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SELF-ADJOINT CYCLICALLY COMPACT OPERATORS AND ITS APPLICATION

Karimbergen Kudaybergenov and Farrukh Mukhamedov

Abstract. The present paper is devoted to self-adjoint cyclically compact operators on Hilbert–Kaplansky module over a ring of bounded measurable functions. The spectral theorem for such a class of operators is given. We use more simple and constructive method, which allowed to apply this result to compact operators relative to von Neumann algebras. Namely, a general form of compact operators relative to a type I von Neumann algebra is given.

1. Introduction

The modern structure theory of AW^*-modules originated with the articles by I. Kaplansky [10,11] and nowadays this theory has many applications in the operator algebras.

One of the important instrument in the theory of operator algebras is a different form spectral theorem. In [18] it was proved an important spectral theorem for Hilbert–Kaplansky modules. Another important concept is the compactness property. Cyclically compact sets and operators in lattice-normed spaces were introduced by Kusraev (see [14]). In [14] (see also [15]) a general form of cyclically compact operators in Hilbert–Kaplansky module, as well as a variant of Fredholm alternative for cyclically compact operators, are given. In [6] it was proved that every cyclically compact operator acting in Banach–Kantorovich space over a ring measurable functions can be represented as a measurable bundle of compact operators acting in Banach spaces. Recently, some properties of cyclically compact operators on Hilbert–Kaplansky modules have been investigated in [7,8]. Certain natural applications of Hilbert–Kaplansky modules appeared in [1]. Namely, it has been shown that the algebra of all locally measurable operators with respect to a type I von Neumann algebra can be represented as an algebra of all bounded module-linear operators acting on a Hilbert–Kaplansky module over the ring of measurable function on a measure

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space. This result played a crucial role in the description of derivations on algebras of locally measurable operators with respect to type I von Neumann algebras and their subalgebras (see for example, [1–4]).

On the other hand, it is well-known that one of the important notions in the theory of operator algebras is compact operators relative to von Neumann algebras with faithful semi-finite trace introduced by M. G. Sonis (see [17]). Later V. Kaftal [9] has shown that Sonis's definition of compact operator relative to von Neumann algebras with faithful semi-finite trace is viable for general von Neumann algebras too and he obtained most of the classical characterizations of compact operators.

In this paper we are going to investigate self-adjoint cyclically compact operators on Hilbert–Kaplansky module over a ring of bounded measurable functions and their applications. Note that similar kind of result has been obtained (independently) in [8] using different tools, but ours is more simple and constructive. This allowed us to provide some applications of it.

The paper is organized as follows: in Section 2 we give preliminaries from the theory of Hilbert–Kaplansky module and give a general form of self-adjoint cyclically compact operators on Hilbert–Kaplansky module over a ring measurable functions. In Section 3, we provide a general form of a self-adjoint compact operator relative to a type I von Neumann algebra. We refer to [15] for the whole standard terminology and a detailed exposition of the subject

2. Self-adjoint cyclically compact operators on Hilbert–Kaplansky modules

Let us recall some notions and results from the theory of Hilbert–Kaplansky modules (see [15]).

Let (Ω, Σ, μ) be a measurable space and suppose that the measure μ has the direct sum property, i.e., there is a family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < +\infty$, $i \in J$ such that for any $A \in \Sigma, \mu(A) < +\infty$ there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i\in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^\infty(\Omega)$ the algebra of all (equivalence classes of) complex measurable bounded functions on Ω and let ∇ be the set of all idempotents of the algebra $L^\infty(\Omega)$.

Let X be an $L^\infty(\Omega)$-module. The mapping $\langle\cdot,\cdot\rangle : X \times X \to L^\infty(\Omega)$ is a $L^\infty(\Omega)$-valued inner product, if for all $\xi, \eta, \zeta \in X$ and $a \in L^\infty(\Omega)$ the following are satisfied:

(1) $\langle\xi, \xi\rangle \geq 0, \langle\xi, \xi\rangle = 0 \Leftrightarrow \xi = 0$;
(2) $\langle\xi, \eta\rangle = \overline{\langle\eta, \xi\rangle}$;
(3) $\langle a\xi, \eta\rangle = a\langle\xi, \eta\rangle$;
(4) $\langle\xi + \eta, \zeta\rangle = \langle\xi, \zeta\rangle + \langle\eta, \zeta\rangle$.

Using an $L^\infty(\Omega)$-valued inner product, we may introduce the norm in X by the formula

$$\|\xi\|_\infty = \sqrt{\|\langle\xi,\xi\rangle\|_{L^\infty(\Omega)}},$$

and the vector norm

$$\|\xi\| = \sqrt{\langle\xi,\xi\rangle}.$$

A Hilbert–Kaplansky module over $L^\infty(\Omega)$ is a unitary module over $L^\infty(\Omega)$ such that it is complete with respect to the norm $\|\cdot\|_\infty$ and the following two properties are true:

(1) let ξ be an arbitrary element in X, and let $\{\pi_i\}_{i\in I}$ be a partition of unity in ∇ with $\pi_i\xi = 0$ for all $i \in I$, then $\xi = 0$;

(2) let $\{\xi_i\}_{i\in I}$ be a norm-bounded family in X, and let $\{\pi_i\}_{i\in I}$ be a partition of unity in ∇, then there exists an element $\xi \in X$ such that $\pi_i\xi = \pi_i\xi_i$ for all $i \in I$.

An orthogonal basis in a Hilbert–Kaplansky module X over $L^\infty(\Omega)$ is an orthogonal set whose orthogonal complement is $\{0\}$. A Hilbert–Kaplansky module X is said to be *α-homogeneous,* if α is a cardinal and X has a basis of cardinality α.

A Hilbert–Kaplansky module X is said to be *σ-finite-generated* if there exists a partition of unity $\{\pi_n\}_{n\in F}$ in ∇, where $F \subseteq \mathbb{N}$, such that $\pi_n X$ is an n-homogeneous module over $\pi_n L^\infty(\Omega)$ for all $n \in F$.

Let C be a subset in X. Denote by $\mathrm{mix}(C)$ the set of all vectors ξ from X for which there is a partition of unity $\{\pi_i\}_{i\in I}$ in ∇ such that $\pi_i\xi \in C$ for all $i \in I$, i.e.,

$$\mathrm{mix}(C) = \left\{\xi \in X : \exists\, \pi_i \in \nabla, \pi_i\pi_j = 0, i \neq j, \bigvee_{i\in I} \pi_i = \mathbf{1}, \pi_i\xi \in C, i \in I\right\}.$$

In other words $\mathrm{mix}(C)$ is the set of all mixings obtained by families $\{\xi_i\}_{i\in I}$ taken from C.

A subset C is said to be *cyclic* if $C = \mathrm{mix}(C)$.

For a nonempty set A by $\nabla(A)$ denotes the set of all partitions of unity in ∇ with the index set A, i.e.,

$$\nabla(A) = \left\{\nu : A \to \nabla : (\forall\, \alpha, \beta \in A)(\alpha \neq \beta \to \nu(\alpha) \wedge \nu(\beta) = 0) \wedge \bigvee_{\alpha\in A} \nu(\alpha) = \mathbf{1}\right\}.$$

If A is a partially ordered set, then we can order the set $\nabla(A)$ as:

$$\nu \leq \mu \leftrightarrow (\forall\, \alpha, \beta \in A)(\nu(\alpha) \wedge \mu(\beta) \neq 0 \to \alpha \leq \beta)\ (\nu, \mu \in \nabla(A)).$$

Then this relation is a partial order in $\nabla(A)$, in particularly, if A is directed upward or downward, then so does $\nabla(A)$.

Take any net $(\xi_\alpha)_{\alpha\in A}$ in X. For each $\nu \in \nabla(A)$ put $\xi_\nu = \mathrm{mix}_{\alpha\in A}\nu(\alpha)\xi_\alpha$. If all the mixings exist, then we have a net $(\xi_\nu)_{\nu\in\nabla(A)}$ in X. Every subnet of the net $(\xi_\nu)_{\nu\in\nabla(A)}$ is called a cyclical subnet of the original net $(\xi_\alpha)_{\alpha\in A}$.

Recall [15] that a subset $C \subset X$ is said to be *cyclically compact* if C is cyclically and any sequence in C has a cyclic subsequence that norm converges to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set.

An operator T on X is called $L^\infty(\Omega)$*-linear* if $T(a\xi+b\eta) = aT(\xi)+bT(\eta)$ for all $a, b \in L^\infty(\Omega)$, $\xi, \eta \in X$. A $L^\infty(\Omega)$-linear operator T on X is called *cyclically compact* if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact.

For every $\xi, \eta \in X$ we define an operator $\xi \otimes \eta$ on X by the rule

$$(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi,\ \zeta \in X.$$

It is well-known [15, Theorem 8.5.6] (see also [16]) that if T is a cyclically compact operator on X then there exists a partition of unity $\{\pi_0, \pi_1, \ldots, \pi_k, \ldots, \pi_\infty\}$ in ∇ and orthonormal system $\{\xi_{k,n}\}_{n=1}^k$, $\{\eta_{k,n}\}_{n=1}^k$ in $\pi_k X$ and families $\{f_{k,n}\}_{n=1}^k$ in $\pi_k L^\infty(\Omega)$, where $k = 1, \ldots, n, \ldots, \infty$, such that the followings are true:

(1) $\pi_0 T = 0$;
(2) $f_{\infty,n} \downarrow 0$;
(3) the representation is valid

$$T = \sum_{n=1}^{\infty} f_{\infty,n}\, \xi_{\infty,n} \otimes \eta_{\infty,n} + \sum_{k=1}^{\infty} \pi_k \sum_{n=1}^{k} f_{k,n}\, \xi_{k,n} \otimes \eta_{k,n}. \tag{2.1}$$

Using the same argument as in [13, Theorem 2.1] one can prove the following result.

Theorem 2.1. *Let T be a self-adjoint cyclically compact operator on X. Then there is a partition of unity $\{\pi_0, \pi_1, \ldots, \pi_k, \ldots, \pi_\infty\}$ in ∇ and orthonormal families $\{\xi_{k,n}\}_{n=1}^k$ in $\pi_k X$ and families $\{f_{k,n}\}_{n=1}^k$ in $\pi_k L^\infty(\Omega)$, where $k = 1, \ldots, n, \ldots, \infty$, such that the following hold*:

(i) $\pi_0 T = 0$;
(ii) $|f_{\infty,n}| \downarrow 0$;
(iii) *the representation is valid*

$$T = \sum_{n=1}^{\infty} f_{\infty,n}\, \xi_{\infty,n} \otimes \xi_{\infty,n} + \sum_{k=1}^{\infty} \pi_k \sum_{n=1}^{k} f_{k,n}\, \xi_{k,n} \otimes \xi_{k,n}. \tag{2.2}$$

Remark 2.2. We notice that in [8] it has been obtained a representation of cyclically compact self-adjoint operators which generalizes the spectral theorem for compact self-adjoint operators on Hilbert spaces. Description of the global eigenvalues of such operators by a sequence consisting of their global eigenvalues taken in the corresponding representation were also given. Our representation in the form (2.2) gives more detailed information about a self-adjoint cyclically compact operator T. Namely, the first summand in (2.2) can be interpreted as an infinite-dimensional part and the second summand as the finite-dimensional part of T. Moreover, in [8] it was used different tools, than ours. Our method (see [13]) is more simple and constructive, which gives a possible application to compact operators relative to von Neumann algebras (see next section). Note that an application of Theorem 2.1 to partial integral equations have been studied in [13].

3. Self-adjoint compact operators in type I von Neumann algebras

In this section, we are going to apply Theorem 2.1 to compact operators relative von Neumann algebras of type I. For more information about the compact operators relative to von Neumann algebras we refer to [9].

Recall that an operator $y \in M$ is *finite* in M, if its left support $l(y) = \inf\{p \in P(M) : py = y\}$ is finite. An operator $x \in M$ is *compact relative* to M, if it is the limit in the norm of finite operators in M.

Theorem 3.1. *Let M be a type I von Neumann algebra and x be a self-adjoint, compact operator relative to M. Then there is a sequence of mutually orthogonal central projections $\{z_0, z_1, \ldots, z_k, \ldots, z_\infty\}$ in M and families of mutually orthogonal abelian projections $\{p_{k,n}\}_{n=1}^k$ in z_kM and central elements $\{f_{k,n}\}_{n=1}^k$ in z_kM, where $k = 1, \ldots, n, \ldots, \infty$, such that the following hold*:

(i) $z_0x = 0$;
(ii) $z_\infty|f_{\infty,n}| \downarrow 0$;
(iii) *the representation holds*

$$x = z_\infty \sum_{n=1}^{\infty} f_{\infty,n}\, p_{\infty,n} + \sum_{k=1}^{\infty} z_k \sum_{n=1}^{k} f_{k,n}p_{k,n}. \tag{3.1}$$

For the proof of this theorem we need several auxiliary notions and lemmas.

Let us consider a Hilbert space H. A mapping $s : \Omega \to H$ is called *simple,* if $s(\omega) = \sum_{k=1}^{n} \chi_{A_k}(\omega)c_k$, where $A_k \in \Sigma, A_i \cap A_j = \emptyset$, $i \neq j$, $c_k \in H$, $k = \overline{1,n}$, $n \in \mathbb{N}$. A mapping $u : \Omega \to H$ is said to be *measurable,* if for each $A \in \sum$ with $\mu(A) < \infty$ there is a sequence $\{s_n\}$ of simple maps such that $\|s_n(\omega) - u(\omega)\| \to 0$ almost everywhere on A.

Denote by $\mathcal{B}(\Omega, H)$ the set of all bounded measurable mappings from Ω into H, and let $L^\infty(\Omega, H)$ denote the space of all equivalence classes with respect to the equality almost everywhere. The equivalence class from $L^\infty(\Omega, H)$ which contains the measurable map $\xi \in \mathcal{B}(\Omega, H)$ is denoted by $\widehat{\xi}$. In what follows, we identify the element $\xi \in \mathcal{B}(\Omega, H)$ and the class $\widehat{\xi}$. It is clear that the function $\omega \to \|\xi(\omega)\|$ is measurable for all $\xi \in \mathcal{B}(\Omega, H)$. Denote by $\|\widehat{\xi}\|$ the equivalence class containing the function $\|\xi(\omega)\|$. The algebraic operations on $L^\infty(\Omega, H)$ are defined by the usual way: $\widehat{\xi} + \widehat{\eta} = \widehat{\xi + \eta}, a\widehat{\xi} = \widehat{a\xi}$ for all $\widehat{\xi}, \widehat{\eta} \in L^\infty(\Omega, H), a \in L^\infty(\Omega)$. Let us consider $L^\infty(\Omega)$-valued inner product

$$\langle \xi, \eta \rangle = \langle \xi(\omega), \eta(\omega) \rangle_H,$$

where $\langle\cdot,\cdot\rangle_H$ is the inner product in H. Then $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$.

It is known [1] that α-dimensional Hilbert space H the Hilbert–Kaplansky module $L^\infty(\Omega, H)$ is α-homogeneous.

Let $B(L^\infty(\Omega, H))$ be the algebra of all bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$. Taking into account that $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$ we get that $B(L^\infty(\Omega, H))$ is an AW^*-algebra of type I with the center $*$-isomorphic to $L^\infty(\Omega)$. Suppose that $\dim H = \alpha$. Then $L^\infty(\Omega, H)$ is α-homogeneous and by [11, Theorem 7] the algebra $B(L^\infty(\Omega, H))$ has the type I_α. The center $Z(B(L^\infty(\Omega, H)))$ of this AW^*-algebra isomorphic with the algebra $L^\infty(\Omega)$ which is a von Neumann algebra, and thus by [10, Theorem 2] $B(L^\infty(\Omega, H))$ is also a von Neumann algebra. Consequently, if $\dim H = \alpha$ then $B(L^\infty(\Omega, H))$ is a of type I_α von Neumann algebra.

Now let us consider an arbitrary type I_α homogeneous von Neumann algebra M with the center isomorphic to $L^\infty(\Omega)$. Taking into account that two von Neumann algebras of the same type I_α with isomorphic centers are mutually $*$-isomorphic, we conclude that the algebra M is $*$-isomorphic to the algebra $B(L^\infty(\Omega, H))$, where $\dim H = \alpha$.

A projection $p \in B(L^\infty(\Omega, H))$ is called σ-finite-generated if $p(L^\infty(\Omega, H))$ is a σ-finite-generated module.

Lemma 3.2. *Let $p \in M \cong B(L^\infty(\Omega, H))$ be a finite projection. Then p is σ-finitely-generated.*

Proof. Let $p \in M$ be a finite projection.

Case 1. Let p be a projection such that pMp is a n-homogeneous von Neumann algebra. Then $p \in pMp \equiv B(p(L^\infty(\Omega, H)))$ is also n-homogeneous algebra. This implies that $p(L^\infty(\Omega, H))$ is a n-homogeneous Hilbert–Kaplansky module. This means that p is a n-homogeneous projection. In particular, p is a σ-finite-generated projection.

Case 2. Let p be an arbitrary finite projection. Then there exists a system of mutually orthogonal central projections $(q_n)_{n\in F} \subset \mathcal{P}(pMp)$, where $F \subseteq \mathbb{N}$, with $\sum_{n\in F} q_n = p$ such that $q_n pMp$ is a homogeneous von Neumann algebra of type I_n. By case 1 we get that $q_n p$ is n-homogeneous for all $n \in F$. Thus $p = \sum_{n\in F} q_n p$ is σ-finite-generated. The proof is complete. $\square$

Lemma 3.3. *Let $M \equiv B(L^\infty(\Omega, H))$. If $x \in M$ is a compact operator relative M, then it is cyclically compact.*

Proof. Let $x \in M$ be a compact operator relative M. By [9, Theorem 1.3] for every $n \in \mathbb{N}$ there exists a projection $p_n \in M$ such that $\|xp_n\| < 1/n$ and $\mathbf{1} - p_n$ is finite. By Lemma 3.2, $\mathbf{1} - p_n$ is σ-finite-generated. Therefore $\mathbf{1} - p_n$ is a cyclically compact operator on $L^\infty(\Omega, H)$. Thus $x(\mathbf{1} - p_n)$ is also a cyclically compact operator on $L^\infty(\Omega, H)$. Since $\|x - x(\mathbf{1} - p_n)\| = \|xp_n\| < 1/n$ we obtain that x is also a cyclically compact operator on $L^\infty(\Omega, H)$. The proof is complete. $\square$

Proof of Theorem 3.1. First, we consider a homogeneous von Neumann algebra. In this case by Lemma 3.3 x is a cyclically compact operator. Therefore by Theorem 2.1 we can assume that without lost of generality it has the following form

$$x = \sum_{k=1}^{\infty} f_k \, \xi_k \otimes \xi_k.$$

According to [11, Lemma 13] we obtain that $p_k = \xi_k \otimes \xi_k$ is an abelian projection for all k. So

$$x = \sum_{k=1}^{\infty} f_k \, p_k.$$

Now if M is an arbitrary von Neumann algebra of type I, then we can consider the decomposition of the algebra M to homogeneous summands and apply the above assertion. The proof is complete. □

Remark 3.4. If M is a type I_n, $n < \infty$, von Neumann algebra which represented as $n \times n$-matrix algebra over its center, then Theorem 3.1 gives us that any self-adjoint element from M can be represented as diagonal matrix (cf. [5]).

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Karimbergen Kudaybergenov
Department of Mathematics
Karakalpak State University
230113 Nukus, Uzbekistan
E-mail address: karim2006@mail.ru

Farrukh Mukhamedov
Department of Mathematical Sciences
College of Science
The United Arab Emirates University
Al Ain, Abu Dhabi, 15551, UAE
E-mail address: far75m@yandex.ru, farrukh.m@uaeu.ac.ae