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CESÀRO OPERATORS IN THE BERGMAN SPACES WITH EXPONENTIAL WEIGHT ON THE UNIT BALL

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Abstract. Let $A^2_{\alpha,\beta}(\mathbb{B}_n)$ denote the space of holomorphic functions that are L^2 with respect to a weight of form $\omega_{\alpha,\beta}(z) = (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}$, where $\alpha \in \mathbb{R}$ and $\beta > 0$ on the unit ball $\mathbb{B}_n$. We obtain some results for the boundedness and compactness of Cesàro operator on $A^2_{\alpha,\beta}(\mathbb{B}_n)$.

1. Introduction

Let $\mathbb{B}_n$ be the unit ball of the complex n-space $\mathbb{C}^n$ and $\mathbb{S}_n$ be the unit sphere in $\mathbb{C}^n$. Let dV denote the ordinary volume measure. If $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are points on $\mathbb{B}_n$, we write

$$\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j, \quad |z| = \langle z, z\rangle^{1/2}.$$

We let $A^2_{\alpha,\beta}(\mathbb{B}_n)$ denote the space of holomorphic functions that are L^2 on $\mathbb{B}_n$ with respect to a rapidly decreasing weight of form

$$\omega_{\alpha,\beta}(z) = (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}, \quad \alpha \in \mathbb{R}, \quad \beta > 0.$$

In this case, we give the norm of the space $A^2_{\alpha,\beta}(\mathbb{B}_n)$ as

$$\|f\|_{2,\alpha,\beta} = \left[\int_{\mathbb{B}_n} |f(z)|^2 (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}\, dV(z)\right]^{1/2}.$$

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Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions in $\mathbb{B}_n$. Given a function $g \in H(\mathbb{B}_n)$ we define the radial derivative Rg of g by

$$Rg(z) = \sum_{j=1}^{n} z_j \frac{\partial g}{\partial z_j}(z).$$

For $m = 1, 2, \dots$, we set $R^m f = R(R^{m-1} f)$. The extended Cesàro operator T_g is defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}, \quad z \in \mathbb{B}_n,$$

where $f, g \in H(\mathbb{B}_n)$.

We obtain some results for the boundedness and compactness of Cesàro operator on $A^2_{\alpha,\beta}(\mathbb{B}_n)$.

Theorem 1.1. *Let $g \in H(\mathbb{B}_n)$ and $\alpha \in \mathbb{R}, \beta > 0$. Then*

(1) *T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|)^2 |Rg(z)| < \infty.$$

Moreover,

$$\|T_g\| \approx \sup_{z \in \mathbb{B}_n} (1 - |z|)^2 |Rg(z)|.$$

(2) *T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if*

$$\lim_{|z| \to 1^-} (1 - |z|)^2 |Rg(z)| = 0.$$

While many works ([1], [2], [3], [4], [5], [6], [7], [8], [9]) on the one-variable theory of Bergman spaces with rapidly decreasing weights have been established, the several-variable theory has not been yet. Our work may be thought of as a prototype for Bergman spaces with exponential weight in the several-variable theory.

On the unit disk $\mathbb{D}$, the boundedness and compactness of T_g have been charaterized for a large class of weights which satisfy certain conditions in terms of the symbol function g ([1], [4], [7]). Recently, in [7], Pau and Peláez completed the characterizations of T_g on Bergman spaces with rapidly decreasing weights.

In Section 2, we extend the equivalence of norms to the unit ball. In Section 3, we estimate the Bergman kernel for $A^2_{\alpha,\beta}(\mathbb{B}_n)$ on the diagonal and norms of our test functions. As an application, we prove some results for the characterization of the boundedness and compactness of T_g in Section 4.

Constants. In the rest of the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will be often specified. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \leq CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Bergman spaces with rapidly decreasing weights

Given a positive differentiable weight function $\omega(z) = \omega(|z|)$ which is integrable in $\mathbb{B}_n$, the weighted Bergman space $A^2_\omega(\mathbb{B}_n)$ consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\|f\|^2_{2,\omega} = \int_{\mathbb{B}_n} |f(z)|^2 \omega(z)\, dV(z) < \infty.$$

We note that $A^2_\omega(\mathbb{B}_n)$ is the standard Bergman spaces when $\omega(r) = (1-r)^\alpha$, $\alpha > -1$.

In the unit disk, we have the norm equivalence completed by Pavlović and Peláz [6]. Prior to stating the result, we introduce the distortion function ψ_ω of the weight function ω. As following Siskakis [9] we define the distortion function as follows:

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(t)\, dt, \quad 0 \le r < 1.$$

Lemma 2.1 ([9]). *If there is a constant $A < \infty$ such that*

$$\frac{\omega'(r)}{\omega(r)^2} \int_r^1 \omega(t)\, dt \le A, \quad 0 < r < 1, \tag{2.1}$$

then for all $f \in H(\mathbb{D})$,

$$\int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dA(z) \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \psi_\omega(z)^2 \omega(z)\, dA(z),$$

where $dA(z)$ denotes the normalized Lebesque area measure in $\mathbb{D}$.

Proof. You can refer to Lemma 2.1 and Lemma 2.2 in [9]. □

Here, we notice that all positive differentiable decreasing functions satisfy (2.1). Owing to the integration of slice, we can extend Lemma 2.1 to the unit ball.

Theorem 2.2. *Suppose $\omega(r)$ satisfies condition* (2.1)*. Then*

$$\int_{\mathbb{B}_n} |f(z)|^2 \omega(z)\, dV(z) \approx |f(0)|^2 + \int_{\mathbb{B}_n} |Rf(z)|^2 \psi_\omega(z)^2 \omega(z)\, dV(z)$$

for all $f \in H(\mathbb{B}_n)$.

Proof. Since the measure dS is a rotation invariant measure and

$$\|f\|^2_{2,\omega} \approx \int_0^1 \int_{\mathbb{S}_n} |f(r\zeta)|^2 dS(\zeta)\, \omega(r) r^k\, dr$$

for some $k \ge 0$ we have ([10], Lemma 1.10)

$$\int_{\mathbb{B}_n} |f(z)|^2 \omega(z)\, dV(z) \approx \int_{\mathbb{S}_n} \int_0^1 \int_0^{2\pi} |f(re^{i\theta}\zeta)|^2 \frac{d\theta}{2\pi}\, \omega(r) r\, dr\, dS(\zeta).$$

Now, we consider the slice function $f_\zeta(re^{i\theta}) := f(re^{i\theta}\zeta)$. Applying $f_\zeta(z)$ to Lemma 2.1, Tonelli's Theorem follows that

$$\begin{aligned}&\int_{\mathbb{B}_n} |f(z)|^2\omega(z)\,dV(z)\\ &\approx |f_\zeta(0)|^2 + \int_{\mathbb{S}_n}\int_0^1\int_0^{2\pi} |f_\zeta'(re^{i\theta})|^2\,\frac{d\theta}{2\pi}\,r\psi_\omega(r)^2\omega(r)\,dr\,dS(\zeta).\end{aligned}$$

Since $zf_\zeta'(z) = Rf(z\zeta)$, we obtain

$$\begin{aligned}\|f\|^2_{2,\omega} &\approx |f(0)|^2 + \int_{\mathbb{S}_n}\int_0^1\int_0^{2\pi} |Rf(re^{i\theta}\zeta)|^2\,\frac{d\theta}{2\pi}\,\psi_\omega(r)^2\omega(r)\,dr\,dS(\zeta)\\ &\approx |f(0)|^2 + \int_{\mathbb{S}_n}\int_0^1 |Rf(r\zeta)|^2\psi_\omega(r)^2\omega(r)r^{2n-1}\,dr\,dS(\zeta).\end{aligned}$$

Thus we complete our proof. □

In [9], Siskakis shows that distortion function of $\omega_{\alpha,\beta}$ is

$$\psi_{\omega_{\alpha,\beta}}(r) \approx (1-r)^2.$$

Therefore we can notice that the weight function

$$\widetilde{\omega}_{\alpha,\beta}(r) = \psi^2_{\omega_{\alpha,\beta}}(r)\omega_{\alpha,\beta}(r) \approx \omega_{\alpha+4,\beta}(r)$$

and it also satisfies condition (2.1) since $\widetilde{\omega}_{\alpha,\beta}(r)$ is still a decreasing function. Thus if we repeat the same work to function Rf in Theorem 2.2, then we obtain the following result.

Corollary 2.3. *Let $k \geq 1$. Then*

$$\|f\|^2_{2,\alpha,\beta} \approx \sum_{m=0}^{k-1}\sum_{|\gamma|=m} |\partial^\gamma f(0)|^2 + \|R^k f\|^2_{2,\alpha+4k,\beta}.$$

Here, we are using the standard multi-index notation. Namely, given an n-tuple $\gamma = (\gamma_1,\ldots,\gamma_n)$ of nonnegative integers,

$$|\gamma| = \sum_{j=1}^n \gamma_j,\quad \gamma! = \gamma_1!\cdots\gamma_n!,\quad z^\gamma = z_1^{\gamma_1}\cdots z_n^{\gamma_n},\quad \partial^\gamma = \partial_1^{\gamma_1}\cdots\partial_n^{\gamma_n},$$

where ∂_j denotes the partial differentiation with respect to the j-th component.

3. Reproducing kernel estimates and test functions

3.1. Reproducing kernel estimates

Since each point evaluation is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, there exists the reproducing kernel $K_{\alpha,\beta}(z,w)$ for $A^2_{\alpha,\beta}(\mathbb{B}_n)$. We know that $K_{\alpha,\beta}(z,w)$ is given by

$$K_{\alpha,\beta}(z,w) = \sum_\gamma \frac{z^\gamma\overline{w}^\gamma}{\|z^\gamma\|^2_{2,\alpha,\beta}}.$$

Unfortunately, the explicit form of $K_{\alpha,\beta}(z,w)$ is unknown, but we are going to calculate the reproducing kernel on the diagonal using the following useful calculations.

Proposition 3.1 ([4]). *Let $\alpha \in \mathbb{R}$ and $\beta, s > 0$. Then*

$$\int_0^1 (1-r)^\alpha \, e^{-\frac{\beta}{1-r}} \, r^s \, dr \approx s^{-\frac{2\alpha+3}{4}} e^{-2\sqrt{\beta s}}, \quad s \to \infty.$$

Let us consider the function

$$z \mapsto (1-z)^\alpha e^{\frac{\beta}{1-z}}, \quad \alpha, \beta \in \mathbb{R} \quad \text{and} \quad z \in \mathbb{D}.$$

We can notice that the function is analytic in the unit disk. Therefore we have its Taylor expansion with coefficients $L_m(\alpha,\beta)$ such that

$$(1-z)^\alpha e^{\frac{\beta}{1-z}} = \sum_{m=0}^{\infty} L_m(\alpha,-\beta) z^m. \tag{3.1}$$

Lemma 3.2 ([4]). *Let $\alpha \in \mathbb{R}$, $\beta > 0$. Then*

$$L_m(\alpha,-\beta) \approx m^{-\frac{2\alpha+3}{4}} e^{2\sqrt{\beta m}}.$$

The area of the unit sphere $\mathbb{S}_n$ in $\mathbb{C}^n$ is given by

$$\sigma_{2n-1} = \frac{2\pi^n}{(n-1)!}.$$

Now, we calculate the size of the Bergman kernel for $A^2_{\alpha,\beta}(\mathbb{B}_n)$ on the diagonal.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$ and $\beta > 0$. Then*

$$K_{\alpha,\beta}(z,z) \approx (1-|z|^2)^{-2n-\alpha-1} e^{\frac{2\beta}{1-|z|^2}}, \quad z \in \mathbb{B}_n.$$

Proof. We first estimate the size of the monomial in $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Proposition 3.1 gives

$$\begin{aligned}
\|z^\gamma\|^2_{2,\alpha,\beta} &= \int_{\mathbb{S}_n} |\zeta^\gamma|^2 \, dS(\zeta) \int_0^1 (1-r)^\alpha \, r^{2|\gamma|+2n-1} \, e^{-\frac{\beta}{1-r}} \, dr \\
&= \sigma_{2n-1} \frac{(n-1)! \, \gamma!}{(n-1+|\gamma|)!} \int_0^1 (1-r)^\alpha \, r^{2|\gamma|+2n-1} \, e^{-\frac{\beta}{1-r}} \, dr \\
&\approx \frac{\gamma!}{(n-1+|\gamma|)!} (2|\gamma|+2n-1)^{-\frac{2\alpha+3}{4}} e^{-2\sqrt{\beta}\sqrt{2|\gamma|+2n-1}}
\end{aligned}$$

for $|\gamma|$ sufficiently large. By Stirling's formula, we have

$$(n-1+|\gamma|)! \approx |\gamma|^{n+|\gamma|-\frac{1}{2}} e^{-|\gamma|}$$

for $|\gamma|$ sufficiently large. Thus we have

$$K_{\alpha,\beta}(z,z) = \sum_\gamma \frac{|z^\gamma|^2}{\|z^\gamma\|^2_{2,\alpha,\beta}} \approx \sum_\gamma \frac{|\gamma|^{n+|\gamma|-\frac{1}{2}} e^{-|\gamma|}}{\gamma!} |\gamma|^{\frac{2\alpha+3}{4}} e^{2\sqrt{2\beta|\gamma|}} |z^\gamma|^2$$

$$= \sum_{m=0}^{\infty} \sum_{|\gamma|=m} \frac{m^{n+m+\frac{2\alpha+1}{4}} e^{-m}}{\gamma!} e^{2\sqrt{2\beta m}} |z^\gamma|^2$$
$$= \sum_{m=0}^{\infty} \frac{m^{n+m+\frac{2\alpha+1}{4}} e^{-m}}{m!} e^{2\sqrt{2\beta m}} \sum_{|\gamma|=m} \frac{m!}{\gamma!} |z^\gamma|^2.$$

Note that

$$\sum_{|\gamma|=m} \frac{m!}{\gamma!} |z^\gamma|^2 = |z|^{2m}$$

and by Stirling's formula, we have $m! e^m \approx m^{m+\frac{1}{2}}$. Thus by Lemma 3.2 and (3.1) we have

$$\begin{aligned} K_{\alpha,\beta}(z,z) &\approx \sum_{m=0}^{\infty} m^{n+\frac{2\alpha-1}{4}} e^{2\sqrt{2\beta m}} |z|^{2m} \\ &\approx \sum_{m=0}^{\infty} L_m(-2n-\alpha-1, -2\beta) |z|^{2m} \\ &= (1-|z|^2)^{-(2n+1)-\alpha} e^{\frac{2\beta}{1-|z|^2}}. \end{aligned}$$

□

3.2. Test functions

In [4], Dostanić proved the boundedness and compactness of Cesàro operators on the unit disc by using the test function. In our paper, the test function also plays an important role for studying Cesàro operators on the unit ball. Now, we calculate the size of our test function.

Lemma 3.4. *Let $\alpha \in \mathbb{R}$ and $\beta > 0$. Then for fixed $a \in \mathbb{B}_n$,*

$$\int_{\mathbb{B}_n} \left| e^{\frac{2\beta}{1-\langle z,a\rangle}} \right|^2 (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}\, dV(z) \approx (1-|a|^2)^{2n+\alpha+1}\, e^{\frac{2\beta}{1-|a|^2}}.$$

Proof. Let $z = r\zeta$ and $a = R\xi$. Then by the polar coordinate, we have

$$\begin{aligned} &\int_{\mathbb{B}_n} \left| e^{\frac{2\beta}{1-\langle z,a\rangle}} \right|^2 (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}\, dV(z) \\ &= \int_0^1 \int_{\mathbb{S}_n} \left| e^{\frac{2\beta}{1-rR\langle \zeta,\xi\rangle}} \right|^2 dS(\zeta)\, (1-r)^\alpha e^{-\frac{\beta}{1-r}} r^{2n-1}\, dr. \end{aligned}$$

When a function f depends only one complex variable, we have the following formula [10],

$$\text{(3.2)} \qquad \frac{1}{\sigma_{2n-1}} \int_{\mathbb{S}_n} f(\langle \zeta, \xi\rangle)\, dS(\zeta) = \frac{n-1}{\pi} \int_0^{2\pi} \int_0^1 (1-\rho^2)^{n-2} f(\rho e^{i\theta}) \rho\, d\rho\, d\theta$$

for any $\xi \in \mathbb{B}_n$. Thus, if we apply the formula (3.2) to our test function, then we obtain the one variable integration,

$$\frac{1}{\sigma_{2n-1}} \int_{\mathbb{S}_n} \left| e^{\frac{2\beta}{1-rR\langle\zeta,\xi\rangle}} \right|^2 dS(\zeta) = \frac{n-1}{\pi} \int_0^{2\pi} \int_0^1 \left| e^{\frac{2\beta}{1-rR\rho e^{i\theta}}} \right|^2 (1-\rho^2)^{n-2} \rho \, d\rho \, d\theta.$$

On the other hand, (3.1) gives that

$$e^{\frac{2\beta}{1-rR\rho e^{i\theta}}} = \sum_{m\geq 0} L_m(0,-2\beta) \, r^m R^m \rho^m e^{im\theta},$$

and by the Parseval equality, we get

$$\int_0^{2\pi} \left| e^{\frac{2\beta}{1-rR\rho e^{i\theta}}} \right|^2 d\theta = 2\pi \sum_{m\geq 0} |L_m(0,-2\beta)|^2 \, r^{2m} R^{2m} \rho^{2m}.$$

Now, by the following approximation

$$\begin{aligned} \int_0^1 (1-\rho^2)^{n-2} \, \rho^{2m+1} \, d\rho &\approx \int_0^1 (1-t)^{n-2} t^m \, dt \\ &= \frac{\Gamma(n-1)\Gamma(m+1)}{\Gamma(n+m)} \approx m^{-n+1}, \end{aligned}$$

together with Lemma 3.2 we obtain

$$\begin{aligned} \int_{\mathbb{S}_n} \left| e^{\frac{2\beta}{1-rR\langle\zeta,\xi\rangle}} \right|^2 dS(\zeta) &\approx \sum_{m\geq 0} |L_m(0,-2\beta)|^2 \, r^{2m} R^{2m} m^{-n+1} \\ &\approx \sum_{m\geq 0} r^{2m} R^{2m} m^{-n-\frac{1}{2}} e^{4\sqrt{2\beta m}}. \end{aligned}$$

Finally, from Lemma 3.1, we have

$$\begin{aligned} &\int_0^1 \int_{\mathbb{S}_n} \left| e^{\frac{2\beta}{1-rR\langle\zeta,\xi\rangle}} \right|^2 dS(\zeta) \, (1-r)^\alpha e^{-\frac{\beta}{1-r}} r^{2n-1} \, dr \\ \approx &\sum_{m\geq 0} R^{2m} m^{-n-\frac{1}{2}} e^{4\sqrt{2\beta m}} \int_0^1 (1-r)^\alpha e^{-\frac{\beta}{1-r}} r^{2n+2m-1} \, dr \\ \approx &\sum_{m\geq 0} m^{-n-\frac{\alpha}{2}-\frac{5}{4}} \, e^{2\sqrt{2\beta m}} \, R^{2m}. \end{aligned}$$

Hence by applying (3.1) again, we obtain the desired estimate,

$$\begin{aligned} \int_{\mathbb{B}_n} \left| e^{\frac{2\beta}{1-\langle z,a\rangle}} \right|^2 (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}} \, dV(z) &\approx \sum_{m\geq 0} m^{-n-\frac{\alpha}{2}-\frac{5}{4}} \, e^{2\sqrt{2\beta m}} \, |a|^{2m} \\ &\approx \sum_{m\geq 0} L_m(2n+\alpha+1,-2\beta) \, |a|^{2m} \\ &\approx (1-|a|^2)^{2n+\alpha+1} \, e^{\frac{2\beta}{1-|a|^2}}. \end{aligned}$$

□

4. Cesàro operators on $A^2_{\alpha,\beta}(\mathbb{B}_n)$

In this section, we characterize the boundedness and compactness of Cesàro operators on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ by using test functions.

Theorem 4.1. *Let $g \in H(\mathbb{B}_n)$ and $\alpha \in \mathbb{R}$, $\beta > 0$. Then T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if*

$$\sup_{z\in\mathbb{B}_n} (1-|z|)^2|Rg(z)| < \infty. \tag{4.1}$$

Moreover,

$$\|T_g\| \approx \sup_{z\in\mathbb{B}_n} (1-|z|)^2|Rg(z)|.$$

Proof. Suppose g satisfies the condition (4.1). We note that $R(T_g f) = fRg$ and $(T_g f)(0) = 0$. Then by the case $k = 1$ in Corollary 2.3, we have

$$\begin{aligned}
\|T_g(f)\|^2_{2,\alpha,\beta} &\approx \int_{\mathbb{B}_n} |R(T_g f)(z)|^2 (1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z) \\
&= \int_{\mathbb{B}_n} |f(z)Rg(z)|^2 (1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z) \\
&\le \sup_{z\in\mathbb{B}_n} \left[(1-|z|)^2|Rg(z)|\right]^2 \|f\|^2_{2,\alpha,\beta}. \qquad (4.2)
\end{aligned}$$

Thus T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Conversely, suppose T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Since $f(z)Rg(z) \in A^2_{\alpha+4,\beta}(\mathbb{B}_n)$ for $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$, we have the following reproducing formula

$$f(a)Rg(a) = \int_{\mathbb{B}_n} f(z)Rg(z)K_{\alpha+4,\beta}(a,z)(1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z), \tag{4.3}$$

where $K_{\alpha+4,\beta}(a,z)$ is the reproducing kernel for $A^2_{\alpha+4,\beta}(\mathbb{B}_n)$. If $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$ with $f(a) \neq 0$ then Corollary 2.3, (4.3) and Hölder's inequality follow that

$$\begin{aligned}
&(1-|a|)^2|Rg(a)| \\
&\le \frac{(1-|a|)^2}{|f(a)|} \int_{\mathbb{B}_n} |f(z)Rg(z)K_{\alpha+4,\beta}(a,z)|(1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z) \\
&\le \frac{(1-|a|)^2}{|f(a)|} \left(\int_{\mathbb{B}_n} |f(z)Rg(z)|^2 (1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z)\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{B}_n} |K_{\alpha+4,\beta}(a,z)|^2 (1-|z|)^{\alpha+4} e^{-\frac{\beta}{1-|z|}}\, dV(z)\right)^{1/2} \\
&\le C\frac{(1-|a|)^2}{|f(a)|} \|T_g f\|_{2,\alpha,\beta} \|K_{\alpha+4,\beta}(a,\cdot)\|_{2,\alpha+4,\beta}.
\end{aligned}$$

Now, we consider the function,

$$f_a(z) = \left[(1-|a|)^{-(2n+1)} e^{-\frac{2}{1-|a|^2}} e^{\frac{4\beta}{1-\langle z,a\rangle}}\right]^{1/2}. \tag{4.4}$$

Since T_g is bounded, by Theorem 3.3 and Lemma 3.4 there is a constant $C' > 0$ such that

$$(1-|a|)^2|Rg(a)| \le C' \frac{(1-|a|)^2}{|f_a(a)|}\|T_g\|\|f_a\|_{2,\alpha,\beta}\sqrt{K_{\alpha+4,\beta}(a,a)} \tag{4.5}$$
$$\lesssim \frac{\|f_a\|_{2,\alpha,\beta}}{|f_a(a)|}(1-|a|)^{-\frac{2n+\alpha+1}{2}} e^{\frac{\beta}{1-|a|^2}}\|T_g\|$$
$$\lesssim \|T_g\|.$$

Furthermore, we obtain the operator norm of T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ with (4.2),

$$\|T_g\| \approx \sup_{z\in\mathbb{B}_n}(1-|z|)^2|Rg(z)|.$$

□

Proposition 4.2. *T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if whenever $\{f_m\}$ is a bounded sequence in $A^p_{\alpha,\beta}(\mathbb{B}_n)$ such that $f_m \to 0$ on compact subsets of $\mathbb{B}_n$, then $T_g f_m \to 0$ in $A^2_{\alpha,\beta}(\mathbb{B}_n)$.*

Proof. For this proof, you can refer to Proposition 3.11 in [3]. □

Theorem 4.3. *Let $g \in H(\mathbb{B}_n)$ and $\alpha \in \mathbb{R}, \beta > 0$. Then T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if*

$$\lim_{|z|\to 1^-}(1-|z|)^2|Rg(z)| = 0. \tag{4.6}$$

Proof. Let $\{f_m\}$ be a sequence in $A^2_{\alpha,\beta}(\mathbb{B}_n)$ such that $\|f_m\|_{2,\alpha,\beta} < M$ and $f_m(z)$ converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. Now, we assume that for sufficiently small $\epsilon > 0$, there is $\delta > 0$ such that

$$(1-|z|^2)^2|Rg(z)| < \epsilon \quad \text{for} \quad \delta < |z| < 1.$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{B}_n}|T_g f_m(z)|^2(1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}\, V(z)\\
&\le C\int_{\mathbb{B}_n}|f_m(z)Rg(z)|^2(1-|z|)^{\alpha+4}e^{-\frac{\beta}{1-|z|}}\, dV(z)\\
&\lesssim \int_{\delta\mathbb{B}_n}+\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}|f_m(z)Rg(z)|^2(1-|z|)^{\alpha+4}e^{-\frac{\beta}{1-|z|}}\, dV(z)\\
&\lesssim \epsilon^2\|f_m\|^2_{2,\alpha,\beta}.
\end{aligned}$$

Thus the condition (4.6) implies that T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$.

We can see that (4.6) can be a necessary condition for the compactness of T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. First, let us show that f_a uniformly converges to 0 as $|a| \to 1^-$ on compact subsets of $\mathbb{B}_n$. Put

$$F_a(z) = \frac{f_a(z)}{\|f_a\|_{2,\alpha,\beta}}. \tag{4.7}$$

Let K be a compact subset of $\mathbb{B}_n$. Then for $z \in K$ we have

$$|F_a(z)| \lesssim (1-|a|)^{-\frac{2n+\alpha+1}{2}} e^{-\frac{\beta}{1-|a|^2}} e^{\frac{2\beta}{1-\max\{|z|:z\in K\}}} \to 0, \quad |a| \to 1^-.$$

By Proposition 4.2, if T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, then $\|T_gF_a\|_{2,\alpha,\beta} \to 0$ as $|a| \to 1^-$. Now, from (4.5) in the proof of Theorem 4.1 we have

$$\begin{aligned}(1-|a|)^2|Rg(a)| &\lesssim \frac{(1-|a|)^2}{|F_a(a)|}\|T_gF_a\|_{2,\alpha,\beta}\|K_{\alpha+4,\beta}(a,\cdot)\|_{2,\alpha+4,\beta} \\ &\lesssim \|T_gF_a\|_{2,\alpha,\beta} \to 0 \quad \text{as} \quad |a| \to 1^-.\end{aligned}$$

Thus (4.6) is a necessary condition for the compactness of T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. $\square$

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