler test (p-value<0.0001), and Royston test (p-value<0.0001), it can be said that this data does not follow a normal distribution.

The parametric bootstrap goodness-of-fit test is used to select the best fitted function among the three copula functions. As shown in Table 4.4, the Frank copula showed a better fit than the Clayton and Gumbel copula, and the optimum Frank copula parameter is 2.94. The $\overline{CTE_\alpha^+}$ and $\underline{CTE_\alpha^-}$ of the three companies according to the risk value are summarized in Table 4.5.

Table 4.4 Parametric bootstrap goodness-of-fit test

	statistics	p-value
Clayton Copula	0.11843	0.0005
Gumbel Copula	0.17984	0.0005
Frank Copula	0.04062	0.1304

The expected value of the loss ratio from applying the Frank copula is estimated to be from 2.59% to 2.68% for S-OIL, 2.44% to 2.52% for LGChem, and 2.57% to 2.65% for NHIS when the risk value is 95%. However, as a result of using the multivariate normal distribution, the expected value of the loss ratio is 1.93% to 2.01% in the case of S-OIL, 1.96% to 2.04% in the case of LGChem, and 1.85% to 1.92% in NHIS's case. We can find that the copula $\underline{CTE_\alpha}$ is larger than that of the normal distribution, and the difference has an increasing trend as the risk increases, as in Section 4.1. Therefore, when a portfolio consisting of the stocks of the three companies is considered, it might be dangerous to invest actively with existing $\underline{CTE_\alpha}$ risk information and it would be better to use the copula $\underline{CTE_\alpha}$ proposed in this work to avoid an imprudent financial loss.

Table 4.5 Comparison of loss rate between Normal and Copula

	Normal 89.5%	Copula 89.5%	Normal 90.5%	Copula 90.5%
S-OIL	1.66%	2.25%	1.71%	2.31%
LGChem	1.68%	2.13%	1.73%	2.18%
NHIS	1.59%	2.27%	1.63%	2.32%
	Normal 94.5%	Copula 94.5%	Normal 95.5%	Copula 95.5%
S-OIL	1.93%	2.59%	2.01%	2.68%
LGChem	1.96%	2.44%	2.04%	2.52%
NHIS	1.85%	2.57%	1.92%	2.65%
	Normal 98.5%	Copula 98.5%	Normal 99.5%	Copula 99.5%
S-OIL	2.39%	3.06%	2.69%	3.34%
LGChem	2.42%	2.90%	2.71%	3.19%
NHIS	2.29%	3.01%	2.58%	3.28%

5. Conclusion

One of the important aspects for the accurate estimation of the CTE is solving the problem associated with the fitting of the loss-rate joint distribution. For real market data, it is problematic to assign the loss-rate joint distribution to a normal distribution because of the data's thick tail and asymmetry. Therefore, in this study, we extended the study on multivariate CTEs by using the copula function to understand the dependency structure of random variables and create a joint distribution function with the properties of a non-normal distribution. We applied the Clayton, Gumbel, and Frank copula to examine the characteristics of multivariate binomial and multivariate CTEs. By applying the Clayton, Gumbel, and Frank copula functions to the multivariate quantile vector and CTE, we analyzed their characteristics and estimated the CTE by fitting the copula to the loss-rate distribution of the actual stock market data. In addition, we examined the features and advantages of a new multivariate CTE fitting the copula function through a comparison with the existing multivariate CTEs using the normal distribution.

The multivariate CTE that applies to the normal distribution explores the dependency structure of the loss-rate joint distribution in a manner that depends only on the variance and correlation coefficient. However, in this study, we used three copula distributions, parameters, and the marginal distribution of each variable. Therefore, we could find various joint distributions that exist in the real market and apply them to the multivariate CTE.

Through simulations and illustrative examples, we found that a multivariate CTE using a copula function has a larger value than the CTE using a normal distribution. Since the multivariate CTE using the normal distribution cannot consider non-normal characteristics, such as skewness and kurtosis, the expected value of the risk loss-rate could be underestimated. Additionally, as the number of dimensions increases, the differences between the multivariate copula CTE and the normal CTE progressively increase. We might say that when estimating the risk of a portfolio in the real world, the use of a multivariate copula CTE can lead to accurate risk estimates without underestimation. Therefore, it is recommended that one may make use of the multivariate copula CTE as well as the normal CTE for more stable asset management.

References

Acerbi, C. and Tasche, D. (2002). Expected shortfall: A natural coherent alternative to VaR. *Economic Notes*, **31**, 379-388.

Andersson, F., Mausser, H., Rosen, D. and Uryasev, S. (2001). Credit risk optimization with conditional value-at-risk criterion. *Mathematical Programming*, **89**, 273-291.

Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**, 203-228.

Cai, J. and Li, H. (2005). Conditional tail expectations for multivariate phase-type distributions. *Journal of Applied Probability*, **42**, 810-825.

Cousin, A. and Di Bernardino, E. (2013). On multivariate extensions of value-at-risk. *Journal of Multivariate Analysis*, **119**, 32-46.

Cousin, A. and Di Bernardino, E. (2014). On multivariate extensions of conditional-tail-expectation. *Insurance: Mathematics and Economics*, **55**, 272-282.

Embrechts, P., Mcneil, A.J. and Straumann, D. (1999). Correlation: Pitfalls and alternatives. *Risk*, **5**, 69-71.

Embrechts, P. and Puccetti, G. (2006). Bounds for functions of multivariate risks. *Journal of Multivariate Analysis*, **97**, 526-547.

Genest, C., Ghoudi, K. and Rivest, L. P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, **82**, 543-552.

Hong, C. S. and Kim, T. W. (2016). Multivariate conditional tail expectations. *Korean Journal of Applied Statistics*, **29**, 1201-1212.

Hong, C. S. and Kwon, T. W. (2010). Distribution fitting for the rate of return and value at risk. *Journal of the Korean Data & Information Science Society*, **21**, 219-229.

Hong, C. S., Han, S. J. and Lee, G. P. (2016). Value at risk and alternative value at risk. *Korean Journal of Applied Statistics*, **29**, 689-697.

Hong, C. S. and Lee, J. H. (2011a). VaR estimation of multivariate distribution using copula functions. *Korean Journal of Applied Statistics*, **24**, 523-533.

Hong, C. S. and Lee, W. Y. (2011b). VaR estimation with multiple copula functions. *Korean Journal of Applied Statistics*, **24**, 809-820.

Jorion, P. (2006). *Value at risk, the new benchmark for market risk*, 3rd Ed., McGraw-Hill, New York.

Ko, K. Y. and Son, Y. S. (2015). Optimal portfolio and VaR of KOSPI200 using One-factor model. *Journal of the Korean Data & Information Science Society*, **26**, 323-334.

Li, D. X. (1999). *Value at Risk based on the volatility skewness and kurtosis*, RiskMetrics Group, New York.

Nelsen, R. B. (2006). *An Introduction to Copulas*, Springer, New York.

Park, S. and Baek, C. (2014). On multivariate GARCH model selection based on risk management. *Journal of the Korean Data & Information Science Society*, **25**, 1333-1343.

Rockafellar, R. T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, **2**, 21-41.

Rockafellar, R. T. and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of banking & finance*, **26**, 1443-1471.

Shih, J. H. and Louis, T. A. (1995). Inferences on the association parameter in copula models for bivariate survival data. *Biometrics*, **51**, 1384-1399.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *l'Institut de Statistique de l'Université de Paris*, **8**, 229-231.

Zangari, P. (1996). An improved methodology for measuring VaR. *RiskMetrics Monitor*, **2**, 7-25.