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WEAK AND QUADRATIC HYPONORMALITY OF 2-VARIABLE WEIGHTED SHIFTS AND THEIR EXAMPLES

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Abstract. Recently, Curto, Lee and Yoon considered the properties (such as, hyponormality, subnormality, and flatness, etc.) for 2-variable weighted shifts and constructed several families of commuting pairs of subnormal operators such that each family can be used to answer a conjecture of Curto, Muhly and Xia negatively. In this paper, we consider the weak and quadratic hyponormality of 2-variable weighted shifts (W_1, W_2). In addition, we detect the weak and quadratic hyponormality with some interesting 2-variable weighted shifts.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. For $S, T \in \mathcal{L}(\mathcal{H})$, we denote the commutator of S and T by $[S,T] := ST - TS$. Let $\mathbb{N}$ (resp., $\mathbb{Z}_+, \mathbb{R}_+, \mathbb{C}$) be the set of positive integers (resp., nonnegative integers, nonnegative real numbers, complex numbers). For $n \geq 1$, we write $\mathcal{H}^{(n)}$ for the orthogonal direct sum of $\mathcal{H}$ with itself n times. For n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$, we write $[\mathbb{T}^*, \mathbb{T}] \in \mathcal{L}\left(\mathcal{H}^{(n)}\right)$ for the self-commutator of $\mathbb{T}$, where (i,j)-entry $[\mathbb{T}^*, \mathbb{T}]_{ij}$ of $[\mathbb{T}^*, \mathbb{T}]$ is $\left[T_j^*, T_i\right]$. We say that an n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ is (*jointly hyponormal*) if the operator matrix

$$[\mathbb{T}^*, \mathbb{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on $\mathcal{H}^{(n)}$ ([5]). The n-tuple $\mathbb{T}$ is said to be *normal* if $\mathbb{T}$ is commuting and each T_i is normal. And $\mathbb{T}$ is *subnormal* if $\mathbb{T}$ is restriction of a normal n-tuple to a common invariant subspace. It is obvious that normal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal ([6]). The n-tuple $\mathbb{T}$ is (*weakly*) *hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$

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is hyponormal for every $\lambda_i \in \mathbb{C}, i = 1, \ldots, n$ ([8]). Because the structure of n-tuple operators can be extended from the study of 2-tuple operators, many operator theorists have concentrated their studies to the structure of n-tuple operators (see [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], etc.). Curto, Lee and Yoon considered the properties (such as, hyponormality, subnormality, flatness, etc.) for 2-variable weighted shifts and constructed several families of commuting pairs of subnormal operators such that each family can be used to answer a conjecture of Curto, Muhly and Xia negatively (see [3], [4], [5], [6], [7], etc.). The present author considered the subnormal completion problem by using the moment theory in [11]. In [12], one considered the expansivity of 2-variable weighted shifts and obtained some related results. In this paper, we discuss the weak and quadratic hyponormality of 2-variable weighted shifts with some interesting examples.

Let $\mathbb{C}[z, w]$ be the set of two variables complex polynomials. A 2-tuple commuting operator (T_1, T_2) is *weakly k-hyponormal* if $(p_1(T_1, T_2), p_2(T_1, T_2))$ is hyponormal for all polynomials $p_1, p_2 \in \mathbb{C}[z, w]$ with $\deg p_1, \deg p_2 \leq k \in \mathbb{N}$. And 2-tuple commuting operator (T_1, T_2) is *mono-weakly k-hyponormal* if $p(T_1, T_2)$ is hyponormal for all polynomials $p \in \mathbb{C}[z, w]$ with $\deg p \leq k$ (see [9], [10]). Thus, for 2-tuple commuting operator (T_1, T_2), we know that mono-weakly 1-hyponormal is just weakly hyponormal. For simplicity, we call mono-weakly 2-hyponormal is *quadratically hyponormal.* For 2-tuple commuting operator (T_1, T_2), it is well known that the hyponormality implies the weak hyponormality (cf. [5]). Obviously, if (T_1, T_2) is quadratically hyponormal, then (T_1, T_2) is weakly hyponormal. In terms of above discussions it is worthwhile studying the weak and quadratic hyponormality of (T_1, T_2).

We now discuss some basic construction for our purpose. Let $\mathcal{H} = \ell^2(\mathbb{Z}_+^2)$ be the usual Hardy space of square-summable complex sequences, where $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$. Consider a canonical orthonormal basis $\left\{e_{(i,j)}\right\}_{(i,j)\in\mathbb{Z}_+^2}$ of $\ell^2(\mathbb{Z}_+^2)$. Let $\alpha = \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}, \beta = \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ be two double-indexed positive bounded sequences. The 2-variable weighted shift $\mathbf{W} = (W_\alpha, W_\beta)$ is defined by

$$W_\alpha e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\epsilon_1} \quad \text{and} \quad W_\beta e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\epsilon_2}, \qquad \forall \mathbf{k} \in \mathbb{Z}_+^2, \tag{1.1}$$

where $\epsilon_1 := (1, 0), \epsilon_2 := (0, 1)$ (see [6], [7]). Clearly, $W_\alpha W_\beta = W_\beta W_\alpha$ if and only if $\beta_{\mathbf{k}+\epsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\epsilon_2}\beta_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. We now consider the ordered orthonormal basis $\boldsymbol{E}$ with the lexicographic order (i.e., $(0,0), (0,1), (1,0), (0,2), (1,1), \ldots$) in the indices of $e_{(i,j)}, (i,j) \in \mathbb{Z}_+^2$. According to (1,1), the shift W_α on $\ell^2(\mathbb{Z}_+^2)$ can be represented by a matrix form

$$W_\alpha \cong \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & \cdots \\ 0 & \mathbf{0} & 0 & 0 & \cdots \\ \alpha_{00} & 0 & \mathbf{0} & 0 & \cdots \\ 0 & 0 & 0 & \mathbf{0} & \cdots \\ 0 & \alpha_{01} & 0 & 0 & \cdots \\ 0 & 0 & \alpha_{10} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_{02} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

corresponding to the ordered basis $\boldsymbol{E}$ of $\ell^2\left(\mathbb{Z}_+^2\right)$ (cf. [12]); note that the diagonal entries of the above matrix is zero and we denote if by "**0**" for reader's convenience. Similarly, the matrix form associated to the shift W_β with respect to $\boldsymbol{E}$ is

$$W_\beta \cong \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & \cdots \\ \beta_{00} & \mathbf{0} & 0 & 0 & \cdots \\ 0 & 0 & \mathbf{0} & 0 & \cdots \\ 0 & \beta_{01} & 0 & \mathbf{0} & \cdots \\ 0 & 0 & \beta_{10} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \beta_{02} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

The hyponormality of (W_α, W_β) was characterized in [7], namely, it is hyponormal if and only if

$$\begin{pmatrix} \alpha_{k_1+1,k_2}^2 - \alpha_{k_1,k_2}^2 & \alpha_{k_1,k_2+1}\beta_{k_1+1,k_2} - \alpha_{k_1,k_2}\beta_{k_1,k_2} \\ \alpha_{k_1,k_2+1}\beta_{k_1+1,k_2} - \alpha_{k_1,k_2}\beta_{k_1,k_2} & \beta_{k_1,k_2+1}^2 - \beta_{k_1,k_2}^2 \end{pmatrix} \geq 0$$

for all $(k_1, k_2) \in \mathbb{Z}_+^2$; this test for hyponormality is called "Six-point Test" which will be used in this paper.

The paper consists of as following. In Section 2, we give the criteria of weak and quadratic hyponormality for a pair of 2-variable weighted shifts. In Section 3 we detect the weak and quadratic hyponormality with useful 2-variable weighted shifts which have been studied by several operator theorists.

2. The criteria of weak and quadratic hyponormality

Let $\alpha = \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}, \beta = \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ be two double-indexed positive bounded sequences and let (W_α, W_β) be 2-variable weighted shifts. In this section, we characterize the weak and quadratic hyponormality of (W_α, W_β). Firstly, we discuss the weak hyponormality of (W_α, W_β). By a direct computation, we get that

$$\left[(W_\alpha + \lambda W_\beta)^*, W_\alpha + \lambda W_\beta\right] = \operatorname{diag}\{M_j\}_{j=0}^{\infty},$$

where $M_0 = (z_{00})$ and

$$M_k = \begin{pmatrix} z_{01} & h_{01} & & \\ h_{01} & z_{1,k-1} & \ddots & \\ & \ddots & \ddots & h_{k-1,1} \\ & & h_{k-1,1} & z_{k,0} \end{pmatrix}, \quad k \in \mathbb{N}, \tag{2.1}$$

and

$$z_{ij} = \left(\alpha_{ij}^2 - \alpha_{i-1,j}^2\right) + |\lambda|^2\left(\beta_{ij}^2 - \beta_{i,j-1}^2\right),$$
$$h_{ij} = \begin{cases} 0 & \text{if} \quad j = 0, \\ \lambda\left(\alpha_{ij}\beta_{i+1,j-1} - \alpha_{i,j-1}\beta_{i,j-1}\right) & \text{if} \quad j \geq 1. \end{cases}$$

Note that $\alpha_{ij} = 0$ and $\beta_{ij} = 0$ for $i < 0$ or $j < 0$. We now obtain the following proposition by some direct computations.

Proposition 2.1. *Let (W_α, W_β) be a 2-variable weighted shifts with weight sequences α and β. Then (W_α, W_β) is weakly hyponormal if and only if $M_k \geq 0$ for all $k \in \mathbb{Z}_+$.*

Next we consider the quadratic hyponormality. Recall that 2-tuple commuting operator (T_1, T_2) is quadratically hyponormal if $\left(T_1, T_2, T_1^2, T_1T_2, T_2^2\right)$ is weakly hyponormal. We denote $\mathbf{T} := \lambda_1 T_1 + \lambda_2 T_2 + \mu_1 T_1^2 + \mu_2 T_2^2 + \mu_3 T_1 T_2$. Then (T_1, T_2) is quadratically hyponormal

$$
\begin{aligned}
&\Longleftrightarrow \mathbf{T} \text{ is hyponormal} \\
&\Longleftrightarrow [\mathbf{T}^*, \mathbf{T}] \geq 0 \\
&\Longleftrightarrow \left[\left(T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2 + \lambda_3 T_1 T_2\right)^*, T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2 + \lambda_3 T_1 T_2\right] \geq 0 \\
&\Longrightarrow \left[\left(T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2\right)^*, T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2\right] \geq 0
\end{aligned}
$$

for any $\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$.

We remark that the above necessary conditions can be replaced by the necessary and sufficient conditions for our key example in Section 3. So, in this paper, we just simply say that (T_1, T_2) is quadratically hyponormal, if

$$\left[\left(T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2\right)^*, T_1 + \lambda_1 T_1^2 + \lambda_2 T_2^2\right] \geq 0, \qquad \forall \lambda_1, \lambda_2 \in \mathbb{C}.$$

Now we consider 2-variable weighted shifts (W_α, W_β). By direct computations, we have

$$
\begin{aligned}
M &:= \left[\left(W_\alpha + \lambda_1 W_\alpha^2 + \lambda_2 W_\beta^2\right)^*, W_\alpha + \lambda_1 W_\alpha^2 + \lambda_2 W_\beta^2\right] \\
&= \begin{pmatrix}
q_{00} & 0 & r_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & q_{01} & 0 & 0 & r_{01} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\bar{r}_{00} & 0 & q_{10} & \eta_{02} & 0 & r_{10} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \bar{\eta}_{02} & q_{02} & 0 & \delta_{02} & 0 & r_{02} & 0 & 0 & 0 & \cdots \\
0 & \bar{r}_{01} & 0 & 0 & q_{11} & 0 & \eta_{03} & 0 & r_{11} & 0 & 0 & \cdots \\
0 & 0 & \bar{r}_{10} & \bar{\delta}_{02} & 0 & q_{20} & 0 & \eta_{12} & 0 & r_{20} & 0 & \cdots \\
0 & 0 & 0 & 0 & \bar{\eta}_{03} & 0 & q_{03} & 0 & \delta_{03} & 0 & 0 & \cdots \\
0 & 0 & 0 & \bar{r}_{02} & 0 & \bar{\eta}_{12} & 0 & q_{12} & 0 & \delta_{12} & \eta_{04} & \cdots \\
0 & 0 & 0 & 0 & \bar{r}_{11} & 0 & \bar{\delta}_{03} & 0 & q_{21} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \bar{r}_{20} & 0 & \bar{\delta}_{12} & 0 & q_{30} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\eta}_{04} & 0 & 0 & q_{04} & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
q_{ij} = {} & \left(\alpha_{ij}^2 - \alpha_{i-1,j}^2\right) + |\lambda_1|^2 \left(\alpha_{ij}^2 \alpha_{i+1,j}^2 - \alpha_{i-2,j}^2 \alpha_{i-1,j}^2\right) \\
& + |\lambda_2|^2 \left(\beta_{ij}^2 \beta_{i,j+1}^2 - \beta_{i,j-2}^2 \beta_{i,j-1}^2\right), \\
r_{ij} = {} & \lambda_1 \alpha_{ij} \left(\alpha_{i+1,j}^2 - \alpha_{i-1,j}^2\right),
\end{aligned}
$$

$$\delta_{ij} = \lambda_2 \left(\alpha_{ij}\beta_{i+1,j-2}\beta_{i+1,j-1} - \alpha_{i,j-2}\beta_{i,j-2}\beta_{i,j-1}\right) \quad (j \geq 2)\,,$$
$$\eta_{ij} = \lambda_1\lambda_2 \left(\alpha_{ij}\alpha_{i+1,j}\beta_{i+2,j-2}\beta_{i+2,j-1} - \alpha_{i,j-2}\alpha_{i+1,j-2}\beta_{i,j-2}\beta_{i,j-1}\right) \quad (j \geq 2)\,.$$

Note that $\alpha_{ij} = 0$ and $\beta_{ij} = 0$ for $i < 0$ or $j < 0$.

Let $d_{ij} = (i+1) + \frac{(j+1)(j+2)}{2}$ and M_{ij} be the upper-left $d_{ij} \times d_{ij}$-submatrix of M and let $\Delta_{ij} := \det M_{ij}$. Let $M_{i,j}^{[1]}$ be the submatrix of M_{ij} such that its entries are $q_{\star,k}, r_{\star,k}$ and $\eta_{\star,k} = 0$, where k's are odd numbers and $\star$ means a nonnegative integer. And $M_{i,j}^{[2]}$ be the submatrix of M_{ij} such that its entries are $q_{\star,k}, r_{\star,k}$ and η_{02}, where k's are even numbers. For example, if $j = 2k+1$, then

$$M_{m,2k+1}^{[1]} = \begin{pmatrix} q_{01} & r_{01} & & & & \\ \bar{r}_{01} & q_{11} & \eta_{03} & r_{11} & & \\ & \bar{\eta}_{03} & q_{03} & \ddots & \ddots & \\ & \bar{r}_{11} & \ddots & \ddots & \ddots & * \\ & & \ddots & \ddots & \ddots & * \\ & & & * & * & q_{m,2k+1} \end{pmatrix},$$

and if $j = 2k$, then

$$M_{m,2k}^{[2]} = \begin{pmatrix} q_{00} & r_{00} & & & & \\ \bar{r}_{00} & q_{10} & \eta_{02} & r_{10} & & \\ & \bar{\eta}_{02} & q_{02} & \delta_{02} & \ddots & \\ & \bar{r}_{10} & \bar{\delta}_{02} & \ddots & \ddots & * \\ & & \ddots & \ddots & \ddots & * \\ & & & * & * & q_{m,2k} \end{pmatrix}.$$

We now give the following key lemma.

Lemma 2.2. *Under the above notation, we get*

(i) $\Delta_{i,2k} = \det M_{i,2k}^{[2]} \cdot \det M_{i+1,2k-1}^{[1]}$ *and* $\Delta_{i,2k+1} = \det M_{i+1,2k}^{[2]} \cdot \det M_{i,2k+1}^{[1]}$,

(ii) *it holds that*

$$\det M_{m,2k+1}^{[1]} = \begin{cases} \left(q_{31}q_{41}\cdot\dots\cdot q_{2k,1}\right)\cdot g_{2k+1}^{[0]}\cdot\left(g_3^{[1]}\cdot\dots\cdot g_{2k-1}^{[1]}\right) & \text{if } m=1, \\ \left(q_{31}q_{41}\cdot\dots\cdot q_{m,2k+1-m}\right)\cdot g_{2k-1}^{[0]}\cdot\left(g_3^{[1]}\cdot\dots\cdot g_{2k-1}^{[1]}\right) & \text{if } m\neq 1, \end{cases}$$

for $k \geq 2$, *where*

$$g_l^{[0]} := q_{0,l}q_{1,l} - |r_{0,l}|^2 \quad \text{and } g_l^{[1]} := -q_{0,l}\,|r_{1,l}|^2 - q_{2,l}\,|r_{0,l}|^2 + q_{0,l}q_{1,l}q_{2,l},$$

and

$$\det M_{m,2k}^{[2]} = \begin{cases} \left(q_{30}q_{40}\cdot\dots\cdot q_{2k,0}\right)\cdot g_{2k}^{[0]}\cdot\left(g_4^{[1]}\cdot\dots\cdot g_{2k-2}^{[1]}\right)\cdot\rho & \text{if } m=1, \\ \left(q_{30}q_{40}\cdot\dots\cdot q_{m,2k-m}\right)\cdot g_{2k-2}^{[0]}\cdot\left(g_4^{[1]}\cdot\dots\cdot g_{2k-2}^{[1]}\right)\cdot\rho & \text{if } m\neq 1, \end{cases}$$

for $k \geq 3$, *where*

$$(2.2) \qquad \rho := \det \begin{pmatrix} q_{00} & r_{00} & 0 & 0 & 0 & 0 \\ \bar{r}_{00} & q_{10} & \eta_{02} & 0 & 0 & 0 \\ 0 & \bar{\eta}_{02} & q_{02} & \delta_{02} & r_{02} & 0 \\ 0 & 0 & \bar{\delta}_{02} & q_{20} & 0 & 0 \\ 0 & 0 & \bar{r}_{02} & 0 & q_{12} & r_{12} \\ 0 & 0 & 0 & 0 & \bar{r}_{12} & q_{22} \end{pmatrix}.$$

We give a necessary condition of quadratic hyponormality of 2-variable weighted shifts as following.

Proposition 2.3. *Let* (W_α, W_β) *be a* 2*-variable weighted shifts with weight sequences* α *and* β*. If* (W_α, W_β) *is quadratically hyponormal, then*

$$F(t_1, t_2) := q_{02}q_{20} - |\delta_{02}|^2, \qquad \forall t_1 \geq 0, t_2 \geq 0$$

with $t_i = |\lambda_i|^2, i = 1, 2.$

Proof. In fact, $F(t_1, t_2) = \det M_{[4,6]} = q_{02}q_{20} - |\delta_{02}|^2$. If (W_α, W_β) is quadratically hyponormal, then $F(t_1, t_2) \geq 0, \forall t_1 \geq 0, t_2 \geq 0$. $\square$

According to Lemma 2.2, we obtain the following proposition.

Proposition 2.4. *Let* (W_α, W_β) *be a* 2*-variable weighted shifts with weight sequences* α *and* β*. Then* (W_α, W_β) *is quadratically hyponormal if and only if* $M^{[1]}_{m,2k+1} \geq 0$ *and* $M^{[2]}_{m,2k} \geq 0$ *for all* $m, k \in \mathbb{Z}_+$*.*

3. Examples

For $x, y \in (0, 1]$, and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2_+$, let

$$(3.1) \qquad \alpha(\mathbf{k}) := \begin{cases} x & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ y & \text{if } k_1 = 0 \text{ and } k_2 \geq 1, \\ 1 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 0, \end{cases}$$

and

$$(3.2) \qquad \beta(\mathbf{k}) := \begin{cases} x & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ y & \text{if } k_1 \geq 1 \text{ and } k_2 = 0, \\ 1 & \text{if } k_1 \geq 0 \text{ and } k_1 \geq 1. \end{cases}$$

We now let (W_1, W_2) be the pair of 2-variable weighted shifts on $\ell^2(\mathbb{Z}^2_+)$ defined by (3.1) and (3.2), whose weight sequence is given by Fig. 1 as following (cf. [7]).

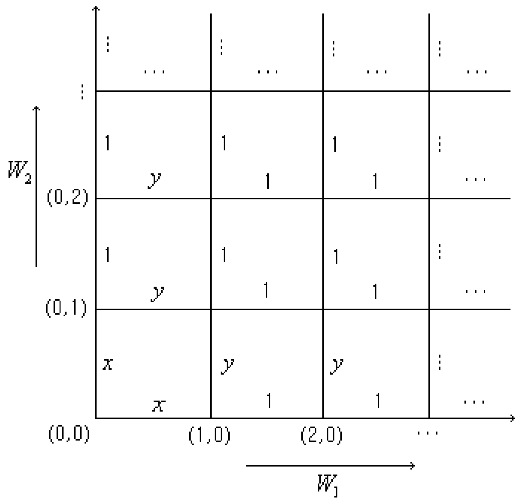

Fig. 1. Weight diagram of 2-variable weighted shifts defined by (3.1) and (3.2).

Obviously $W_1W_2 = W_2W_1$. Furthermore, we have the following results.

Proposition 3.1. *Let* (W_1, W_2) *be the 2-variable weighted shift with weight sequences defined by* (3.1) *and* (3.2)*. Then the following assertions hold.*

(i) (W_1, W_2) *is hyponormal if and only if* $1 - 2x^2 + y^2 \geq 0$.

(ii) (W_1, W_2) *is subnormal if and only if* $x^2y^2 - 2x^2 + 1 \geq 0$.

(iii) (W_1, W_2) *is weakly hyponormal if and only if* $2x^2y^2 - 2x^2 + 1 \geq 0$ *or* $2x^2y^2 - 2x^2 + 1 < 0$ *and*

$$\left(2y^2 - 1\right)\left(2x^2 - 1\right)\left(1 - 2x^2 + 2y^2\right) \leq 0.$$

Proof. (i) According to the Six-point Test, we need to check the positivity of the following four kinds 2×2 matrices,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1-y \end{pmatrix}, \quad \begin{pmatrix} 1-y & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1-x^2 & y^2-x^2 \\ y^2-x^2 & 1-x^2 \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} 1-x^2 & y^2-x^2 \\ y^2-x^2 & 1-x^2 \end{pmatrix} = \left(1-y^2\right)\left(1-2x^2+y^2\right),$$

thus, all four matrices are positive if and only if $1 - 2x^2 + y^2 \geq 0$.

(ii) See [7, Proposition 4.9].

(iii) Observe that (see Appendix)

$$M_0 = \left(x^2\left|\lambda\right|^2 + x^2\right), \quad M_1 = \begin{pmatrix} \left(1-x^2\right)\left|\lambda\right|^2 + y^2 & \left(y^2-x^2\right)\lambda \\ \left(y^2-x^2\right)\bar{\lambda} & 1-x^2+y^2\left|\lambda\right|^2 \end{pmatrix},$$

$$M_2 = \operatorname{diag}\left\{y^2, \left(\left|\lambda\right|^2+1\right)\left(1-y^2\right), y^2\left|\lambda\right|^2\right\},$$

$$M_3 = \operatorname{diag}\left\{y^2, 1-y^2, |\lambda|^2\left(1-y^2\right), y^2|\lambda|^2\right\},$$

and for $k \geq 4$,

$$M_k = \operatorname{diag}\left\{y^2, 1-y^2, 0, \ldots, 0, |\lambda|^2\left(1-y^2\right), y^2|\lambda|^2\right\}.$$

If (W_1, W_2) is weakly hyponormal, then

$$\det M_1 = \left(y^2 - x^2y^2\right)|\lambda|^4 + \left(2x^2y^2 - 2x^2 + 1\right)|\lambda|^2 + \left(y^2 - x^2y^2\right) \geq 0, \quad \forall \lambda \in \mathbb{C}.$$

Hence $\det M_1 \geq 0$ if and only if $2x^2y^2 - 2x^2 + 1 \geq 0$, or $2x^2y^2 - 2x^2 + 1 < 0$, and $\Delta \leq 0$, where Δ is the discriminant for quadratic polynomial $\det M_1$ in $t = |\lambda|^2$, i.e.,

$$\Delta := \left(2y^2 - 1\right)\left(2x^2 - 1\right)\left(1 - 2x^2 + 2y^2\right). \tag{3.3}$$

Conversely, if $\Delta \leq 0$, then the block matrices M_k $(k \in \mathbb{Z}_+)$ are all positive. By Proposition 2.1, we know that (W_1, W_2) is weakly hyponormal. □

We now discuss the quadratic hyponormality of 2-variable weighted shift (W_1, W_2) as above.

Theorem 3.2. *The 2-variable weighted shift* (W_1, W_2) *with weight sequences defined by* (3.1) *and* (3.2) *is quadratically hyponormal if and only if* $2x^2y^2 - 2x^2 + 1 \geq 0$ *or* $2x^2y^2 - 2x^2 + 1 < 0$ *and* $\Delta \leq 0$*, where* Δ *is as* (3.3).

Proof. $(\Rightarrow)$ Since (W_1, W_2) is quadratically hyponormal, by Proposition 2.3, we have (see Appendix)

$$\begin{aligned} F(t_1, t_2) &:= q_{02}q_{20} - |\delta_{02}|^2 \\ &= y^2t_1^2 + c_1t_1 + c_2 \geq 0, \quad \forall t_1 \geq 0, t_2 \geq 0, \end{aligned}$$

where

$$\begin{aligned} c_1 &= \left(1 - x^2 + y^4\right)t_2 + y^2\left(2 - x^2\right) \geq 0, \\ c_2 &= \left(y^2 - x^2y^2\right)t_2^2 + \left(2x^2y^2 - 2x^2 + 1\right)t_2 + \left(y^2 - x^2y^2\right). \end{aligned}$$

Thus $c_2 \geq 0$ if and only if $2x^2y^2 - 2x^2 + 1 \geq 0$, or $2x^2y^2 - 2x^2 + 1 < 0$ and $\Delta \leq 0$, where Δ is as (3.3).
$(\Leftarrow)$ To check the positivity of matrices $M^{[1]}_{m,2k+1}$ and $M^{[2]}_{m,2k}$ (see Appendix), we first consider the matrices $M^{[1]}_{m,2k+1}$. Let $M^{[*,l]}_{\dagger,\natural}$ be the truncations to the first l rows and columns of matrix $M^{[*]}_{\dagger,\natural}$ and let $\Delta^{[*,l]}_{\dagger,\natural} = \det\left(M^{[*,l]}_{\dagger,\natural}\right)$. If $t_1 = t_2 = 0$, then $M^{[1]}_{m,2k+1}$ and $M^{[2]}_{m,2k}$ are all diagonal matrices and positive. So we can

assume that $t_1 > 0$ or $t_2 > 0$. We observe that

$$M_{2,3}^{[1]} = \begin{pmatrix} q_{01} & r_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{r}_{01} & q_{11} & 0 & r_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{03} & 0 & r_{03} & 0 & 0 & 0 \\ 0 & \bar{r}_{11} & 0 & q_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{r}_{03} & 0 & q_{13} & 0 & 0 & r_{13} \\ 0 & 0 & 0 & 0 & 0 & q_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{05} & 0 \\ 0 & 0 & 0 & 0 & \bar{r}_{13} & 0 & 0 & q_{23} \end{pmatrix}.$$

By Appendix, we know that $\Delta_{2,3}^{[1,l]} > 0$ for all $l = 1, 2, \ldots, 8$. Thus $M_{2,3}^{[1]}$ is positive. Similarly, for the matrix $M_{m,2k+1}^{[1]}$ $(k \geq 2)$, we have

$$\begin{aligned} g_1^{[0]} &= y^2 t_1^2 + \left(y^2 t_2 - y^4 + y^2 + t_2\right) t_1 + \left(y^2 - y^4 + t_2^2 + t_2\right) > 0, \\ g_3^{[0]} &= y^2 t_1^2 + \left(y^2 - y^4\right) t_1 + \left(y^2 - y^4\right) \geq 0, \\ g_{2k+1}^{[0]} &= y^2 t_1^2 + \left(y^2 - y^4\right) t_1 + \left(y^2 - y^4\right) \geq 0, \\ g_1^{[1]} &= \left(y^2 - y^4\right) t_1^3 + t_2 \left(y^2 - y^4 + 1\right) t_1^2 + 2t_2 \left(t_2 + y^2 - y^4\right) t_1 \\ &\quad + t_2 \left(t_2 + y^2\right) \left(t_2 + 1 - y^2\right) \geq 0, \\ g_{2k+1}^{[1]} &= y^2 \left(1 - y^2\right) t_1^3 \geq 0 \quad (k \geq 1), \end{aligned}$$

by Lemma 2.2 and using the Nested determinant test ([1] or [2]), the matrices $M_{m,2k+1}^{[1]}$ are all positive.

Next, we consider the matrices $M_{m,2k}^{[2]}$. Observe that

$$\rho(t_1, t_2) = x^2 \left(1 - y^2\right) f(t_1, t_2)$$

as in (2.2), where $f(t_1, t_2) := \sum_{i=0}^{6} b_i t_1^{6-i}$ with

$$\begin{aligned} b_0 &= y^2 \left(1 - x^2\right), \\ b_1 &= \left(-x^2 y^2 - 2x^2 + 3y^2 + 1\right) t_2, \\ b_2 &= \left(x^2 y^6 - x^2 y^4 + 2x^2 y^2 - 6x^2 - y^6 + y^4 + 4y^2 + 3\right) t_2^2 + 2y^2 \left(1 - x^2\right) t_2, \\ b_3 &= \left(y^2 - x^2 y^2\right) t_2 + \left(2x^2 y^4 - 6x^2 - 2y^6 + y^4 + 5y^2 + 3\right) t_2^3 \\ &\quad + \left(4x^2 y^6 - 2x^2 y^4 - 3x^2 y^2 - 2x^2 - 6y^6 + 5y^4 + 4y^2 + 1\right) t_2^2, \\ b_4 &= \left(6x^2 y^4 - 2x^2 y^6 - 5x^2 y^2 - 2x^2 - 2y^6 + 5y^2 + 1\right) t_2^4 \\ &\quad + \left(5y^2 - 5y^4 + 1\right) \left(1 - 2x^2 + 2y^2\right) t_2^3 \\ &\quad + \left(6x^2 y^6 - 4x^2 y^4 - 3x^2 y^2 - 8y^6 + 7y^4 + 2y^2\right) t_2^2, \\ b_5 &= \left(2x^2 y^4 - 3x^2 y^2 - 2y^6 + y^4 + 2y^2\right) t_2^5 \\ &\quad + \left(14x^2 y^4 - 4x^2 y^6 - 11x^2 y^2 - 6y^6 + y^4 + 6y^2\right) t_2^4 \\ &\quad + 4y^2 \left(1 - y^2\right) \left(1 - 2x^2 + 2y^2\right) t_2^3 + 4y^4 \left(1 - x^2\right) \left(1 - y^2\right) t_2^2, \end{aligned}$$

$$\begin{aligned} b_6 = {} & y^4\left(1-x^2\right)\left(1-y^2\right)t_2^6 + y^2\left(1-y^2\right)\left(1-2x^2+2y^2\right)t_2^5 \\ & + 2y^2\left(1-y^2\right)\left(x^2y^2-2x^2+y^2+1\right)t_2^4 \\ & + y^2\left(1-y^2\right)\left(1-2x^2+2y^2\right)t_2^3 + y^4\left(1-x^2\right)\left(1-y^2\right)t_2^2. \end{aligned}$$

It is not difficult to show that $b_i \geq 0$ $(i = 0,1,2,3,4,5,6)$ if the conditions in the hypothesis are satisfied. Thus, $\rho(t_1,t_2) \geq 0$ for all $t_1,t_2 \in \mathbb{R}_+$. Furthermore,

$$\begin{aligned} g_0^{[0]} = {} & x^2t_1^2 + \left(x^2y^2t_2 - x^4 + x^2t_2 + x^2\right)t_1 + x^2\left(t_2+1\right)\left(y^2t_2 - x^2 + 1\right), \\ g_2^{[1]} = {} & \left(1-y^2\right)y^2t_1^3 + \left(1-y^2\right)\left(2y^2 - y^4 - x^2 + 1\right)t_2t_1^2 \\ & + t_2\left(1-y^2\right)\left(\left(x^2y^2 - 2x^2 - y^4 + 2\right)t_2 + \left(2y^2 - 2y^4\right)\right)t_1 \\ & + t_2\left(t_2+1\right)\left(1-y^2\right)^2\left(-x^2t_2 + t_2 + y^2\right), \\ g_{2k}^{[1]} = {} & t_1^3y^2\left(1-y^2\right), \end{aligned}$$

we know that $g_0^{[0]}, g_2^{[1]}, \ldots, g_{2k}^{[1]}$ are all positive. By Lemma 2.2 and using the Nested determinant test, we can show that the matrices $M_{m,2k}^{[2]}$ are all positive. By Proposition 2.4, we know that (W_1,W_2) is quadratically hyponormal. □

In particular, if we let $x = 1, y = a \in (0,1]$, then the weight sequence is the following (cf. [5])

$$\alpha(\mathbf{k}) := \begin{cases} 1 & \text{if } k_1 \geq 1 \quad \text{or} \quad k_2 = 0, \\ a & \text{if } k_1 = 0 \quad \text{and} \quad k_2 \geq 1, \end{cases} \tag{3.4}$$

and

$$\beta(\mathbf{k}) := \begin{cases} 1 & \text{if } k_1 = 0 \quad \text{or} \quad k_2 \geq 1, \\ a & \text{if } k_1 \geq 1 \quad \text{and} \quad k_2 = 0. \end{cases} \tag{3.5}$$

Let (W_1,W_2) be the pair of 2-variable weighted shifts on $\ell^2\left(\mathbb{Z}_+^2\right)$ defined by (3.4) and (3.5). By Proposition 3.1 and Theorem 3.2, we have the following results.

Corollary 3.3. *Let (W_1,W_2) be the 2-variable weighted shift with weight sequences defined by (3.4) and (3.5). Then the following assertions hold.*
(i) (W_1,W_2) *is hyponormal if and only if* $a = 1$.
(ii) (W_1,W_2) *is subnormal if and only if* $a = 1$.
(iii) (W_1,W_2) *is weakly hyponormal if and only if* $\frac{\sqrt{2}}{2} \leq a \leq 1$.
(iv) (W_1,W_2) *is quadratically hyponormal if and only if* $\frac{\sqrt{2}}{2} \leq a \leq 1$.

Remark. The following two figures Fig. 2 and Fig. 3 provide the regions of subnormality, hyponormality, weak hyponormality and quadratic hyponormality for the 2-variable weighted shift with weight sequences defined by (3.1) and (3.2), from which we know the following relationship

$$\text{subnormal} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{hyponormal} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{weakly or quadratically hyponormal,}$$

and we know that the regions of weak hyponormality and quadratic hyponormality for the 2-variable weighted shift with weight sequences defined by (3.1) and (3.2) are same. However, in general, the former contains the latter. So we may try to find a 2-variable weighted shifts that the regions of weak and quadratic hyponormality are different. We leave it to interesting readers.

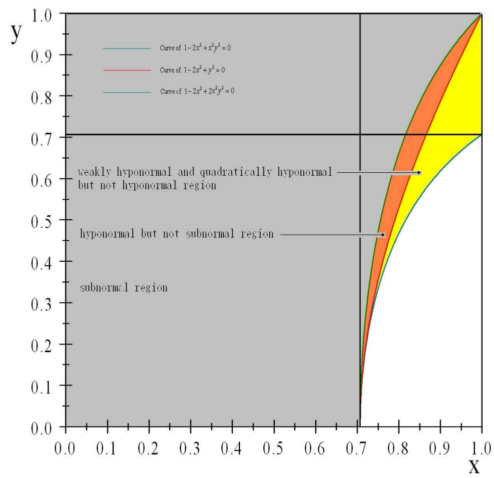

Fig. 2. Case of $2x^2y^2 - 2x^2 + 1 \geq 0$.

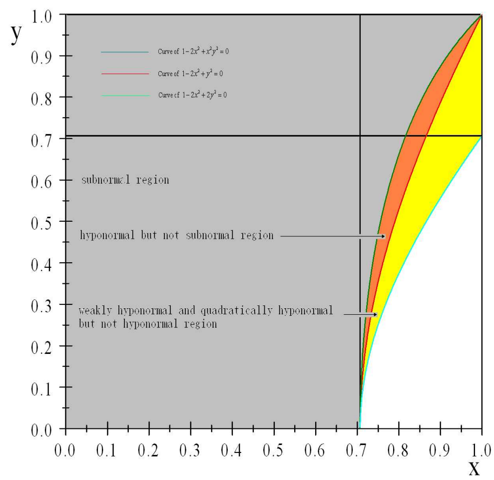

Fig. 3. Case of $2x^2y^2 - 2x^2 + 1 < 0$.

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4. Appendix

We give some exact values appeared in this paper for reader's convenience.

I. $\boldsymbol{z_{ij}, h_{ij}, q_{ij}, r_{ij}, \delta_{ij}, \eta_{ij}}$ **in Section 3**

$$\begin{aligned}
&z_{00} = x^2 |\lambda|^2 + x^2,\\
&z_{0k} = y^2 \quad (\forall k \geq 1),\\
&z_{10} = -x^2 + y^2 |\lambda|^2 + 1,\\
&z_{11} = \left(|\lambda|^2 + 1\right)\left(1 - y^2\right),\\
&z_{1k} = 1 - y^2 \quad (\forall k \geq 2),\\
&z_{20} = y^2 |\lambda|^2,\\
&z_{21} = |\lambda|^2 \left(1 - y^2\right),\\
&z_{2k} = 0 \quad (\forall k \geq 2),\\
&z_{m0} = y^2 |\lambda|^2,\\
&z_{m1} = |\lambda|^2 \left(1 - y^2\right) \quad (\forall m \geq 2),\\
&z_{mk} = 0 \quad (\forall m \geq 2, \forall k \geq 2).
\end{aligned}$$

$$h_{ij} = \begin{cases} \lambda\left(y^2 - x^2\right) & \text{for } i = 0, j = 1,\\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
&q_{00} = x^2 (1 + t_1 + t_2),\\
&q_{01} = y^2 + t_1 y^2 + t_2,\\
&q_{02} = y^2 (1 + t_1) + \left(1 - x^2\right) t_2,\\
&q_{0j} = y^2 + t_1 y^2 \quad (j \geq 3),\\
&q_{10} = t_1 + t_2 y^2 + 1 - x^2,\\
&q_{11} = \left(1 - y^2\right) + t_1 + t_2,\\
&q_{12} = \left(1 - y^2\right) + t_1 + t_2 \left(1 - y^2\right),\\
&q_{1j} = \left(1 - y^2\right) + t_1 \quad (j \geq 3),\\
&q_{20} = t_2 y^2 + \left(1 - x^2\right) t_1,\\
&q_{21} = t_1 \left(1 - y^2\right) + t_2,\\
&q_{22} = \left(1 - y^2\right)(t_1 + t_2),\\
&q_{2j} = t_1 \left(1 - y^2\right) \quad (j \geq 3),\\
&q_{i0} = t_2 y^2,\\
&q_{i1} = t_2,\\
&q_{i2} = t_2 \left(1 - y^2\right), \quad (i \geq 3)\\
&q_{ij} = 0 \quad (i \geq 3 \text{ and } j \geq 3).
\end{aligned}$$

$$\begin{aligned}
&r_{00} = \lambda_1 x,\\
&r_{0j} = y \lambda_1 \quad (j \geq 1),\\
&r_{10} = \left(1 - x^2\right) \lambda_1,\\
&r_{1j} = \lambda_1 \left(1 - y^2\right) \quad (j \geq 1),\\
&r_{ij} = 0 \quad (i \geq 2 \text{ and } j \geq 0).
\end{aligned}$$

$$\eta_{ij} = \begin{cases} \lambda_2\left(y^2 - x^2\right) & \text{for } i = 0 \text{ and } j = 2,\\ 0 & \text{otherwise.} \end{cases}$$

$$\delta_{ij} = \begin{cases} \lambda_1\lambda_2\left(y^2 - x^2\right) & \text{for } i = 0 \text{ and } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

II. The determinants $\boldsymbol{\Delta_{2,3}^{[1,l]}}$ of matrix $\boldsymbol{M_{2,3}^{[1]}}$ in Section 3

$$\begin{aligned}
\Delta_{2,3}^{[1,1]} &= y^2 + t_2 y^2 + t_2, \\
\Delta_{2,3}^{[1,2]} &= y^2 t_1^2 + \left(t_2 + y^2 t_2 + y^2 - y^4\right) t_1 + \left(t_2^2 + t_2 + y^2 - y^4\right), \\
\Delta_{2,3}^{[1,3]} &= \left(1 + t_1\right) y^2 \Delta_{2,3}^{[1,2]}, \\
\Delta_{2,3}^{[1,4]} &= \left(1 + t_1\right) y^2 g_1^{[1]}, \\
\Delta_{2,3}^{[1,5]} &= y^2 \left(t_1^2 + \left(1 - y^2\right)\left(1 + t_1\right)\right) g_1^{[1]}, \\
\Delta_{2,3}^{[1,6]} &= t_2 \Delta_{2,3}^{[1,5]}, \\
\Delta_{2,3}^{[1,7]} &= t_2 \left(1 + t_1\right) y^2 \Delta_{2,3}^{[1,5]}, \\
\Delta_{2,3}^{[1,8]} &= t_1^3 t_2 \left(1 + t_1\right)\left(1 - y^2\right) y^4 g_1^{[1]},
\end{aligned}$$

where

$$\begin{aligned}
g_1^{[1]} = {} & \left(y^2 - y^4\right) t_1^3 + t_2\left(y^2 - y^4 + 1\right) t_1^2 + 2t_2\left(t_2 + y^2 - y^4\right) t_1 \\
& + t_2\left(t_2 + y^2\right)\left(t_2 + 1 - y^2\right).
\end{aligned}$$

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