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A CONDITIONAL FOURIER-FEYNMAN TRANSFORM AND CONDITIONAL CONVOLUTION PRODUCT WITH CHANGE OF SCALES ON A FUNCTION SPACE I

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Abstract. Using a simple formula for conditional expectations over an analogue of Wiener space, we calculate a generalized analytic conditional Fourier-Feynman transform and convolution product of generalized cylinder functions which play important roles in Feynman integration theories and quantum mechanics. We then investigate their relationships, that is, the conditional Fourier-Feynman transform of the convolution product can be expressed in terms of the product of the conditional Fourier-Feynman transforms of each function. Finally we establish change of scale formulas for the generalized analytic conditional Fourier-Feynman transform and the conditional convolution product. In this evaluation formulas and change of scale formulas we use multivariate normal distributions so that the orthonormalization process of projection vectors which are essential to establish the conditional expectations, can be removed in the existing conditional Fourier-Feynman transforms, conditional convolution products and change of scale formulas.

1. Introduction

Let $C_0[0,T]$ denote the Wiener space, that is, the space of real-valued continuous functions x on the closed interval $[0,T]$ with $x(0)=0$. On the space $C_0[0,T]$, the analytic conditional Fourier-Feynman transform and conditional convolution product are introduced by Chang and Skoug [3]. In that paper they also investigated the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Im and Ryu [9] introduced an analogue of Wiener space $C[0,T]$, the space of real-valued continuous functions on $[0,T]$, which generalizes $C_0[0,T]$. The author [4] introduced a generalized conditional Wiener integral with drift on $C[0,T]$ and then, derived two simple formulas which calculate the conditional expectations in terms of ordinary expectations, that is,

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non-conditional expectations. Using the simple formulas on $C[0,T]$, the author and his coauthors [5, 6, 7] established a conditional analytic Fourier-Feynman transform, a conditional convolution product which has no drift, and change of scale formulas for conditional Wiener integrals which simplify the evaluations of the analytic conditional Feynman integrals, because the measure used on $C[0,T]$ is not scale-invariant [1, 2].

Let a be in $C[0,T]$ and let h be of bounded variation with $h \neq 0$ a.e. on $[0,T]$. Define a stochastic process $Z : C[0,T] \times [0,T] \to \mathbb{R}$ by

$$Z(x,t) = \int_0^t h(s)dx(s) + x(0) + a(t)$$

for $x \in C[0,T]$ and for $t \in [0,T]$, where the integral denotes a generalized Paley-Wiener-Zygmund stochastic integral. For a partition $t_0 = 0 < t_1 < \cdots < t_n = T$ of $[0,T]$, define a random vector $Z_n : C[0,T] \to \mathbb{R}^{n+1}$ by

$$Z_n(x) = (Z(x,t_0), Z(x,t_1), \ldots, Z(x,t_n)).$$

Using a simple formula for a generalized conditional Wiener integral on $C[0,T]$ with the conditioning function Z_n [4], we evaluate a generalized analytic conditional Fourier-Feynman transform and conditional convolution product of the following generalized cylinder function

$$F_Z(x) = f\bigg(\int_0^T v_1(s)dZ(x,s), \ldots, \int_0^T v_r(s)dZ(x,s)\bigg),$$

where $f \in L_p(\mathbb{R}^r)$ with $1 \le p \le \infty$ and $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,T]$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions. In fact we show that the L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[[(F_Z * G_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n]$ of the conditional convolution product for the cylinder functions F_Z and G_Z, can be expressed by the formula

$$\begin{aligned} &T_q^{(p)}[[(F_Z * G_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n) \\ &= \bigg[T_q^{(p)}[F_Z|Z_n]\bigg(\frac{1}{\sqrt{2}}y + (\sqrt{2}-1)a, \frac{1}{\sqrt{2}}(\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2}-1)a\bigg)\bigg] \\ &\quad\times \bigg[T_q^{(p)}[G_Z|Z_n]\bigg(\frac{1}{\sqrt{2}}y - a, \frac{1}{\sqrt{2}}(\vec{\eta}_n - \vec{\xi}_n) + a\bigg)\bigg] \end{aligned}$$

for a nonzero real q, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, where P_{Z_n} is the probability distribution of Z_n on the Borel class of $\mathbb{R}^{n+1}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the cylinder functions, can be interpreted as the product of the conditional analytic Fourier-Feynman transforms of each function. Finally we establish various change of scale formulas for the analytic conditional Fourier-Feynman transforms and the conditional convolution products.

In this evaluation formulas and change of scale formulas we use multivariate normal distributions so that Gram-Schmidt orthonormalization process of $\{\mathcal{P}^{\perp}(hv_1), \ldots, \mathcal{P}^{\perp}(hv_r)\}$ can be removed in the existing conditional Fourier-Feynman transforms, conditional convolution products and change of scale formulas for a suitable orthogonal projection $\mathcal{P}^{\perp}$ on $L_2[0,T]$.

2. An analogue of Wiener space and preliminary results

We begin this section with introducing an analogue of Wiener space which is our underlying space.

For a positive real T let $C[0,T]$ denote the space of real-valued continuous functions on the time interval $[0,T]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le T$ let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For $B_j (j = 0, 1, \ldots, n)$ in the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $\mathcal{B}(\mathbb{R})$, let

$$m_\varphi\left[J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right] = \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^n B_j} W_n(\vec{t}, \vec{u}, u_0) d\vec{u} d\varphi(u_0),$$

where for $\vec{u} = (u_1, \ldots, u_n)$

$$W(\vec{t}, \vec{u}, u_0) = \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}.$$

$\mathcal{B}(C[0,T])$, the Borel σ-algebra of $C[0,T]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $C[0,T]$ such that $w_\varphi(I) = m_\varphi(I)$ for all $I \in \mathcal{I}$. This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ [9].

Let $\{e_k : k = 1, 2, \ldots\}$ be a complete orthonormal subset of $L_2[0,T]$ such that each e_k is of bounded variation. For $v \in L_2[0,T]$ and x in $C[0,T]$ let

$$(v, x) = \lim_{n\to\infty} \sum_{k=1}^n \int_0^T \langle v, e_k\rangle e_k(t) dx(t)$$

if the limit exists, where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,T]$. (v,x) is called the Paley-Wiener-Zygmund integral of v according to x.

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $F : C[0,T] \to \mathbb{C}$ be integrable and X be a random vector on $C[0,T]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory [12]. Furthermore there exists a P_X integrable complex-valued function ψ on the value space of X such that

$$E[F|X](x) = (\psi \circ X)(x) \text{ for } w_\varphi \text{ a.e. } x \in C[0,T],$$

where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \cdots < t_n = T$ of a partition of $[0,T]$, where n is a fixed nonnegative integer. Let h be of bounded variation with $h \neq 0$ a.e. on $[0,T]$. Let a be absolutely continuous on $[0,T]$ and define stochastic processes $X, Z : C[0,T] \times [0,T] \to \mathbb{R}$ by

$$X(x,t) = (h\chi_{[0,t]}, x) \text{ and } Z(x,t) = X(x,t) + x(0) + a(t)$$

for $x \in C[0,T]$ and for $t \in [0,T]$, where χ denotes an indicator function. Define a random vector $Z_n : C[0,T] \to \mathbb{R}^{n+1}$ by

$$Z_n(x) = (Z(x,t_0), Z(x,t_1), \ldots, Z(x,t_n))$$

for $x \in C[0,T]$. For $t \in [0,T]$ let $b(t) = \int_0^t [h(s)]^2 ds$ and for any function f on $[0,T]$ define a polygonal function $P_{b,n}(f)$ of f by

$$\begin{aligned} P_{b,n}(f)(t) = \sum_{j=1}^{n} &\left[\frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})} f(t_{j-1}) + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} f(t_j) \right] \\ &\times \chi_{(t_{j-1}, t_j]}(t) + f(0)\chi_{\{0\}}(t) \end{aligned} \tag{1}$$

for $t \in [0,T]$. For $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ define a polygonal function $P_{b,n}(\vec{\xi_n})$ of $\vec{\xi_n}$ by (1), where $f(t_j)$ is replaced by ξ_j. For $x \in C[0,T]$ and for $t \in [0,T]$ let

$$A(t) = a(t) - P_{b,n}(a)(t), \tag{2}$$

$$X_{b,n}(x,t) = X(x,t) - P_{b,n}(X(x,\cdot))(t) \tag{3}$$

and

$$Z_{b,n}(x,t) = Z(x,t) - P_{b,n}(Z(x,\cdot))(t). \tag{4}$$

For a function $F : C[0,T] \to \mathbb{C}$ let

$$F_Z(x,y) = F(Z(x,\cdot) + y) \text{ for } x, y \in C[0,T].$$

By Theorem 6 in [4], we have the following theorem.

Theorem 2.1. *Let F be a complex valued function on $C[0,T]$ and F_Z be integrable with respect to x. Then for $y \in C[0,T]$ and for P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$*

$$E[F_Z(\cdot,y)|Z_n](\vec{\xi_n}) = \int_{C[0,T]} F(Z_{b,n}(x,\cdot) + y + P_{b,n}(\vec{\xi_n}))\,dw_\varphi(x),$$

where $Z_{b,n}$ is given by (4), P_{Z_n} is the probability distribution of Z_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

For $\lambda > 0$ and $x, y \in C[0,T]$, let $F_Z^\lambda(x,y) = F_Z(\lambda^{-\frac{1}{2}}x, y)$ and $Z_n^\lambda(x) = Z_n(\lambda^{-\frac{1}{2}}x)$. Suppose that $E[F_Z^\lambda(\cdot, y)]$ exists, where the expectation is taken over the first variable. By Theorem 2.1 and Lemma 2.1 in [11] we have for $y \in C[0,T]$

$$(5) \quad E[F_Z^\lambda(\cdot,y)|Z_n^\lambda](\vec{\xi_n}) = \int_{C[0,T]} F(\lambda^{-\frac{1}{2}}X_{b,n}(x,\cdot) + y + A + P_{b,n}(\vec{\xi_n}))dw_\varphi(x)$$

for $P_{Z_n^\lambda}$ a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$, where A and $X_{b,n}$ are given by (2) and (3), respectively, and $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

For an extended real number p with $1 < p \le \infty$ suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let $q \in \mathbb{R} - \{0\}$, F_λ and F be measurable functions on $C[0,T]$ for $\lambda \in \mathbb{C}_+$ such that

$$(6) \qquad \lim_{\lambda \to -iq} \int_{C[0,T]} |F_\lambda(x) - F(x)|^{p'} dw_\varphi(x) = 0.$$

Then we write

$$\underset{\lambda \to -iq}{\text{l.i.m.}} (w^{p'})(F_\lambda) = F.$$

Let $I_{F_Z}^\lambda(y, \vec{\xi_n})$ be the right-hand side of (5). If, for w_φ a.e. $y \in C[0,T]$ and for P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $I_{F_Z}^\lambda(y,\vec{\xi_n})$ has an analytic extension $J_\lambda^*(F_Z)(y,\vec{\xi_n})$ on $\mathbb{C}_+$, then it is called a generalized analytic conditional Fourier-Wiener transform of F given Z_n with the parameter λ and denoted by

$$T_\lambda[F_Z|Z_n](y,\vec{\xi_n}) = J_\lambda^*(F_Z)(y,\vec{\xi_n})$$

for $\vec{\xi_n} \in \mathbb{R}^{n+1}$. Moreover if $T_\lambda[F_Z|Z_n](y,\vec{\xi_n})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called a generalized L_1-analytic conditional Fourier-Feynman transform of F given Z_n with the parameter q and denoted by

$$T_q^{(1)}[F_Z|Z_n](y,\vec{\xi_n}) = \lim_{\lambda\to -iq} T_\lambda[F_Z|Z_n](y,\vec{\xi_n}).$$

For $1 < p \le \infty$, define a generalized L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[F_Z|Z_n]$ of F given Z_n by the formula

$$T_q^{(p)}[F_Z|Z_n](\cdot,\vec{\xi_n}) = \underset{\lambda \to -iq}{\text{l.i.m.}} (w^{p'})(T_\lambda[F_Z|Z_n](\cdot,\vec{\xi_n})) \text{ (if exists).}$$

For $j = 1, \dots, n$ let

$$\alpha_j = \frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h,$$

let V be the subspace of $L_2[0,T]$ generated by $\{\alpha_1,\dots,\alpha_n\}$ and let $V^\perp$ be the orthogonal complement of V. Let $\mathcal{P} : L_2[0,T] \to V$ be the orthogonal projection given by

$$\mathcal{P}v = \sum_{j=1}^n \langle v, \alpha_j\rangle \alpha_j$$

and let $\mathcal{P}^\perp : L_2[0,T] \to V^\perp$ be an orthogonal projection.

The following lemma is useful to prove the results in the next sections [11].

Lemma 2.2. *Let $v \in L_2[0,T]$. Then for w_φ a.e. $x \in C[0,T]$*

$$(v, X(x,\cdot)) = (M_h v, x) \text{ and } (v, P_{b,n}(X(x,\cdot))) = (\mathcal{P}M_h v, x),$$

where $M_h : L_2[0,T] \to L_2[0,T]$ is the multiplication operator defined by

$$M_h u = hu \text{ for } u \in L_2[0,T].$$

For simplicity let

$$(\vec{v}, x) = ((v_1, x), \ldots, (v_r, x))$$

for $x \in C[0,T]$ and for $\{v_1, \ldots, v_r\} \subseteq L_2[0,T]$. For $\vec{a}, \vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$ and for any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$, let

$$\Psi_r(\lambda, \vec{a}, A_r, \vec{u}) = \left[\frac{\lambda^r}{(2\pi)^r |A_r|}\right]^{\frac{1}{2}} \exp\left\{-\frac{\lambda}{2}\langle A_r^{-1}(\vec{u}-\vec{a}), \vec{u}-\vec{a}\rangle_{\mathbb{R}}\right\}, \tag{7}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denotes the dot product on $\mathbb{R}^r$. Let I_r be the $r \times r$ identity matrix.

The following lemma is useful to prove the results in the next sections [5].

Lemma 2.3. *Let $\{v_1, \ldots, v_r\}$ be a subset of $L_2[0,T]$ such that $\{M_h v_1, \ldots, M_h v_r\}$ is an independent set. Then the random vector $(\vec{v}, Z(x,\cdot))$ has the multivariate normal distribution* [12] *with mean vector $(\vec{v}, a)$ and covariance matrix $\Sigma_{M_h} = [\langle M_h v_i, M_h v_j\rangle]_{r\times r}$. Moreover, for any Borel measurable function $f : \mathbb{R}^{r+1} \to \mathbb{C}$, we have*

$$\begin{aligned}
&\int_{C[0,T]} f(x(0), (\vec{v}, Z(x,\cdot)))\,dw_\varphi(x) \\
&\overset{*}{=} \int_{\mathbb{R}^r}\int_{\mathbb{R}} f(u_0, \vec{u})\Psi_r(1, (\vec{v}, a), \Sigma_{M_h}, \vec{u})\,d\varphi(u_0)\,d\vec{u} \\
&\overset{*}{=} \int_{\mathbb{R}^r}\int_{\mathbb{R}} f(u_0, \Sigma_{M_h}^{\frac{1}{2}}\vec{u} + (\vec{v}, a))\Psi_r(1, \vec{0}, I_r, \vec{u})\,d\varphi(u_0)\,d\vec{u},
\end{aligned}$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Remark 2.4. (1) If φ is the Dirac measure δ_0 concentrated at 0, then we can obtain the definition of the conditional Fourier-Feynman transform on the classical Wiener space [3].

(2) Because the Borel sets of $C[0,T]$ are always scale-invariant measurable and we use the Borel class of $C[0,T]$ on which w_φ is defined, the scale-invariant measurability is not essential in (6).

3. Generalized conditional Fourier-Feynman transforms

Let $1 \le p \le \infty$, let r be any fixed positive integer, let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,T]$ such that both $\{M_h v_1, \ldots, M_h v_r\}$ and

$\{\mathcal{P}^\perp M_h v_1, \ldots, \mathcal{P}^\perp M_h v_r\}$ are independent sets. Let $\mathcal{A}^{(p)}$ be the space of cylinder functions F having the form

$$F(x) = f((\vec{v}, x)) \tag{8}$$

for w_φ a.e. $x \in C[0,T]$, where $f \in L_p(\mathbb{R}^r)$. Without loss of generality we can take f to be Borel measurable.

Theorem 3.1. *Let* $1 \le p \le \infty$ *and let* $F(\in \mathcal{A}^{(p)})$ *be given by* (8)*. Then for* $\lambda \in \mathbb{C}_+$

$$\begin{aligned} &T_\lambda[F_Z|Z_n](y, \vec{\xi}_n) \\ &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\xi}_n)), \Sigma_{\mathcal{P}^\perp}, \vec{u}) d\vec{u} \\ &= (f_1(\vec{\xi}_n, A, \cdot) * \Psi_r(\lambda, \vec{0}, I_r, \cdot))(\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v}, y)) \end{aligned} \tag{9}$$

for w_φ *a.e.* $y \in C[0,T]$ *and a.e.* $\vec{\xi}_n \in \mathbb{R}^{n+1}$*, where* $\Sigma_{\mathcal{P}^\perp} = [\langle \mathcal{P}^\perp M_h v_i, \mathcal{P}^\perp M_h v_j\rangle]$,

$$f_1(\vec{\xi}_n, A, \vec{u}) = f(\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} \vec{u} + (\vec{v}, A + P_{b,n}(\vec{\xi}_n))) \tag{10}$$

and A *is given by* (2)*. Moreover* $T_\lambda[F_Z|Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(p)}$.

Proof. For $j = 1, \ldots, r$ and w_φ a.e. $x \in C[0,T]$ we have by Lemma 2.2

$$(v_j, X_{b,n}(x, \cdot)) = (\mathcal{P}^\perp M_h v_j, x)$$

so that for $\lambda > 0$, $y \in C[0,T]$ and $\vec{\xi}_n \in \mathbb{R}^{n+1}$ we have by Lemma 2.3

$$\begin{aligned} I_{F_Z}^\lambda(y, \vec{\xi}_n) &= \int_{C[0,T]} f(\lambda^{-\frac{1}{2}}((\mathcal{P}^\perp M_h v_1, x), \ldots, (\mathcal{P}^\perp M_h v_r, x)) + (\vec{v}, y) \\ &\quad + (\vec{v}, A + P_{b,n}(\vec{\xi}_n))) dw_\varphi(x) \\ &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\xi}_n)), \Sigma_{\mathcal{P}^\perp}, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} \vec{u} + (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\xi}_n))) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u} \\ &= (f_1(\vec{\xi}_n, A, \cdot) * \Psi_r(\lambda, \vec{0}, I_r, \cdot))(\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v}, y)), \end{aligned}$$

where Ψ_r is given by (7). We note that if $1 \le p < \infty$, then by the change of variable theorem

$$\|f_1(\vec{\xi}_n, y + A, \cdot)\|_p^p = |\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}| \, \|f\|_p^p < \infty. \tag{11}$$

Now, by the Morera's theorem with aids of the Hölder's inequality and the dominated convergence theorem, we have (9) for $\lambda \in \mathbb{C}_+$. Since $f_1(\vec{\xi}_n, A, \cdot) \in L_p(\mathbb{R}^r)$ and $\Psi_r(\lambda, \vec{0}, I_r, \cdot) \in L_1(\mathbb{R}^r)$, the final result follows by the change of variable theorem and the Young's inequality [8]. □

Theorem 3.2. *Let $F(\in \mathcal{A}^{(p)})$ be given by* (8) *with $1 \le p \le 2$. Then for a nonzero real q, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(p)}[F_Z|Z_n](y,\vec{\xi}_n)$ exists and it is given by the right-hand side of* (9)*, where λ is replaced by $-iq$. Furthermore $T_q^{(p)}[F_Z|Z_n](\cdot,\vec{\xi}_n) \in \mathcal{A}^{(p')}$, where $\frac{1}{p}+\frac{1}{p'}=1$ if $1<p\le 2$ and $p'=\infty$ if $p=1$.*

Proof. Let $T_q^{(p)}[F_Z|Z_n](y,\vec{\xi}_n)$ be given by the right-hand side of (9) with $\lambda = -iq$, formally. By the change of variable theorem and an application of Lemma 1.1 in [10], we have $T_q^{(p)}[F_Z|Z_n](\cdot,\vec{\xi}_n) \in \mathcal{A}^{(p')}$. When $p=1$, the results follow by the Hölder's inequality, the Morera's theorem and the dominated theorem. Suppose that $1<p\le 2$. By (11), Lemma 2.3, Theorems 2.1 and 3.1, and the change of variable theorem we have

$$\begin{aligned}
&\int_{C[0,T]} |T_\lambda[F|Z_n(y,\vec{\xi}_n) - T_q^{(p)}[F|Z_n](y,\vec{\xi}_n)|^{p'} dw_\varphi(y)\\
&= \int_{C[0,T]} |f_1(\vec{\xi}_n,A,\cdot) * \Psi_r(\lambda,\vec{0},I_r,\cdot))(\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v},y)) - (f_1(\vec{\xi}_n,A,\cdot) * \Psi_r(-iq,\\
&\quad \vec{0},I_r,\cdot))(\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v},y))|^{p'} dw_\varphi(y)\\
&\le |\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}| \int_{\mathbb{R}^r} |(f_1(\vec{\xi}_n,A,\cdot) * \Psi_r(\lambda,\vec{0},I_r,\cdot))(\vec{u}) - (f_1(\vec{\xi}_n,A,\cdot) * \Psi_r(-iq,\vec{0},I_r,\\
&\quad \cdot))(\vec{u})|^{p'} du,
\end{aligned}$$

which converges to 0 as λ approaches $-iq$ through $\mathbb{C}_+$ by Lemma 1.2 of [10]. Now the proof is completed. □

Theorem 3.3. *Let $F(\in \mathcal{A}^{(p)})$ be given by* (8) *with $1 \le p \le \infty$. For w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, let $F_1(y,\vec{\xi}_n,\vec{\eta}_n) = f_1(\vec{\xi}_n + \vec{\eta}_n, 2A, \Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v},y))$, where f_1 is given by* (10)*. Then, for a nonzero real q,*

$$\int_{C[0,T]} |T_{\bar{\lambda}}[T_\lambda[F_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n) - F_1(y,\vec{\xi}_n,\vec{\eta}_n)|^p dw_\varphi(y) \to 0$$

for $1 \le p < \infty$, and for $1 \le p \le \infty$

$$T_{\bar{\lambda}}[T_\lambda[F_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n) \longrightarrow F_1(y,\vec{\xi}_n,\vec{\eta}_n)$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

Proof. By Theorem 3.1, $T_{\bar{\lambda}}[T_\lambda[F_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$ is well-defined. By repeated applications of Theorem 3.1, we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
&T_{\bar{\lambda}}[T_\lambda[F_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)\\
&= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} f_1(\vec{\xi}_n + \vec{\eta}_n, y+2A, \vec{u}+\vec{z})\Psi_r(\lambda,\vec{0},I_r,\vec{u})\Psi_r(\bar{\lambda},\vec{0},I_r,\vec{z})d\vec{u}d\vec{z}
\end{aligned}$$

$$= \left(\frac{|\lambda|}{2\pi}\right)^r \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} f_1(\vec{\xi}_n+\vec{\eta}_n, y+2A, \vec{u}) \exp\left\{-\frac{\lambda}{2}\|\vec{z}\|_{\mathbb{R}}^2 - \frac{\bar{\lambda}}{2}\|\vec{z}-\vec{u}\|_{\mathbb{R}}^2\right\} d\vec{z}d\vec{u}$$
$$= \int_{\mathbb{R}^r} f_1(\vec{\xi}_n+\vec{\eta}_n, y+2A, \vec{u})\Psi_r\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda}, \vec{0}, I_r, \vec{u}\right) d\vec{u}$$
$$= \left(f_1(\vec{\xi}_n+\vec{\eta}_n, 2A, \cdot) * \Psi_r\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda}, \vec{0}, I_r, \cdot\right)\right)(\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v}, y)).$$

Using similar method as used in the proof of Theorem 2.4 in [6], we have the theorem. □

4. Generalized conditional convolution products

Let F and G be defined on $C[0,T]$. For $y \in C[0,T]$ and for $\lambda > 0$, redefine $F_Z^\lambda(\frac{\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}})G_Z^\lambda(-\frac{\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}})] = F_{Z/\sqrt{2}}^\lambda(\cdot, \frac{y}{\sqrt{2}})G_{-Z/\sqrt{2}}^\lambda(\cdot, \frac{y}{\sqrt{2}})$ and suppose that $E[F_Z^\lambda(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})G_Z^\lambda(-\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})]$ exists over the variable x. By Theorem 2.1 and Lemma 2.1 in [11] we have for $y \in C[0,T]$

$$\begin{aligned}(12)\qquad & E\left[F_Z^\lambda\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)G_Z^\lambda\left(-\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)\middle| Z_n^\lambda\right](\vec{\xi}_n) \\ &= \int_{C[0,T]} F\left(\frac{1}{\sqrt{2}}[y+\lambda^{-\frac{1}{2}}X_{b,n}(x,\cdot)+A+P_{b,n}(\vec{\xi}_n)]\right) \\ &\quad\times G\left(\frac{1}{\sqrt{2}}[y-\lambda^{-\frac{1}{2}}X_{b,n}(x,\cdot)-A-P_{b,n}(\vec{\xi}_n)]\right)dw_\varphi(x)\end{aligned}$$

for $P_{Z_n^\lambda}$ a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where A and $X_{b,n}$ are given by (2) and (3), respectively. Let $K_{F_Z,G_Z}^\lambda(y,\vec{\xi}_n)$ be the right-hand side of (12). If, for w_φ a.e. $y \in C[0,T]$ and for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $K_{F_Z,G_Z}^\lambda(y,\vec{\xi}_n)$ has an analytic extension $J_\lambda^*(F_Z,G_Z)(y,\vec{\xi}_n)$ on $\mathbb{C}_+$, then it is called a generalized conditional convolution product of F and G given Z_n with the parameter λ and denoted by

$$[(F_Z * G_Z)_\lambda|Z_n](y,\vec{\xi}_n) = J_\lambda^*(F_Z,G_Z)(y,\vec{\xi}_n)$$

for a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$. Moreover if, for a nonzero real q, $[(F_Z * G_Z)_\lambda|Z_n](y,\vec{\xi}_n)$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called a generalized conditional convolution product of F and G given Z_n with the parameter q and denoted by

$$[(F_Z * G_Z)_q|Z_n](y,\vec{\xi}_n) = \lim_{\lambda\to -iq}[(F_Z * G_Z)_\lambda|Z_n](y,\vec{\xi}_n).$$

Theorem 4.1. *Let $F(\in \mathcal{A}^{(p_1)})$, $G(\in \mathcal{A}^{(p_2)})$ and f, g be related by (8), respectively, where $1 \le p_1, p_2 \le \infty$. Furthermore let $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $\frac{1}{p_2}+\frac{1}{p_2'}=1$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[(F_Z * G_Z)_\lambda|Z_n](y,\vec{\xi}_n)$ exists and is given by*

$$[(F_Z * G_Z)_\lambda|Z_n](y,\vec{\xi}_n)$$

$$= \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\vec{v},y)+\vec{u}]\right)g\left(\frac{1}{\sqrt{2}}[(\vec{v},y)-\vec{u}]\right)\Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\vec{u})d\vec{u}$$

$$= \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\vec{v},y)+\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}\vec{u}+(\vec{v},A+P_{b,n}(\vec{\xi_n}))]\right)g\left(\frac{1}{\sqrt{2}}[(\vec{v},y)-\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}\vec{u}-(\vec{v},A+P_{b,n}(\vec{\xi_n}))]\right)\Psi_r(\lambda,\vec{0},I_r,\vec{u})d\vec{u}.$$

Moreover for a.e. $\vec{\xi_n}\in\mathbb{R}^{n+1}$, *we have* $[(F_Z*G_Z)_\lambda|Z_n](\cdot,\vec{\xi_n})\in\mathcal{A}^{(1)}$ *if either* $p_2\le p_1'$ *or* $p_1\le p_2'$, $[(F_Z*G_Z)_\lambda|Z_n](\cdot,\vec{\xi_n})\in\mathcal{A}^{(p_2)}$ *if* $p_2\ge p_1'$ *and* $[(F_Z*G_Z)_\lambda|Z_n](\cdot,\vec{\xi_n})\in\mathcal{A}^{(p_1)}$ *if* $p_1\ge p_2'$.

Proof. Using similar method as used in the proof of Theorem 3.1, we have for $\lambda>0$, w_φ a.e. $y\in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi_n}\in\mathbb{R}^{n+1}$,

$$K^\lambda_{F_Z,G_Z}(y,\vec{\xi_n}) \overset{*}{=} \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\vec{v},y)+\vec{u}]\right)g\left(\frac{1}{\sqrt{2}}[(\vec{v},y)-\vec{u}]\right) \times \Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\vec{u})d\vec{u}. \tag{13}$$

Now, for $\lambda\in\mathbb{C}_+$ and $\vec{z}\in\mathbb{R}^r$, let $\phi(\lambda,\vec{z})$ be given by the right-hand side of (13) with replacing $(\vec{v},y)$ by $\vec{z}$ and suppose that $p_2\le p_1'$. Let $\vec{\alpha}=\frac{1}{\sqrt{2}}(\vec{z}+\vec{u})$ and $\vec{\beta}=\frac{1}{\sqrt{2}}(\vec{z}-\vec{u})$. Then we have by the change of variable theorem,

$$\begin{aligned}&\int_{\mathbb{R}^r}|\phi(\lambda,\vec{z})|d\vec{z}\\ &\le \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\left|f(\vec{\alpha})g(\vec{\beta})\Psi_r\left(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\frac{1}{\sqrt{2}}(\vec{\alpha}-\vec{\beta})\right)\right|d\vec{\beta}d\vec{\alpha}\\ &= \int_{\mathbb{R}^r}|f(\vec{\alpha})|(|g|*|\Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\cdot/\sqrt{2})|)(\vec{\alpha})d\vec{\alpha}.\end{aligned}$$

Now take a real number q_1 satisfying $\frac{1}{p_2}+\frac{1}{q_1}=\frac{1}{p_1'}+1$. Then we have $1\le q_1\le\infty$ for $1\le p_1,p_2\le\infty$ and $\Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\cdot/\sqrt{2})\in L_{q_1}(\mathbb{R}^r)$. Now by the general form of Young's inequality [8] and the Hölder's inequality

$$\begin{aligned}\int_{\mathbb{R}^r}|\phi(\lambda,\vec{z})|d\vec{z} &\le \|f\|_{p_1}\|(|g|*|\Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\cdot/\sqrt{2})|)\|_{p_1'}\\ &\le \|f\|_{p_1}\|g\|_{p_2}\|\Psi_r(\lambda,(\vec{v},A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\cdot/\sqrt{2})\|_{q_1}<\infty,\end{aligned}$$

which shows that $\phi(\lambda,\cdot)\in L_1(\mathbb{R}^r)$ so that $[(F_Z*G_Z)_\lambda|Z_n](\cdot,\vec{\xi_n})\in\mathcal{A}^{(1)}$. Similarly $[(F_Z*G_Z)_\lambda|Z_n](\cdot,\vec{\xi_n})\in\mathcal{A}^{(1)}$ if $p_1\le p_2'$. Applying similar method as used in the proof of Theorem 3.2 in [6] with minor modifications we can establish the remainder part of the proof. □

Applying similar method as used in the proof of Theorem 3.3 of [6] with minor modifications we can prove the following theorem.

Theorem 4.2. *Let q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = q$, and $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have the followings*:

(1) *if $F, G \in \mathcal{A}^{(1)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(1)}$,*
(2) *if $F, G \in \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$,*
(3) *if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(2)}$,*
(4) *if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(1)} \cap \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(1)} \cap \mathcal{A}^{(2)}$, and*
(5) *if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(\infty)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.*

Theorem 4.3. *Let $F, G \in \cup_{1 \le p \le \infty} \mathcal{A}^{(p)}$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, we have*

$$\begin{aligned}
&T_\lambda[[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) \\
&= \left[T_\lambda[F_Z|Z_n]\left(\frac{1}{\sqrt{2}} y + (\sqrt{2}-1)a, \frac{1}{\sqrt{2}}(\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2}-1)a \right) \right] \\
&\quad \times \left[T_\lambda[G_Z|Z_n]\left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}}(\vec{\eta}_n - \vec{\xi}_n) + a \right) \right].
\end{aligned}$$

Proof. We note that $T_\lambda[[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n)$ is well-defined by Theorems 3.1 and 4.1. By those theorems as stated above we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
&T_\lambda[[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f\bigg(\frac{1}{\sqrt{2}} [\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} (\vec{z} + \vec{u}) + (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\eta}_n)) + (\vec{v}, A + \\
&\quad P_{b,n}(\vec{\xi}_n))] \bigg) g\bigg(\frac{1}{\sqrt{2}} [\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} (\vec{z} - \vec{u}) + (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\eta}_n)) - (\vec{v}, A + \\
&\quad P_{b,n}(\vec{\xi}_n))] \bigg) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Psi_r(\lambda, \vec{0}, I_r, \vec{z}) d\vec{u} d\vec{z} \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f\bigg(\frac{1}{\sqrt{2}} [\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} (\vec{z} + \vec{u})] + \frac{1}{\sqrt{2}} (\vec{v}, y) + (\vec{v}, A + (\sqrt{2}-1)A) + \frac{1}{\sqrt{2}} (\vec{v}, \\
&\quad P_{b,n}(\vec{\eta}_n + \vec{\xi}_n)) \bigg) g\bigg(\frac{1}{\sqrt{2}} [\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} (\vec{z} - \vec{u})] + \frac{1}{\sqrt{2}} (\vec{v}, y) + (\vec{v}, A) - (\vec{v}, A) + \\
&\quad \frac{1}{\sqrt{2}} (\vec{v}, P_{b,n}(\vec{\eta}_n - \vec{\xi}_n)) \bigg) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Psi_r(\lambda, \vec{0}, I_r, \vec{z}) d\vec{u} d\vec{z}.
\end{aligned}$$

Let $\vec{\alpha} = \frac{1}{\sqrt{2}}(\vec{z} + \vec{u})$ and $\vec{\beta} = \frac{1}{\sqrt{2}}(\vec{z} - \vec{u})$. Then we have by the change of variable theorem

$$\begin{aligned}
&T_\lambda[[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) \\
&= \left(\frac{\lambda}{2\pi} \right)^r \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f\bigg(\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}} \vec{\alpha} + \left(\vec{v}, \frac{1}{\sqrt{2}} y + (\sqrt{2}-1)a \right) + \bigg(\vec{v}, A + P_{b,n}\bigg(\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\times(\vec{\eta}_n+\vec{\xi}_n)-(\sqrt{2}-1)a\Big)\Big)\Big)g\Big(\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}\vec{\beta}+\Big(\vec{v},\frac{1}{\sqrt{2}}y-a\Big)+\Big(\vec{v},A+$$

$$P_{b,n}\Big(\frac{1}{\sqrt{2}}(\vec{\eta}_n-\vec{\xi}_n)+a\Big)\Big)\Big)\exp\Big\{-\frac{\lambda}{4}[\|\vec{\alpha}+\vec{\beta}\|_{\mathbb{R}}^2+\|\vec{\alpha}-\vec{\beta}\|_{\mathbb{R}}^2]\Big\}d\vec{\alpha}d\vec{\beta}$$

$$=\Big[T_\lambda[F_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y+(\sqrt{2}-1)a,\frac{1}{\sqrt{2}}(\vec{\eta}_n+\vec{\xi}_n)-(\sqrt{2}-1)a\Big)\Big]$$

$$\times\Big[T_\lambda[G_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}(\vec{\eta}_n-\vec{\xi}_n)+a\Big)\Big],$$

which completes the proof. □

We now have the following relationships between the conditional Fourier-Feynman transforms and the conditional convolution products from Theorems 3.1, 4.1, 4.2 and 4.3.

Theorem 4.4. *Let q be a nonzero real number. Then we have the followings*:

(1) *if $F,G\in\mathcal{A}^{(1)}$, then we have for w_φ a.e. $y\in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n,\vec{\eta}_n\in\mathbb{R}^{n+1}$*

$$T_q^{(1)}[[(F_Z*G_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$$
$$=\Big[T_q^{(1)}[F_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y+(\sqrt{2}-1)a,\frac{1}{\sqrt{2}}(\vec{\eta}_n+\vec{\xi}_n)-(\sqrt{2}-1)a\Big)\Big]$$
$$\times\Big[T_q^{(1)}[G_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}(\vec{\eta}_n-\vec{\xi}_n)+a\Big)\Big],$$

(2) *if $F\in\mathcal{A}^{(1)}$ and $G\in\mathcal{A}^{(2)}$, then we have for w_φ a.e. $y\in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n,\vec{\eta}_n\in\mathbb{R}^{n+1}$*

$$T_q^{(2)}[[(F_Z*G_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$$
$$=\Big[T_q^{(1)}[F_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y+(\sqrt{2}-1)a,\frac{1}{\sqrt{2}}(\vec{\eta}_n+\vec{\xi}_n)-(\sqrt{2}-1)a\Big)\Big]$$
$$\times\Big[T_q^{(2)}[G_Z|Z_n]\Big(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}(\vec{\eta}_n-\vec{\xi}_n)+a\Big)\Big].$$

5. Evaluation formulas for bounded cylinder functions

Let ψ be the function on $\mathbb{R}^r$ defined by

$$\psi(\vec{u})=\int_{\mathbb{R}^r}\exp\{i\langle\vec{u},\vec{z}\rangle_{\mathbb{R}}\}d\rho(\vec{z})\text{ for }\vec{u}\in\mathbb{R}^r,\tag{14}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. For w_φ a.e. $x\in C[0,T]$, let Φ be given by

$$\Phi(x)=\psi((\vec{v},x)).\tag{15}$$

Applying similar method as used in the proof of Theorem 4.1 in [6] with minor modifications we can prove the following theorem.

Theorem 5.1. *Let $1 \le p \le \infty$ and Φ be given by* (15). *Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda[\Phi_Z|Z_n](y,\vec{\xi}_n)$ exists and it is given by*

$$\begin{aligned}(16)\qquad & T_\lambda[\Phi_Z|Z_n](y,\vec{\xi}_n) \\ &= \int_{\mathbb{R}^r} \exp\left\{ i\langle \vec{z}, (\vec{v}, y) + (\vec{v}, A + P_{b,n}(\vec{\xi}_n))\rangle_{\mathbb{R}} - \frac{1}{2\lambda}\langle \Sigma_{\mathcal{P}^\perp}\vec{z}, \vec{z}\rangle_{\mathbb{R}} \right\} d\rho(\vec{z}).\end{aligned}$$

For a nonzero real q, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(p)}[\Phi_Z|Z_n](y,\vec{\xi}_n)$ exists and it is given by the right-hand side of (16), *where λ is replaced by $-iq$. Furthermore $T_q^{(p)}[\Phi_Z|Z_n](\cdot,\vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.*

Theorem 5.2. *Under the assumptions as given in Theorem 5.1, we have for P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$\|T_{\bar{\lambda}}[T_\lambda[\Phi_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](\cdot,\vec{\eta}_n) - \psi((\vec{v}, \cdot + 2A + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n)))\|_p \to 0$$

and for w_φ a.e. $y \in C[0,T]$

$$T_{\bar{\lambda}}[T_\lambda[\Phi_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n) \longrightarrow \psi((\vec{v}, y + 2A + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n)))$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

Proof. By Theorem 5.1, $T_{\bar{\lambda}}[T_\lambda[\Phi_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$ is well-defined so that we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}& T_{\bar{\lambda}}[T_\lambda[\Phi_Z|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n) \\ &= \int_{\mathbb{R}^r} \exp\left\{ i\langle \vec{z}, (\vec{v}, y) + (\vec{v}, 2A + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n))\rangle_{\mathbb{R}} - \frac{1}{2\lambda}\langle \Sigma_{\mathcal{P}^\perp}\vec{z}, \vec{z}\rangle_{\mathbb{R}} \right. \\ &\quad \left. - \frac{1}{2\bar{\lambda}}\langle \Sigma_{\mathcal{P}^\perp}\vec{z}, \vec{z}\rangle_{\mathbb{R}} \right\} d\rho(\vec{z}).\end{aligned}$$

Applying similar method as used in the proof of Theorem 4.2 in [6] with minor modifications we can obtain the remainder part of the proof. □

Applying similar method as used in the proof of Theorem 4.3 in [6] with minor modifications we can prove the following theorem.

Theorem 5.3. *Let ψ_1, ψ_2 and ρ_1, ρ_2 be related by* (14), *respectively. Let $\Phi_1(x) = \psi_1((\vec{v}, x))$ and $\Phi_2(x) = \psi_2((\vec{v}, x))$ for w_φ a.e. $x \in C[0,T]$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[((\Phi_1)_Z * (\Phi_2)_Z)_\lambda|Z_n](y,\vec{\xi}_n)$ exists and it is given by*

$$\begin{aligned}& [((\Phi_1)_Z * (\Phi_2)_Z)_\lambda|Z_n](y,\vec{\xi}_n) \\ &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\left\{ \frac{i}{\sqrt{2}}[\langle (\vec{v}, y), \vec{u} + \vec{w}\rangle_{\mathbb{R}} + \langle (\vec{v}, A + P_{b,n}(\vec{\xi}_n)), \vec{u} - \vec{w}\rangle_{\mathbb{R}}] \right. \\ &\quad \left. - \frac{1}{4\lambda}\|\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}(\vec{u} - \vec{w})\|_{\mathbb{R}}^2 \right\} d\rho_1(\vec{u})\, d\rho_2(\vec{w}).\end{aligned}$$

For a nonzero real q, $[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](y,\vec{\xi}_n)$ is given by the above equation, where λ is replaced by $-iq$. Furthermore, $[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)\in\mathcal{A}^{(\infty)}$.

Theorem 5.4. *Let q be a nonzero real number and $1\le p\le\infty$. Furthermore let Φ_1 and Φ_2 be as given in Theorem 5.3. Then we have for w_φ a.e. $y\in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n,\vec{\eta}_n\in\mathbb{R}^{n+1}$*

$$\begin{aligned}&T_q^{(p)}[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)\\&=\bigg[T_q^{(p)}[(\Phi_1)_Z|Z_n]\bigg(\frac{1}{\sqrt{2}}y+(\sqrt{2}-1)a,\frac{1}{\sqrt{2}}(\vec{\eta}_n+\vec{\xi}_n)-(\sqrt{2}-1)a\bigg)\bigg]\\&\quad\times\bigg[T_q^{(p)}[(\Phi_2)_Z|Z_n]\bigg(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}(\vec{\eta}_n-\vec{\xi}_n)+a\bigg)\bigg].\end{aligned}$$

Proof. By Theorems 5.1 and 5.3 we have for $\lambda\in\mathbb{C}_+$, w_φ a.e. $y\in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n,\vec{\eta}_n\in\mathbb{R}^{n+1}$

$$\begin{aligned}&T_\lambda[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)\\&=\Big(\frac{\lambda}{2\pi}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\bigg\{\frac{i}{\sqrt{2}}[\langle\Sigma_{\mathcal{P}^\perp}^{\frac12}\vec{z}+(\vec{v},y)+(\vec{v},A+P_{b,n}(\vec{\eta}_n)),\vec{u}\\&\quad+\vec{w}\rangle_{\mathbb{R}}+\langle(\vec{v},A+P_{b,n}(\vec{\xi}_n)),\vec{u}-\vec{w}\rangle_{\mathbb{R}}]+\frac{1}{4qi}\|\Sigma_{\mathcal{P}^\perp}^{\frac12}(\vec{u}-\vec{w})\|_{\mathbb{R}}^2-\frac{\lambda}{2}\|\vec{z}\|_{\mathbb{R}}^2\bigg\}\\&\quad d\vec{z}d\rho_1(\vec{u})d\rho_2(\vec{w})\\&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\bigg\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y)+2A+P_{b,n}(\vec{\eta}_n+\vec{\xi}_n),\vec{u}\rangle_{\mathbb{R}}+\langle(\vec{v},y)+P_{b,n}(\vec{\eta}_n\\&\quad-\vec{\xi}_n),\vec{w}\rangle_{\mathbb{R}}]+\frac{1}{4qi}\|\Sigma_{\mathcal{P}^\perp}^{\frac12}(\vec{u}-\vec{w})\|_{\mathbb{R}}^2-\frac{1}{4\lambda}\|\Sigma_{\mathcal{P}^\perp}^{\frac12}(\vec{u}+\vec{w})\|_{\mathbb{R}}^2\bigg\}d\rho_1(\vec{u})d\rho_2(\vec{w}),\end{aligned}$$

since $\Sigma_{\mathcal{P}^\perp}^{\frac12}$ is symmetric. Let $T_q^{(p)}[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$ be given by the right-hand side of the last equality, where λ is replaced by $-iq$. The existence of $T_q^{(1)}[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)$ follows from the dominated convergence theorem. Now let $1<p\le\infty$ and $\frac1p+\frac1{p'}=1$. Then we have by the dominated convergence theorem

$$\begin{aligned}&\int_{C[0,T]}|T_\lambda[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)\\&\quad-T_q^{(p)}[[((\Phi_1)_Z*(\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi}_n)|Z_n](y,\vec{\eta}_n)|^{p'}dw_\varphi(y)\\&\le\bigg[\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\bigg|\exp\bigg\{-\frac{1}{4\lambda}\|\Sigma_{\mathcal{P}^\perp}^{\frac12}(\vec{u}+\vec{w})\|_{\mathbb{R}}^2\bigg\}-\exp\bigg\{\frac{1}{4qi}\|\Sigma_{\mathcal{P}^\perp}^{\frac12}(\vec{u}+\vec{w})\|_{\mathbb{R}}^2\bigg\}\bigg|\\&\quad d|\rho_1|(\vec{u})d|\rho_2|(\vec{w})\bigg]^{p'}\to0\end{aligned}$$

as λ approaches $-iq$ through $\mathbb{C}_+$, which shows the existence of $T_q^{(p)}[[((\Phi_1)_Z * (\Phi_2)_Z)_q|Z_n](\cdot,\vec{\xi_n})|Z_n](y,\vec{\eta}_n)$. Now the equality in the theorem follows from Theorems 4.3 and 5.1. $\square$

Remark 5.5. (1) Without using Theorem 4.3, we can directly prove Theorem 5.4 with aids of Theorems 5.1 and 5.3.

(2) Comparing Theorem 5.4 with Theorem 4.4, the result in Theorem 5.4 holds for $1 \le p \le \infty$ if Φ_1 and Φ_2 are given by (15).

6. Change of scale formulas for the transforms and convolutions

For $\lambda \in \mathbb{C}$, $x \in C[0,T]$ and $\vec{\xi_n} \in \mathbb{R}^{n+1}$ let

$$K(\lambda,\vec{\xi_n},x) = \left(\frac{|\Sigma_{M_h}|}{|\Sigma_{\mathcal{P}^\perp}|}\right)^{\frac{1}{2}} \exp\bigg\{\frac{1}{2}\|\Sigma_{M_h}^{-\frac{1}{2}}(\vec{v},X(x,\cdot))\|_{\mathbb{R}}^2 - \frac{\lambda}{2}\|\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v},Z(x,\cdot)-A-P_{b,n}(\vec{\xi_n}))\|_{\mathbb{R}}^2\bigg\}. \tag{17}$$

Furthermore, for a nonzero real q, let $\{\lambda_m\}$ be any sequence in $\mathbb{C}_+$ with $\lim_{m\to\infty}\lambda_m = -iq$.

Theorem 6.1. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$*

$$T_\lambda[F_Z|Z_n](y,\vec{\xi_n}) = \lambda^{\frac{r}{2}}\int_{C[0,T]} K(\lambda,\vec{\xi_n},x)F_Z(x,y)dw_\varphi(x), \tag{18}$$

where K is given by (17)*. If $p=1$, then*

$$T_q^{(1)}[F_Z|Z_n](y,\vec{\xi_n}) = \lim_{m\to\infty}\lambda_m^{\frac{r}{2}}\int_{C[0,T]} K(\lambda_m,\vec{\xi_n},x)F_Z(x,y)dw_\varphi(x). \tag{19}$$

Proof. Let F be given by (8). For $\lambda > 0$, w_φ a.e. $y \in C[0,T]$ and a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$ we have by Lemma 2.3

$$\begin{aligned}
&\lambda^{\frac{r}{2}}\int_{C[0,T]} K(\lambda,\vec{\xi_n},x)F_Z(x,y)dw_\varphi(x)\\
&= \left(\frac{\lambda^r|\Sigma_{M_h}|}{|\Sigma_{\mathcal{P}^\perp}|}\right)^{\frac{1}{2}}\int_{C[0,T]} f((\vec{v},Z(x,\cdot))+(\vec{v},y))\exp\bigg\{\frac{1}{2}\|\Sigma_{M_h}^{-\frac{1}{2}}((\vec{v},Z(x,\cdot))-\\
&\quad(\vec{v},a))\|_{\mathbb{R}}^2 - \frac{\lambda}{2}\|\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v},Z(x,\cdot)-A-P_{b,n}(\vec{\xi_n}))\|_{\mathbb{R}}^2\bigg\}dw_\varphi(x)\\
&= \left(\frac{\lambda^r|\Sigma_{M_h}|}{|\Sigma_{\mathcal{P}^\perp}|}\right)^{\frac{1}{2}}\int_{\mathbb{R}^r} f(\vec{u}+(\vec{v},y))\Psi_r(1,(\vec{v},a),\Sigma_{M_h},\vec{u})\exp\bigg\{\frac{1}{2}\|\Sigma_{M_h}^{-\frac{1}{2}}(\vec{u}-\\
&\quad(\vec{v},a))\|_{\mathbb{R}}^2 - \frac{\lambda}{2}\|\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{u}-(\vec{v},A+P_{b,n}(\vec{\xi_n})))\|_{\mathbb{R}}^2\bigg\}d\vec{u}\\
&= \int_{\mathbb{R}^r} f(\vec{u})\Psi_r(\lambda,(\vec{v},y+A+P_{b,n}(\vec{\xi_n})),\Sigma_{\mathcal{P}^\perp},\vec{u})d\vec{u}.
\end{aligned}$$

By the analytic continuation, the dominated convergence theorem and Theorem 3.1 we have the theorem. □

Theorem 6.2. *Let $F \in \mathcal{A}^{(p_1)}$ and $G \in \mathcal{A}^{(p_2)}$ with $1 \le p_1, p_2 \le \infty$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$*

$$[(F_Z * G_Z)_\lambda | Z_n](y, \vec{\xi_n}) = \lambda^{\frac{r}{2}} \int_{C[0,T]} K(\lambda, \vec{\xi_n}, x) F_Z\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \times G_Z\left(-\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) dw_\varphi(x), \tag{20}$$

where K is given by (17). *If $p_1 = p_2 = 1$, then*

$$\begin{aligned}&[(F_Z * G_Z)_q | Z_n](y, \vec{\xi_n}) \\ &= \lim_{m\to\infty} \lambda_m^{\frac{r}{2}} \int_{C[0,T]} K(\lambda_m, \vec{\xi_n}, x) F_Z\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) G_Z\left(-\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) dw_\varphi(x).\end{aligned} \tag{21}$$

Proof. Let F, G and f, g be related by (8), respectively. For $\lambda > 0$ and a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$ we have by Lemma 2.3

$$\begin{aligned}&\lambda^{\frac{r}{2}} \int_{C[0,T]} K(\lambda, \vec{\xi_n}, x) F_Z\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) G_Z\left(-\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) dw_\varphi(x) \\ &= \left(\frac{\lambda^r |\Sigma_{M_h}|}{|\Sigma_{\mathcal{P}^\perp}|}\right)^{\frac{1}{2}} \int_{C[0,T]} f\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) + (\vec{v}, Z(x,\cdot))]\right) g\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) - (\vec{v},\right. \\ &\quad \left.Z(x,\cdot))]\right) \exp\left\{\frac{1}{2}\|\Sigma_{M_h}^{-\frac{1}{2}}((\vec{v}, Z(x,\cdot)) - (\vec{v}, a))\|_{\mathbb{R}}^2 - \frac{\lambda}{2}\|\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{v}, Z(x,\cdot) - A \right. \\ &\quad \left. - P_{b,n}(\vec{\xi_n}))\|_{\mathbb{R}}^2\right\} dw_\varphi(x) \\ &= \left(\frac{\lambda^r |\Sigma_{M_h}|}{|\Sigma_{\mathcal{P}^\perp}|}\right)^{\frac{1}{2}} \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) + \vec{u}]\right) g\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) - \vec{u}]\right) \Psi_r(1, (\vec{v}, a), \\ &\quad \Sigma_{M_h}, \vec{u}) \exp\left\{\frac{1}{2}\|\Sigma_{M_h}^{-\frac{1}{2}}(\vec{u} - (\vec{v}, a))\|_{\mathbb{R}}^2 - \frac{\lambda}{2}\|\Sigma_{\mathcal{P}^\perp}^{-\frac{1}{2}}(\vec{u} - (\vec{v}, A + P_{b,n}(\vec{\xi_n})))\|_{\mathbb{R}}^2\right\} \\ &\quad d\vec{u} \\ &= \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) + \vec{u}]\right) g\left(\frac{1}{\sqrt{2}}[(\vec{v}, y) - \vec{u})]\right) \Psi_r(\lambda, (\vec{v}, A + P_{b,n}(\vec{\xi_n})), \Sigma_{\mathcal{P}^\perp}, \\ &\quad \vec{u}) d\vec{u}.\end{aligned}$$

By the analytic continuation, the dominated convergence theorem and Theorem 4.1 we have the theorem. □

By Theorems 5.1 and 6.1, and the dominated convergence theorem, we have the following theorem.

Theorem 6.3. *Let $1 \le p \le \infty$ and let Φ be given by* (15). *Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $T_\lambda[\Phi_Z|Z_n](y,\vec{\xi_n})$ and $T_q^{(p)}[\Phi_Z|Z_n](y,\vec{\xi_n})$ are given by the right-hand sides of* (18) *and* (19)*, respectively, with replacing F_Z by Φ_Z.*

By Theorems 5.3 and 6.2, and the dominated convergence theorem, we have the following theorem.

Theorem 6.4. *Let the assumptions be as given in Theorem 5.3. Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $[((\Phi_1)_Z * (\Phi_2)_Z)_\lambda|Z_n](y,\vec{\xi_n})$ and $[((\Phi_1)_Z * (\Phi_2)_Z)_q|Z_n](y,\vec{\xi_n})$ are given by the right-hand sides of* (20) *and* (21) *with replacing F_Z and G_Z by $(\Phi_1)_Z$ and $(\Phi_2)_Z$, respectively.*

Remark 6.5.
(1) An orthonormal subset $\{v_1, v_2, \dots, v_r\}$ of $L_2[0,T]$ such that both $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ are independent sets, exists [7].
(2) Let $\{e_{11}, \dots, e_{1r}\}$ and $\{e_{21}, \dots, e_{2r}\}$ be the orthonormal sets obtained from $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$, respectively, by the Gram-Schmidt orthonormalization process. For $l = 1, \dots, r$ let $M_h v_l = \sum_{j=1}^r \alpha_{lj} e_{1j}$ and $\mathcal{P}^\perp M_h v_l = \sum_{j=1}^r \beta_{lj} e_{2j}$ be the linear combinations, and let $B_1 = [\alpha_{lj}]_{r\times r}$ and $B_2 = [\beta_{lj}]_{r\times r}$ be the coefficient matrices of the combinations, respectively. Then $M_h^{\frac{1}{2}}$ and $\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}$ can be replaced by B_1 and B_2, respectively, in each expression of the theorems.
(3) It does not mean that $B_1 = \Sigma_{M_h}^{\frac{1}{2}}$ and $B_2 = \Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}}$ in (2). They satisfy only the following equations:

$$B_1 B_1^T = \Sigma_{M_h} = (\Sigma_{M_h}^{\frac{1}{2}})^2 \text{ and } B_2 B_2^T = \Sigma_{\mathcal{P}^\perp} = (\Sigma_{\mathcal{P}^\perp}^{\frac{1}{2}})^2.$$

Remark 6.6.
(1) Letting $\lambda = \gamma^{-2}$ in the theorems of this section, where $\gamma > 0$, we have change of scale formulas for $E[F_Z(\gamma\cdot, y)|Z_n(\gamma\cdot)]$ and $E[F_Z(\frac{\gamma\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}})G_Z(-\frac{\gamma\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}})|Z_n(\gamma\cdot)]$ for $y \in C[0,T]$.
(2) If $y = 0$, then we can obtain the change of scale formulas with Z_{n+1} in [5].
(3) If $a = 0$ and $y = 0$, then we can obtain the results in [7] with cylinder functions.
(4) If $a = 0$ and $h = 1$ a.e., then we can obtain the results in [6].
(5) If $n = 1$ and $\varphi = \delta_0$ which is the Dirac measure concentrated at 0, then we can obtain the results in [3].
(6) The results of this paper are independent of a particular choice of the initial distribution φ.

Remark 6.7. Almost all results of this paper will be extended with the conditioning function $(Z(x,t_0), Z(x,t_1), \dots, Z(x,t_{n-1}))$ which does not contain the present position $Z(x,T)$ of the generalized Wiener path $Z(x,\cdot)$.

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