

EXAMPLES OF SMOOTH SURFACES IN \mathbb{P}^3 WHICH ARE ULRICH–WILD

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ABSTRACT. Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree $3 \leq d \leq 9$ whose equation can be expressed as either the determinant of a $d \times d$ matrix of linear forms, or the pfaffian of a $(2d) \times (2d)$ matrix of linear forms. In this paper we show that F supports families of dimension p of pairwise non-isomorphic, indecomposable, Ulrich bundles for arbitrary large p .

1. Introduction and notation

We work on an algebraically closed field k of characteristic 0 and \mathbb{P}^N will denote the projective space over k of dimension N .

Let $F \subseteq \mathbb{P}^N$ be a subvariety of dimension n and set $\mathcal{O}_F(h) := \mathcal{O}_F \otimes \mathcal{O}_{\mathbb{P}^N}(h)$. We recall that the variety F is called *arithmetically Cohen–Macaulay* (aCM for short) if

$$h^i(F, \mathcal{I}_{F|\mathbb{P}^N}(th)) = 0, \quad i = 1, \dots, n, \quad t \in \mathbb{Z}.$$

Definition 1.1. A vector bundle \mathcal{E} on F is called aCM if $h^i(F, \mathcal{E}(th)) = 0$ for $i = 1, \dots, n - 1$ and $t \in \mathbb{Z}$.

When $F = \mathbb{P}^N$ it is well known that aCM bundles are exactly direct sums of line bundles: this is the so called Horrocks' theorem (see [21]; for a modern treatment see also [26]).

The property of being aCM is invariant up to shifting degrees. For this reason we restrict our attention to *initialized* bundles, i.e., bundles \mathcal{E} such that $h^0(F, \mathcal{E}(-h)) = 0$ and $h^0(F, \mathcal{E}) \neq 0$. Moreover, we are also interested to *indecomposable* bundles, i.e., bundles which do not split in a direct sum of bundles of lower rank.

D. Eisenbud and J. Herzog in [15] proved that the aCM varieties supporting only finitely many aCM bundles are linear spaces, smooth quadrics, rational

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normal curves, the Veronese surface, up to three reduced points and the smooth cubic scrolls in \mathbb{P}^4 .

On the opposite side M. Casanellas and R. Hartshorne proved in [7] and [8] that a smooth cubic surface is endowed with families of arbitrary dimension of non-isomorphic, indecomposable aCM bundles.

Thus a natural way to classify varieties could be according to the complexity of the category of aCM bundles that the varieties support. Inspired by an analogous classification for k -algebras of finite type, T. Drozd and G. M. Greuel proposed in [14] a classification of aCM varieties as

- *of finite representation type* if they are endowed with only a finite number of initialized, indecomposable, aCM bundles;
- *of tame representation type* if they are endowed with either an infinite discrete set of initialized, indecomposable, aCM bundles or, for each r , the initialized, indecomposable, aCM bundles of rank r form a finite number of families of dimension at most n ;
- *of wild representation type* if they are endowed with families of dimension p of pairwise non-isomorphic initialized, indecomposable, aCM bundles for arbitrary large p .

As pointed out in [14] the trichotomy of representation types is exhaustive for curves due to the classical results of Grothendieck and M. F. Atiyah (see [1]). A priori it is not clear if the same holds also for varieties of dimension $n \geq 2$. Indeed this is a non-trivial question which has been answered positively by D. Faenzi and J. Pons-Llopis in [17] under suitable restrictions on F . Indeed, they proved that all the aCM varieties of positive dimension which are not cones are of wild representation type except

- the aforementioned varieties of finite representation type,
- elliptic curves, nodal rational curves and every smooth, rational, quartic scroll in \mathbb{P}^4 .

If \mathcal{E} is an aCM vector bundle of rank r on the variety $F \subseteq \mathbb{P}^N$ of degree d , then it can be viewed as a sheaf on \mathbb{P}^N , thus

$$\Gamma_*(\mathcal{E}) := \bigoplus_{t \in \mathbb{Z}} H^0(F, \mathcal{E}(th))$$

has a natural structure of module over the ring $S := k[x_0, \dots, x_N]$.

As pointed out by B. Ulrich in [28], the minimal number of generators of the S -module $\Gamma_*(\mathcal{E})$ is at most rd . The bundles for which such a maximum is attained are worth of interest.

Definition 1.2. We say that \mathcal{E} is *Ulrich* if it is initialized, aCM and $h^0(F, \mathcal{E}) = rd$.

If \mathcal{E} is Ulrich, then it is globally generated by definition. More in general Ulrich bundles on F of rank r are exactly the bundles \mathcal{E} on F having a linear minimal free resolution over \mathbb{P}^N (see [8]). E.g. for $N = 3$ (which is the case we

are interested in)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus rd} \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus rd} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The problem of the existence of Ulrich bundles on a fixed variety is a changelling problem (see e.g. the aforementioned [8] and also [3], [11], [12], [16], [24]). A particularly important result for us is due to J. Backelin, J. Herzog and H. Sanders, who proved in [2] that each hypersurface supports Ulrich bundles.

It is thus natural to ask whether each variety of wild representation type also supports families of dimension p of pairwise non-isomorphic, indecomposable, Ulrich bundles for arbitrary large p . Varieties enjoying such a property will be briefly called *Ulrich-wild*. In particular Ulrich-wildness is known for the smooth cubic surface (see the aforementioned [8]), all the Segre products but $\mathbb{P}^1 \times \mathbb{P}^1$ (see [13]), del Pezzo surfaces of degree at least 3 (see e.g. [9], [25], [27]).

In order to complete the picture, the author has been informed that J. O. Kleppe and R. M. Miró-Roig proved the Ulrich-wildness of linear determinantal varieties in the unpublished paper [23] as a particular case of a more general study.

In [17], the authors prove the following useful result (see Theorem 1 and Corollary 1). Its proof is based on the construction of Ulrich bundles of arbitrarily high rank as extensions of Ulrich bundles of lower degree, as in [25], [27].

Lemma 1.3. *Let $F \subseteq \mathbb{P}^N$ be a closed subscheme. If \mathcal{A} and \mathcal{B} are simple Ulrich bundles on F satisfying $h^0(F, \mathcal{A} \otimes \mathcal{B}^\vee) = h^0(F, \mathcal{B} \otimes \mathcal{A}^\vee) = 0$ and $h^1(F, \mathcal{A} \otimes \mathcal{B}^\vee) \geq 3$, then X is Ulrich-wild.*

We will prove the following result in Sections 2 and 3 using the above lemma as the main ingredient.

Theorem 1.4. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$.*

- (1) *If the equation of F is the determinant of a $d \times d$ matrix of linear forms (we briefly say in this case that F is linear determinantal), then it is Ulrich-wild.*
- (2) *If the equation of F is the pfaffian of a $(2d) \times (2d)$ matrix of linear forms (we briefly say in this case that F is linear pfaffian) and $d \leq 9$, then it is Ulrich-wild.*

In both the cases, the first step of the proof is to check that F supports at least two non-isomorphic, simple, Ulrich bundles of low rank satisfying some technical conditions. Then we apply the aforementioned result from [17] which guarantees the Ulrich-wildness of F .

The existence of Ulrich bundles of rank 1 and 2 on surfaces is well-understood (see [6]: see also [3], [16], [24]). Very recently, A. Beauville proved in [4] and [5] the existence of rank 2 Ulrich bundles on each abelian, bielliptic and Enriques surface.

It is obvious that line bundles are simple. The main point of our proof for linear determinantal surfaces is that if an Ulrich line bundle \mathcal{L} satisfying the needed conditions actually exists, then there is also another Ulrich line bundle not isomorphic to \mathcal{L} . We recall these easy facts in Section 2.

The existence of simple Ulrich bundles of rank 2 satisfying the needed conditions on a surface F is a little bit less evident: we prove it in Section 3 by dealing with the stability of certain Ulrich bundles whose existence is known (e.g. see the aforementioned papers [3], [16]).

Notice that each linear determinantal surface is trivially linear pfaffian, but the converse is not true. Indeed the general surface of degree $d \geq 4$ is not linear determinantal, but it is linear pfaffian if $d \leq 15$, (see Proposition 7.6 of [3]). It is classically known that each cubic surface is linear determinantal (see [18]). Moreover, E. Coskun, R. S. Kulkarni and Y. Mustopa proved that each quartic surface is linear pfaffian (see Corollary 1.2 of [10]).

Hence the above results imply the following consequences.

Corollary 1.5. *If $F \subseteq \mathbb{P}^3$ is a smooth surface of degree $d = 3, 4$, then it is Ulrich-wild.*

As we already explained above the proof of the Ulrich-wildness for cubic surfaces is already known. Indeed in [8] the authors proved it by constructing explicit families of Ulrich bundles of arbitrarily large dimension.

Corollary 1.6. *If $F \subseteq \mathbb{P}^3$ is a general smooth surface of degree $5 \leq d \leq 9$, then it is Ulrich-wild.*

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1.1. Notation

In what follows $F \subseteq \mathbb{P}^3$ will denote a smooth surface of degree d and h the class of the hyperplane section of F . We have $\omega_F \cong \mathcal{O}_F((d-4)h)$ and $h^1(F, \mathcal{O}_F(th)) = 0$ for $t \in \mathbb{Z}$.

Recall that for each vector bundle \mathcal{F} of rank r on F we have the following the Riemann–Roch formula

$$(1.1) \quad \chi(\mathcal{F}) = r \left(1 + \binom{d-1}{3} \right) + \frac{c_1(\mathcal{F}) \cdot (c_1(\mathcal{F}) - (d-4)h)}{2} - c_2(\mathcal{F}).$$

For all the other notation we refer to [20].

2. Linear determinantal surfaces

Let $F \subseteq \mathbb{P}^3$ be a linear determinantal smooth surface of degree d , i.e., there is a $d \times d$ matrix Φ with linear entries such that $F = \{ \det(\Phi) = 0 \} \subseteq \mathbb{P}^3$. We recall the following results.

Lemma 2.1. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree d . Then F is linear determinantal if and only if it is endowed with an Ulrich line bundle \mathcal{L} .*

Proof. Assume that F supports an Ulrich line bundle \mathcal{L} . Thus \mathcal{L} is initialized by definition, hence $h^0(F, \mathcal{L}(-h)) = 0$. Moreover, Grothendieck duality on F (see Paragraph (6.3) of [3]) yields that $\mathcal{M} := \mathcal{L}^\vee((d-1)h)$ is Ulrich too. In particular it is initialized, hence

$$h^2(F, \mathcal{L}(-2h)) = h^0(F, \mathcal{L}^\vee((d-2)h)) = h^0(F, \mathcal{M}(-h)) = 0.$$

It follows that the hypothesis of Corollary 1.12(a) of [3] are satisfied, hence F is linear determinantal. The converse also follows from the same aforementioned corollary. \square

Lemma 2.2. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree d . The line bundle \mathcal{L} on F is Ulrich if and only if $\mathcal{L} \cong \mathcal{O}_F(\Delta)$, where $\Delta \subseteq F$ is a smooth projectively normal curve in \mathbb{P}^3 with degree and genus given by*

$$(2.1) \quad \delta := \frac{1}{2}d(d-1), \quad \pi := \frac{1}{6}(d-2)(d-3)(2d+1).$$

Proof. See Proposition 6.2 of [3] and the previous lemma. \square

We are now ready to prove Theorem 1.4 in the particular case of linear determinantal smooth surfaces.

2.1. Proof of (1) of Theorem 1.4

If F is linear determinantal, then it is endowed with an Ulrich line bundle $\mathcal{O}_F(\Delta)$ as in Lemma 2.1. Adjunction on F and the equalities (2.1) yield

$$(2.2) \quad \Delta^2 = 2\pi - 2 - (d-4)\delta = \binom{d}{3}.$$

It is easy to check that the line bundle $\mathcal{O}_F((d-1)h - \Delta)$ is Ulrich too (see [3], Paragraph (6.3)). We have

$$\Delta^2 \neq ((d-1)h - \Delta) \cdot \Delta$$

due to the equality (2.2), thus $\mathcal{O}_F(\Delta) \not\cong \mathcal{O}_F((d-1)h - \Delta)$, i.e., $\mathcal{O}_F(2\Delta - (d-1)h) \not\cong \mathcal{O}_F$. Since $\mathcal{O}_F(h)$ is ample and $(2\Delta - (d-1)h) \cdot h = 0$, we deduce that

$$h^0(F, \mathcal{O}_F(2\Delta - (d-1)h)) = 0.$$

The Riemann–Roch theorem on F thus returns

$$h^1(F, \mathcal{O}_F(2\Delta - (d-1)h)) - h^2(F, \mathcal{O}_F(2\Delta - (d-1)h)) = d^2 - 2d.$$

It follows that $w := h^1(F, \mathcal{O}_F(2\Delta - (d-1)h)) \geq 3$ when $d \geq 3$. Since line bundles are trivially simple and both $\mathcal{O}_F(\Delta)$ and $\mathcal{O}_F((d-1)h - \Delta)$ are Ulrich, it follows that F is Ulrich-wild, by Lemma 1.3.

Remark 2.3. Notice that each cubic surface is linear determinantal. Indeed for $d = 3$ the curve Δ in Lemma 2.1 is a rational cubic due to Lemma 2.2. It is easy to check that each smooth cubic surface contains 72 linear systems whose general element is a rational cubic curve.

When $d \geq 4$, the general surface F does not support Ulrich line bundles $\mathcal{O}_F(\Delta)$. Indeed the Noether–Lefschetz theorem (see [19]) implies that the locus inside the projective space associated to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$ of surfaces F with Picard group generated by $\mathcal{O}_F(h)$ is the complement of a countable union of proper closed subsets. It follows that every curve on a general surface F of degree $d \geq 4$ is the complete intersection of F with another surface: looking at the first equality (2.1), we deduce that Δ is cut out on F by a surface of degree $(d-1)/2$. The adjunction formula for Δ in \mathbb{P}^3 and the second equality (2.1) would force the equality

$$\frac{1}{8}(3d^3 - 12d^2 + 9d + 8) = \frac{1}{6}(d-2)(d-3)(2d+1)$$

whose unique integral solutions are 0 and ± 1 , a contradiction.

Assume that $F := \{f = 0\}$ supports an Ulrich line bundle $\mathcal{O}_F(\Delta)$. As we noticed above, it also supports the Ulrich line bundle $\mathcal{O}_F((d-1)h - \Delta) \not\cong \mathcal{O}_F(\Delta)$. Square matrices of order d with linear entries are parameterized by an affine space \mathbb{M} of dimension $4d^2$. The group $\mathrm{GL}_d \times \mathrm{GL}_d$ acts on such an \mathbb{M} by left and right multiplication.

On the one hand, the set $\mathbb{M}_f \subseteq \mathbb{M}$ of classes of matrices Φ such that $f = \det(\Phi)$ is closed, hence it has finitely many irreducible components. On the other hand, the irregularity of F is 0, thus $\mathrm{Pic}(F)$ is discrete.

It follows that the natural morphism $m: \mathbb{M}_f \rightarrow \mathrm{Pic}(F)$ associating to the class containing Φ the cokernel of the morphism $\varphi: \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus d}$ with matrix Φ is locally constant on \mathbb{M}_f . Thus m is constant on each irreducible component, hence $\mathrm{im}(m)$ is finite. Since $\mathrm{im}(m)$ is the set of Ulrich line bundles on F , we conclude that these bundles are finitely many or, equivalently, that there is a finite even number of matrices Φ up to the natural action of $\mathrm{GL}_d \times \mathrm{GL}_d$ such that $f = \det(\Phi)$.

3. Pfaffian surfaces

Let $F \subseteq \mathbb{P}^3$ be a linear pfaffian smooth surface of degree d , i.e., there is a $(2d) \times (2d)$ skew-symmetric matrix Φ with linear entries such that $F = \{\mathrm{pf}(\Phi) = 0\} \subseteq \mathbb{P}^3$. We have the following result.

Lemma 3.1. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree d . Then F is linear pfaffian if and only if it is endowed with an Ulrich rank 2 vector bundle \mathcal{E} with $c_1(\mathcal{E}) = \gamma_1 := (d-1)h$.*

Proof. If there is an Ulrich rank 2 vector bundle \mathcal{E} with $c_1(\mathcal{E}) = \gamma_1 := (d-1)h$, then F is linear pfaffian by Corollary 2.4 of [3].

Conversely let $F = \{ \text{pf}(\Phi) = 0 \}$ for a suitable $(2d) \times (2d)$ skew-symmetric matrix Φ with linear entries. Then Theorem B of [3] implies that the cokernel \mathcal{E} of the morphism $\varphi: \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2d} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d}$ having matrix Φ is aCM and $c_1(\mathcal{E}) = \gamma_1$. Moreover it is easy to check that it is also initialized and $h^0(F, \mathcal{E}) = 2d$, thus \mathcal{E} is actually Ulrich. \square

Computing the above formula for an Ulrich bundle \mathcal{E} of rank 2 on F (so that $\chi(\mathcal{E}) = 2d$) with $c_1(\mathcal{E}) = \gamma_1$, we obtain that

$$(3.1) \quad c_2(\mathcal{E}) = \gamma_2 := \frac{1}{6}d(d-1)(2d-1).$$

We recall now some facts about the notions of stability and μ -stability of bundles. The slope $\mu(\mathcal{F})$ and the reduced Hilbert polynomial $p_{\mathcal{F}}(t)$ of \mathcal{F} of a bundle on F are

$$\mu(\mathcal{F}) := c_1(\mathcal{F})h/\text{rk}(\mathcal{F}), \quad p_{\mathcal{F}}(t) =: \chi(\mathcal{F}(th))/\text{rk}(\mathcal{F}).$$

The bundle \mathcal{E} is μ -semistable if for all subsheaves \mathcal{G} with $0 < \text{rk}(\mathcal{G}) < \text{rk}(\mathcal{E})$

$$\mu(\mathcal{G}) \leq \mu(\mathcal{E}),$$

and μ -stable if the inequality is always strict.

The bundle \mathcal{E} is called semistable if for \mathcal{G} as above

$$p_{\mathcal{G}}(t) \leq p_{\mathcal{E}}(t),$$

and stable if again the inequality is always strict. The following chain of implications is well-known.

$$\mathcal{E} \text{ is } \mu\text{-stable} \Rightarrow \mathcal{E} \text{ is stable} \Rightarrow \mathcal{E} \text{ is semistable} \Rightarrow \mathcal{E} \text{ is } \mu\text{-semistable}.$$

The following remark will be helpful.

Remark 3.2. If \mathcal{E} is Ulrich, then it is semistable (hence μ -semistable) and it is stable if and only if it is μ -stable (see Theorem 2.9 of [8]).

If \mathcal{E} is a vector bundle of rank 2 with reduced Hilbert polynomial $p(t)$, then there exists the coarse moduli space $\mathcal{M}_F^{ss}(p)$ parameterizing S -equivalence classes of semistable rank 2 bundles on F with reduced Hilbert polynomial $p(t)$ (see Section 1.5 of [22] for details about S -equivalence of semistable sheaves). We will denote by $\mathcal{M}_F^s(p)$ the open locus inside $\mathcal{M}_F^{ss}(p)$ of stable bundles.

The scheme $\mathcal{M}_F^{ss}(p)$ is the disjoint union of open and closed subsets $\mathcal{M}_F^{ss}(2; c_1, c_2)$ whose points represent S -equivalence classes of semistable rank 2 bundles with fixed Chern classes c_1 and c_2 . Similarly $\mathcal{M}_F^s(p)$ is the disjoint union of open and closed subsets $\mathcal{M}_F^s(2; c_1, c_2)$.

As in Section 2 of [8] we can define open loci $\mathcal{M}_F^{ss,U}(2; c_1, c_2) \subseteq \mathcal{M}_F^{ss}(2; c_1, c_2)$ and $\mathcal{M}_F^{s,U}(2; c_1, c_2) \subseteq \mathcal{M}_F^s(2; c_1, c_2)$ parameterizing S -equivalence classes of semistable and stable Ulrich bundles respectively.

Proposition 3.3. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree d . If \mathcal{E} is a bundle of rank 2 on F with $c_1(\mathcal{E}) = \gamma_1$, then it is Ulrich and strictly semistable if and only if it fits into an exact sequence of the form*

$$(3.2) \quad 0 \longrightarrow \mathcal{O}_F(\Delta) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F((d-1)h - \Delta) \longrightarrow 0,$$

where $\Delta \subseteq F$ is a smooth projectively normal curve in \mathbb{P}^3 with degree δ and genus π given by the equalities (2.1).

Proof. If \mathcal{E} is Ulrich and strictly semistable, then it is certainly μ -semistable. It follows the existence of an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

for suitable coherent proper subsheaves \mathcal{F} and \mathcal{G} with $\mu(\mathcal{F}) = \mu(\mathcal{E}) = \Delta \cdot h$ and \mathcal{G} torsion-free.

Due to Theorem 2.9 of [8], it follows that both \mathcal{F} and \mathcal{G} are Ulrich line bundles. In particular $\mathcal{F} \cong \mathcal{O}_F(\Delta)$ and $\mathcal{G} \cong \mathcal{O}_F((d-1)h - \Delta)$ for a suitable smooth and projectively normal curve Δ of degree δ and genus π given by the equalities (2.1) (see Lemma 2.2).

Conversely let \mathcal{E} be a bundle fitting into the sequence (3.2): we know from Lemma 2.2 that $\mathcal{O}_F(\Delta)$ is Ulrich. As pointed out in Paragraph (6.3) of [3], by Grothendieck duality we deduce that $\mathcal{O}_F((d-1)h - \Delta)$ is Ulrich too.

Taking the cohomology of the sequence (3.2) suitably twisted, very easy computations show that the bundle \mathcal{E} is Ulrich, hence semistable. Since $\mu(\mathcal{O}_F(\Delta)) = \mu(\mathcal{E})$, it follows that \mathcal{E} cannot be μ -stable, hence stable in view of Remark 3.2. \square

Notice that, once the divisor Δ is fixed, bundles fitting in the sequence (3.2) are parameterized by the projective space $\mathbb{P}(V)$, where $V := H^1(F, \mathcal{O}_F((d-1)h - 2\Delta))$: thus the computation made in Paragraph 2.1 shows that this space has dimension $w \geq d^2 - 2d - 1$.

It is interesting to better understand points represented by semistable Ulrich bundles inside $\mathcal{M}_F^{ss,U}(2; \gamma_1, \gamma_2)$.

Corollary 3.4. *Let $F \subseteq \mathbb{P}^3$ be a smooth surface of degree d . The points in $\mathcal{M}_F^{ss,U}(2; \gamma_1, \gamma_2) \setminus \mathcal{M}_F^{s,U}(2; \gamma_1, \gamma_2)$ are in one-to-one correspondence with the set of unordered pairs $\{\mathcal{O}_F(\Delta), \mathcal{O}_F((d-1)h - \Delta)\}$ where $\Delta \subseteq F$ is a smooth projectively normal curve in \mathbb{P}^3 with degree δ and genus π given by the equalities (2.1).*

Proof. Taking into account Proposition 3.3 we know that \mathcal{E} represents an S -equivalence class in $\mathcal{M}_F^{ss,U}(2; \gamma_1, \gamma_2) \setminus \mathcal{M}_F^{s,U}(2; \gamma_1, \gamma_2)$ if and only if it fits into the sequence (3.2) for some divisor Δ on F which is a smooth projectively normal curve in \mathbb{P}^3 with degree δ and genus π given by equalities (2.1).

Notice that $0 \subseteq \mathcal{O}_F(\Delta) \subseteq \mathcal{E}$ is the Jordan–Hölder filtration for \mathcal{E} . Indeed

$$\mathcal{E}/\mathcal{O}_F(\Delta) \cong \mathcal{O}_F((d-1)h - \Delta), \quad p_{\mathcal{E}}(t) = p_{\mathcal{O}_F(\Delta)}(t) = p_{\mathcal{O}_F((d-1)h - \Delta)}(t).$$

We conclude that $\text{gr}(\mathcal{E}) \cong \mathcal{O}_F(\Delta) \oplus \mathcal{O}_F((d-1)h - \Delta)$. Hence all the bundles fitting in the aforementioned sequence (3.2) are S -equivalent and they correspond to the same point in $\mathcal{M}_F^{ss,U}(2; \gamma_1, \gamma_2)$. If $\bar{\Delta}$ is another divisor on F enjoying the same properties as Δ , it is easy to check that

$$\mathcal{O}_F(\Delta) \oplus \mathcal{O}_F((d-1)h - \Delta) \cong \mathcal{O}_F(\bar{\Delta}) \oplus \mathcal{O}_F((d-1)h - \bar{\Delta})$$

if and only if $\bar{\Delta}$ is linearly equivalent to either Δ , or $(d-1)h - \Delta$ because they are all curves of degree δ . □

We prove Theorem 1.4 for linear pfaffian smooth surfaces.

3.1. Proof of (2) of Theorem 1.4

The statement obviously follows from Paragraph 2.1 if F is linear determinantal: thus we will assume that F is not linear determinantal from now on.

It follows that F does not support any Ulrich line bundle (by Lemma 2.1), but it certainly supports an Ulrich bundle of rank 2 with Chern classes γ_1 and γ_2 (by Lemma 3.1). Proposition 3.3 and the equality (3.1) imply that each such a bundle is necessarily stable, thus simple and indecomposable. As pointed out in [3], Lemma 7.7 and Remark 7.8, we know that each component of the moduli space $\mathcal{M}_F^s(2; \gamma_1, \gamma_2)$ containing an Ulrich bundle is generically smooth of dimension at least (see Theorem 4.5.8 of [22])

$$4\gamma_2 - \gamma_1^2 - 3\chi(\mathcal{O}_F) = -\frac{1}{6}d(d^2 - 18d + 35)$$

which is greater than d in the range $3 \leq d \leq 15$.

Since $\mathcal{M}_F^{s,U}(2; \gamma_1, \gamma_2)$ is open inside $\mathcal{M}_F^s(2; \gamma_1, \gamma_2)$, it follows that there are at least two non-isomorphic stable Ulrich bundles \mathcal{A} and \mathcal{B} of rank 2 on each pfaffian surface of degree $3 \leq d \leq 9$. The reduced Hilbert polynomials of \mathcal{A} and \mathcal{B} coincide, because they depend only on the Chern classes γ_1, γ_2 (more generally the reduced Hilbert polynomials of an Ulrich bundle \mathcal{F} on F is

$$p_{\mathcal{F}}(t) = \chi(\mathcal{F}(th))/\text{rk}(\mathcal{F}) = d \binom{t+2}{2} :$$

see [8], Lemma 2.6 for the computation of the standard Hilbert polynomial).

Since \mathcal{A} and \mathcal{B} are also non-isomorphic and stable, it follows from Proposition 1.2.7 of [22] that

$$h^0(F, \mathcal{A} \otimes \mathcal{B}^\vee) = h^0(F, \mathcal{B} \otimes \mathcal{A}^\vee) = 0,$$

thus

$$h^1(F, \mathcal{A} \otimes \mathcal{B}^\vee) = h^2(F, \mathcal{A} \otimes \mathcal{B}^\vee) - \chi(\mathcal{A} \otimes \mathcal{B}^\vee) \geq -\chi(\mathcal{A} \otimes \mathcal{B}^\vee).$$

The above inequality, equality (1.1) with $\mathcal{F} := \mathcal{A} \otimes \mathcal{B}^\vee$ and the equalities $\text{rk}(\mathcal{A} \otimes \mathcal{B}^\vee) = 4$, $c_1(\mathcal{A} \otimes \mathcal{B}^\vee) = 0$ and $c_2(\mathcal{A} \otimes \mathcal{B}^\vee) = 4\gamma_2 - \gamma_1^2$ imply

$$h^1(F, \mathcal{A} \otimes \mathcal{B}^\vee) \geq 4\gamma_2 - \gamma_1^2 - 4\chi(\mathcal{O}_F) = -\frac{1}{3}d(d^2 - 12d + 23) \geq 3$$

which we already know is at least 3 in the range $4 \leq d \leq 9$. We conclude that F is Ulrich–wild, again by Lemma 1.3.

Remark 3.5. Essentially the same argument used in the above proof and the results proved in [4] and [5] allow to prove the Ulrich–wildness of Abelian, bielliptic and Enriques surfaces.

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