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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE KAEHLER MANIFOLD WITH A NON-METRIC ϕ-SYMMETRIC CONNECTION

Dae Ho Jin

Abstract. We define a new connection on semi-Riemannian manifold, which is called a *non-metric ϕ-symmetric connection*. Semi-symmetric non-metric connection and quarter-symmetric non-metric connection are two impotent examples of this connection. The purpose of this paper is to study the geometry of lightlike hypersurfaces of an indefinite Kaehler manifold with a non-metric ϕ-symmetric connection.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric ϕ-symmetric connection* if it and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\theta(\bar{Y})\phi(\bar{X}, \bar{Z}) - \theta(\bar{Z})\phi(\bar{X}, \bar{Y}), \tag{1.1}$$

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}, \tag{1.2}$$

where ϕ and J are tensor fields of types $(0, 2)$ and $(1, 1)$ respectively, and θ is a 1-form associated with a smooth unit vector field ζ, which is called the *characteristic vector field* of $\bar{M}$, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Throughout this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

It is known [11, 12] that the induced linear connection ∇ on any 1-lightlike submanifold of an indefinite Kaehler manifold with a quarter-symmetric metric connection is an example of non-metric ϕ-symmetric connection. In case $\phi = \bar{g}$ in (1.1), the above connection $\bar{\nabla}$ is reduced to the quarter-symmetric non-metric connection [3, 4, 8, 9, 11, 12, 14]. In case $\phi = \bar{g}$ in (1.1) and $J = I$ in (1.2), where I is the identity tensor field of type $(1, 1)$, the above connection $\bar{\nabla}$ is reduced to so called the semi-symmetric non-metric connection [1, 2].

The subject of this paper is to study lightlike hypersurfaces of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$ with a non-metric ϕ-symmetric connection, in which the tensor field J defined by (1.2) is identical with the indefinite almost complex

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structure J of $\bar{M}$ and the tensor field ϕ in (1.1) is identical with the fundamental 2-form associated with the indefinite almost complex structure J, *i.e.*,

$$\phi(\bar{X},\bar{Y}) = \bar{g}(J\bar{X},\bar{Y}). \tag{1.3}$$

Remark 1.1. Denote $\widetilde{\nabla}$ by the Levi-Civita connection of $(\bar{M},\bar{g},J)$ with respect to the metric $\bar{g}$. We define a linear connection $\bar{\nabla}$ on $\bar{M}$ given by

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})J\bar{X}. \tag{1.4}$$

By directed calculations, we see that $\bar{\nabla}$ is a non-metric ϕ-symmetric connection.

Conversely if $\bar{\nabla}$ is a non-metric ϕ-symmetric connection, then we can write

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \psi(\bar{X},\bar{Y}). \tag{1.5}$$

Substituting (1.5) into (1.1) and using the fact that $\widetilde{\nabla}$ is metric, we have

$$\bar{g}(\psi(\bar{X},\bar{Y}),\bar{Z}) + \bar{g}(\psi(\bar{X},\bar{Z}),\bar{Y}) = \theta(\bar{Y})\phi(\bar{X},\bar{Z}) + \theta(\bar{Z})\phi(\bar{X},\bar{Y}). \tag{1.6}$$

Also, from (1.5) and the fact that $\widetilde{\nabla}$ is torsion-free, it follows that

$$\bar{T}(\bar{X},\bar{Y}) = \psi(\bar{X},\bar{Y}) - \psi(\bar{Y},\bar{X}).$$

Thus, by using (1.2), we obtain

$$\psi(\bar{X},\bar{Y}) - \psi(\bar{Y},\bar{X}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}. \tag{1.7}$$

Exchanging $\bar{X}$ with $\bar{Y}$ and $\bar{Y}$ with $\bar{X}$ to (1.6), we have

$$\bar{g}(\psi(\bar{Y},\bar{X}),\bar{Z}) + \bar{g}(\psi(\bar{Y},\bar{Z}),\bar{X}) = \theta(\bar{X})\phi(\bar{Y},\bar{Z}) + \theta(\bar{Z})\phi(\bar{Y},\bar{X}).$$

Subtracting this equation from (1.6) and using (1.7), we obtain

$$\bar{g}(\psi(\bar{X},\bar{Z}),\bar{Y}) - \bar{g}(\psi(\bar{Y},\bar{Z}),\bar{X}) = 2\theta(\bar{Z})\phi(\bar{X},\bar{Y}). \tag{1.8}$$

Again from (1.7) we get

$$\bar{g}(\psi(\bar{X},\bar{Y}),\bar{Z}) - \bar{g}(\psi(\bar{Y},\bar{X}),\bar{Z}) = \theta(\bar{Y})\phi(\bar{X},\bar{Z}) - \theta(\bar{X})\phi(\bar{Y},\bar{Z}),$$
$$\bar{g}(\psi(\bar{X},\bar{Z}),\bar{Y}) - \bar{g}(\psi(\bar{Z},\bar{X}),\bar{Y}) = \theta(\bar{Z})\phi(\bar{X},\bar{Y}) - \theta(\bar{X})\phi(\bar{Z},\bar{Y}).$$

Adding these two equations and using (1.6), we have

$$\bar{g}(\psi(\bar{Y},\bar{X}),\bar{Z}) + \bar{g}(\psi(\bar{Z},\bar{X}),\bar{Y}) = 0.$$

Using this equation, (1.8) and the fact that $\bar{g}$ is non-degenerate, we obtain

$$\psi(\bar{X},\bar{Y}) = \theta(\bar{Y})J\bar{X}.$$

Thus the non-metric ϕ-symmetric connection $\bar{\nabla}$ satisfies (1.4). It follows that, for a linear connection $\bar{\nabla}$ on an indefinite Kaehler manifold $(\bar{M},\bar{g},J)$, $\bar{\nabla}$ *is a non-metric ϕ-symmetric connection if and only if $\bar{\nabla}$ satisfies* (1.4).

In this paper, we study the geometry of lightlike hypersurfaces of an indefinite Kaehler manifold with a non-metric ϕ-symmetric connection. For the rest of this paper, by saying that the non-metric ϕ-symmetric connection we shall mean the non-metric ϕ-symmetric connection given by (1.4).

2. Lightlike hypersurfaces

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure satisfying

$$J^2\bar{X} = -\bar{X}, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0, \tag{2.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to $\bar{g}$. Let $\bar{\nabla}$ be a non-metric ϕ-symmetric connection on $\bar{M}$ given by (1.4). Then $(2.1)_3$ is reformed as follow:

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \theta(\bar{Y})\bar{X} + \theta(J\bar{Y})J\bar{X}. \tag{2.2}$$

Let (M, g) be a lightlike hypersurface of $\bar{M}$. The normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1, and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others.

A complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. It is known [6] that, for any null section ξ of $Rad(TM)$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.5}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, and τ and σ are 1-forms on TM.

For a lightlike hypersurface M of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g}, J)$, it is known ([6, Section 6.2], [10]) that $J(Rad(TM))$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1 such that $J(Rad(TM)) \cap J(tr(TM)) = \{0\}$.

Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(Rad(TM)) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_o.$$

In this case, the decomposition form of TM is reduced to

$$TM = D \oplus J(tr(TM)). \tag{2.7}$$

Consider two null vector fields U and V, and two 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{2.8}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.9}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying J to (2.9) and using (2.1) and (2.8), we have

$$F^2X = -X + u(X)U. \tag{2.10}$$

As $u(U) = 1$ and $FU = 0$, the set $(F,\, u,\, U)$ defines an indefinite almost contact structure on M and F is called the *structure tensor field* of M. But it is not an indefinite almost contact metric structure on M and satisfies

$$g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X). \tag{2.11}$$

3. Non-metric ϕ-symmetric connections

Let $(\bar{M}, \bar{g}, J)$ be an indefinite Kaehler manifold with a non-metric ϕ-symmetric connection $\bar{\nabla}$. Using (1.1), (1.2), (1.3), (2.3) and (2.9), we obtain

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) - \theta(Y)\phi(X, Z) - \theta(Z)\phi(X, Y), \tag{3.1}$$

$$T(X, Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}$$

$$B(X, Y) - B(Y, X) = \theta(Y)u(X) - \theta(X)u(Y), \tag{3.3}$$

$$\phi(X, Y) = g(FX, Y) + u(X)\eta(Y), \tag{3.4}$$

$$\phi(X, \xi) = u(X), \qquad \phi(X, N) = v(X), \qquad \phi(X, V) = 0, \qquad \phi(X, U) = -\eta(X), \tag{3.5}$$

where T is the torsion tensor with respect to ∇ and η is a 1-form such that

$$\eta(X) = \bar{g}(X, N).$$

Theorem 3.1. *There exist no lightlike hypersurfaces of an indefinite Kaehler manifold with a non-metric ϕ-symmetric connection such that B is symmetric.*

Proof. Assume that B is symmetric. From (3.3), we get $\theta(X)u(Y) = \theta(Y)u(X)$. Replacing Y by U to this equation, we have

$$\theta(X) = \theta(U)u(X).$$

Taking $X = \xi$ and $X = V$ to this equation by turns, we get $b = 0$, *i.e.*, the characteristic vector field ζ is tangent to M, and $\theta(V) = 0$ respectively.

As ζ is tangent to M, we have

$$u(\zeta) = g(\zeta, V) = \theta(V) = 0.$$

Taking $Y = \zeta$ to $\theta(Y)u(X) = \theta(X)u(Y)$, we get $u(X) = u(\zeta)\theta(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u(U) = 1$. Thus there exist no lightlike hypersurfaces of an indefinite Kaehler manifold with a non-metric ϕ-connection such that B is symmetric. □

Definition. A lightlike hypersurface M is called *totally umbilical* [6] if there exists a smooth function β on a coordinate neighborhood $\mathcal{U}$ such that

$$B(X, Y) = \beta g(X, Y).$$

Corollary 3.2. *There exist no totally umbilical lightlike hypersurfaces of an indefinite Kaehler manifold with a non-metric ϕ-connection.*

From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = bu(X), \qquad B(\xi, X) = 0, \tag{3.6}$$

where we set $a = \theta(N)$ and $b = \theta(\xi)$. From (2.3), (2.6) and (3.6), we obtain

$$\bar{\nabla}_X \xi = -A^*_\xi X - \sigma(X)\xi + bu(X)N. \tag{3.7}$$

The local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y) + bg(FX, Y) + u(X)\theta(Y), \tag{3.8}$$

$$C(X, PY) = g(A_N X, PY) + ag(FX, PY) + v(X)\theta(PY), \tag{3.9}$$

$$\bar{g}(A^*_\xi X, N) = 0, \qquad \bar{g}(A_N X, N) = -av(X), \tag{3.10}$$

$$\sigma(X) = \tau(X) - au(X) - bv(X). \tag{3.11}$$

Taking $X = \xi$ to (3.8) and using $(3.5)_1$, $(3.6)_2$ and the facts that ϕ is skew-symmetric and $S(TM)$ is non-degenerate, we obtain

$$A^*_\xi \xi = bV. \tag{3.12}$$

Applying $\bar{\nabla}_X$ to (2.8) and (2.9) and using (2.2), (2.3), (2.4), (2.9), (3.1), (3.4), $(3.5)_{3,4}$, (3.7) and (3.9), we have

$$\begin{aligned} B(X, U) &= u(A_N X) + \theta(U)u(X) \\ &= C(X, V) - \theta(V)v(X) + \theta(U)u(X), \end{aligned} \tag{3.13}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - aX + \theta(U)FX, \tag{3.14}$$

$$\nabla_X V = F(A^*_\xi X) - \sigma(X)V + bu(X)U - bX + \theta(V)FX, \tag{3.15}$$

$$(3.16)\qquad (\nabla_X F)Y = u(Y)A_N X - B(X,Y)U + \theta(Y)X + \theta(JY)FX,$$

$$(3.17)\qquad (\nabla_X u)Y = -u(Y)\tau(X) - B(X,FY) + \theta(JY)u(X),$$

$$(3.18)\qquad (\nabla_X v)Y = v(Y)\tau(X) - g(A_N X, FY) + \theta(Y)\eta(X) - a\{g(X,Y) - u(Y)v(X)\}.$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain two Gauss-Codazzi equations for M and $S(TM)$ such that

$$(3.19)\quad \bar{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) + B(T(X,Y),Z)\}N,$$

$$(3.20)\quad \bar{R}(X,Y)N = -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] + \tau(X)A_N Y - \tau(Y)A_N X + \{B(Y,A_N X) - B(X,A_N Y) + 2d\tau(X,Y)\}N,$$

$$(3.21)\quad R(X,Y)PZ = R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \sigma(X)C(Y,PZ) + \sigma(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi,$$

$$(3.22)\quad R(X,Y)\xi = -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \sigma(X)A^*_\xi Y + \sigma(Y)A^*_\xi X + \{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\sigma(X,Y)\}\xi.$$

4. Recurrent and Lie recurrent lightlike hypersurfaces

Definition. The structure tensor field F of M is said to be *recurrent* [13] if there exists a 1-form ϖ on TM such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.1. *Let M be a recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then*

(1) *the characteristic vector field ζ on $\bar{M}$ is tangent to M, i.e., $b = 0$,*
(2) *F is parallel with respect to the induced connection ∇ on M,*
(3) *the 1-form θ vanishes, i.e., $\theta = 0$, on M,*
(4) *D and $J(tr(TM))$ are parallel distributions on M, and*
(5) *M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of the distribution D.*

Proof. (1) From the above definition and (3.16), we get

$$(4.1)\qquad \varpi(X)FY = u(Y)A_N X - B(X,Y)U + \theta(Y)X + \theta(JY)FX.$$

Replacing Y by ξ and using $(3.6)_1$ and the fact that $F\xi = -V$, we get

$$\varpi(X)V = bu(X)U - bX + \theta(V)FX.$$

Taking the scalar product with N to this equation, we obtain

$$b\eta(X) - \theta(V)v(X) = 0.$$

Taking $X = \xi$ and then $X = V$ to this equation, we have

$$b = 0, \qquad \theta(V) = 0.$$

As $b = 0$, the characteristic vector field ζ on $\bar{M}$ is tangent to M.

(2) As $b = 0$ and $\theta(V) = 0$, we see that $\varpi(X)V = 0$. Taking the scalar product with U to this result, we get $\varpi = 0$. It follows that $\nabla_X F = 0$. Thus F is parallel with respect to the induced connection ∇ on M.

(3) Taking the scalar product with N to (4.1) and using $(3.10)_2$, we have

$$-\,au(Y)v(X) + \theta(Y)\eta(X) + \theta(JY)v(X) = 0.$$

Replacing X by ξ to this equation, we obtain

$$\theta(X) = 0, \qquad \forall\, X \in \Gamma(TM). \tag{4.2}$$

(4) Taking the scalar product with V to (4.1) and using (4.2), we get

$$B(X, Y) = u(Y)u(A_{_N} X).$$

Taking $Y = V$ and $Y = FZ$, $Z \in \Gamma(D_o)$ to this equation by turns and using the fact that $u(FZ) = 0$ as $FZ = JZ \in \Gamma(D_o)$, we have

$$B(X, V) = 0, \qquad B(X, FZ) = 0. \tag{4.3}$$

In general, by using (2.11), (3.1), (3.4), $(3.5)_{1,3}$, (3.8) and (3.15), we derive

$$g(\nabla_X \xi, V) = -B(X, V) + \theta(V)u(X), \qquad g(\nabla_X V, V) = 0,$$
$$g(\nabla_X Z, V) = B(X, FZ) - \theta(FZ)u(X), \qquad \forall\, Z \in \Gamma(D_o),$$

due to $u(Z) = v(Z) = 0$. From these equations, (4.2) and (4.3), we see that

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(D).$$

It follows that D is a parallel distribution on M.

On the other hand, taking $Y = U$ to (4.1) and using (4.2), we have

$$A_{_N} X = B(X, U)U - aFX. \tag{4.4}$$

Applying F to (4.4) and using the facts that $FU = 0$ and $\theta(U) = 0$, we get

$$F(A_{_N} X) = aX - au(X)U.$$

Using this, (3.11) and the fact that $\theta = 0$, (3.14) is reduced to

$$\nabla_X U = \sigma(X)U. \tag{4.5}$$

It follows that $J(tr(TM))$ is also a parallel distribution on M, *i.e.*,

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM).$$

(5) As D and $J(tr(TM))$ are parallel distributions satisfying (2.7), by the decomposition theorem [5], M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* [13] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{4.6}$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then*

(1) *F is Lie parallel,*
(2) *the 1-forms τ and σ satisfy $\tau = au$ and $\sigma = -bv$, and*
(3) *the shape operator A^*_ξ satisfies $A^*_\xi U = A^*_\xi V = 0$.*

Proof. (1) Using the above definition, (2.9), (2.10), (3.2) and (3.16), we get

$$\begin{aligned}\vartheta(X)FY = -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X - B(X,Y)U \\ + au(Y)FX + \theta(Y)u(X)U.\end{aligned} \tag{4.7}$$

Taking $Y = \xi$ to (4.7) and using $(3.6)_1$ and the fact that $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X. \tag{4.8}$$

Taking the scalar product with V to (4.8) and using $g(FX, V) = 0$, we have

$$u(\nabla_V X) = g(\nabla_V X, V) = 0. \tag{4.9}$$

Replacing Y by V to (4.7) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U + \theta(V)u(X)U.$$

Applying F to this equation and using (3.11) and (4.9), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X.$$

Comparing this equation with (4.8), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N to (4.7) and using $(3.10)_2$, we have

$$-\bar{g}(\nabla_{FY}X, N) + \bar{g}(F\nabla_Y X, N) = 0. \tag{4.10}$$

Replacing X by V to (4.10) and using (2.10) and (3.15), we have

$$g(A^*_\xi FY, U) + \sigma(Y) = 0.$$

Taking $X = FY$ and $Y = U$ to (3.8) and using $(3.5)_4$ and (3.11), we have

$$B(FY, U) = -\tau(Y) + au(Y).$$

Replacing Y by U to this and using the fact that $FU = 0$, we obtain

$$\tau(U) = a. \tag{4.11}$$

Replacing X by ξ to (4.10) and using (2.6), (2.8) and (2.10), we have

$$g(A^*_\xi X, U) = \sigma(FX).$$

Taking $Y = U$ to (3.8) and using the last equation, $(3.5)_4$, (3.11), (4.11) and the facts that $v(FX) = -\eta(X)$ and $u(FX) = 0$, we have

$$B(X, U) = \tau(FX) + \theta(U)u(X). \tag{4.12}$$

From this equation and (3.13), we see that

$$u(A_N X) = \tau(FX). \tag{4.13}$$

Replacing X by U to (4.7) and using (2.10), (3.3) and (3.14), we get

$$u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U = 0.$$

Taking the scalar product with V and using (4.13), we get

$$\tau(FY) = 0.$$

Taking $Y = FX$ to this and using (2.10), (4.11) and then, (3.11), we have

$$\tau(X) = au(X), \qquad \sigma(X) = -bv(X).$$

(3) Replacing Y by U to (3.3) and using (4.12), we have

$$B(U, X) = \theta(X). \tag{4.14}$$

Taking $X = U$ to (3.8) and using (4.14), we have $g(A^*_\xi U, X) = 0$, As $S(TM)$ is non-degenerate, we get $A^*_\xi U = 0$. Replacing X by ξ to (4.8) and using (2.6), (3.12) and the facts that $F\xi = V$ and $\sigma(X) = -bv(X)$, we obtain $A^*_\xi V = 0$. □

5. Indefinite complex space forms

An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4}\{&\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} \\ &- \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\}.\end{aligned} \tag{5.1}$$

Comparing the tangential and transversal components of the two equations (3.19) and (5.1), and using (2.9) and (3.2), we get

$$\begin{aligned}R(X, Y)Z = \;& B(Y, Z)A_N X - B(X, Z)A_N Y \\ &+ \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + \bar{g}(JY, Z)FX \\ &- \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ\},\end{aligned} \tag{5.2}$$

$$\begin{aligned}&(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &+ \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ &- \theta(X)B(FY, Z) + \theta(Y)B(FX, Z)\end{aligned} \tag{5.3}$$

$$= \frac{c}{4}\{u(X)g(FY,Z) - u(Y)g(FX,Z) + 2u(Z)\bar{g}(X,JY)\}.$$

Taking the scalar product with N to (3.21) and then, substituting (5.2) into the resulting equation and using (2.9), (3.2) and $(3.10)_2$, we obtain

$$\begin{aligned}(5.4)\qquad & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ & - \sigma(X)C(Y,PZ) + \sigma(Y)C(X,PZ)\\ & - \theta(X)C(FY,PZ) + \theta(Y)C(FX,PZ)\\ & + a\{v(X)B(Y,PZ) - v(Y)B(X,PZ)\}\\ = & \frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + v(X)g(FY,PZ)\\ & - v(Y)g(FX,PZ) + 2v(PZ)\bar{g}(X,JY)\}.\end{aligned}$$

Theorem 5.1. *Let M be a recurrent lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-connection. Then $\bar{M}(c)$ is flat, i.e., $c = 0$, and the 1-form σ is closed, i.e., $d\sigma = 0$ on M. Moreover, there exists a null pair $\{\xi, N\}$ on a coordinate neighborhood $\mathcal{U}$ such that the corresponding 1-forms σ and τ satisfy $\sigma = 0$ and $\tau = au$.*

Proof. Taking the scalar product with U to (4.4) and using (3.9), we have

$$C(X,U) = 0,$$

due to $(3.5)_4$ and $\theta = 0$. Applying ∇_Y to this and using (4.5), we get

$$(\nabla_X C)(Y,U) = 0.$$

Replacing PZ by U to (5.4) and using the last two equations, we obtain

$$(5.5)\qquad a\{B(Y,U)v(X) - B(X,U)v(Y)\} = \frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\}.$$

Taking $Y = V$ to (3.3) and using $(4.3)_1$ and the fact that $\theta = 0$, we have

$$(5.6)\qquad B(V,X) = 0.$$

Taking $X = \xi$ and $Y = V$ to (5.5) and using $(3.6)_2$ and (5.6), we get $c = 0$.

As $c = 0$, taking $Y = V$ to (5.5) and using (5.6), we have

$$aB(X,U) = 0.$$

Substituting (4.4) into (5.2) and using (4.2) and the last equation, we get

$$R(X,Y)U = 0.$$

On the other hand, by directed calculations from (4.5), we obtain

$$R(X,Y)U = 2d\sigma(X,Y)U.$$

Comparing the last two equations, we have $d\sigma = 0$.

As $d\sigma = 0$, there exists a smooth function f on $\mathcal{U}$ such that $\sigma = df$. Thus $\sigma(X) = X(f)$. If we take $\bar{\xi} = \alpha\xi$, then $\bar{N} = (1/\alpha)N$ and $\sigma(X) = \bar{\sigma}(X) + X(\text{In}\,\alpha)$. Setting $\alpha = \exp(f)$ in this equation, we get $\bar{\sigma} = 0$. Therefore, there exists a null pair $\{\xi, N\}$ on $\mathcal{U}$ such that $\sigma = 0$ and τ satisfies $\tau = au$. □

Theorem 5.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-symmetric connection. Then*

$$c = 4\{ab - Ub\}.$$

Proof. Replacing Z by ξ to (5.2) and then, comparing with (3.22), we have

$$\begin{aligned} &- \nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \sigma(X)A^*_\xi Y + \sigma(Y)A^*_\xi X \\ &+ \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\sigma(X,Y)\}\xi \\ = \ & b\{u(Y)A_N X - u(X)A_N Y\} + \frac{c}{4}\{u(Y)FX - u(X)FY - 2\bar{g}(X, JY)V\}, \end{aligned}$$

due to $(3.6)_1$. Taking the scaler product with N, we obtain

$$C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\sigma(X,Y) = \{ab - c/4\}\{u(X)v(Y) - u(Y)v(X)\}.$$

Taking $X = U$ and $Y = V$ to this and using (3) of Theorem 4.2, we get

$$2d\sigma(U,V) = -ab + c/4.$$

By directed calculation from $\sigma(X) = -bv(X)$, (3.2) and (3.18), we derive

$$\begin{aligned} 2d\sigma(X,Y) = \ & -(Xb)v(Y) + (Yb)v(X) + ab\{u(X)v(Y) - u(Y)v(X)\} \\ & + b\{v(X)\tau(Y) - v(Y)\tau(X) + g(A_N X, FY) - g(A_N Y, FX)\}. \end{aligned}$$

Taking $X = U$ and $Y = V$ to this equation and using (4.11), we get

$$2d\sigma(U,V) = -Ub,$$

due to $FV = \xi$ and $FU = 0$. Thus we have $c = 4\{ab - Ub\}$. □

Corollary 5.3. *Let M be a Lie recurrent lightlike hypersurface of $\bar{M}(c)$ with a non-metric ϕ-symmetric connection. If $b = 0$, then $c = 0$ and $\bar{M}(c)$ is flat.*

Definition. A screen distribution $S(TM)$ is called *totally umbilical* [6] in M if there exists a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X, PY) = \gamma g(X,Y).$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 5.4. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-connection. If $S(TM)$ is totally umbilical, then*

$$c = 4\gamma\theta(V).$$

Moreover, if $S(TM)$ is totally geodesic in M, then $c = 0$.

Proof. From (3.3) and (3.13), we see that

$$B(U,V) = 0, \qquad B(V,U) = -\theta(V). \tag{5.7}$$

Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (3.1) and (3.4), we get

$$\begin{aligned} (\nabla_X C)(Y, PZ) = \ & (X\gamma)g(Y, PZ) \\ & + \gamma\{B(X, PZ)\eta(Y) - \theta(Y)g(FX, PZ) - \theta(PZ)\phi(X,Y)\}. \end{aligned}$$

Substituting this equation into (5.4), we obtain

$$\begin{aligned}&\{X\gamma-\gamma\sigma(X)\}g(Y,PZ)-\{Y\gamma-\gamma\sigma(Y)\}g(X,PZ)\\&+\ \{\gamma\eta(Y)-av(Y)\}B(X,PZ)-\{\gamma\eta(X)-av(X)\}B(Y,PZ)\\&-\ 2\gamma\theta(PZ)\phi(X,Y)\\=\ &\frac{c}{4}\{\eta(X)g(Y,PZ)-\eta(Y)g(X,PZ)+v(X)g(FY,PZ)\\&\quad-\ v(Y)g(FX,PZ)+2v(PZ)\bar{g}(X,JY)\}.\end{aligned}$$

Replacing X by ξ to this and using $(3.5)_1$ and $(3.6)_2$, we have

$$\begin{aligned}&\{\xi\gamma-\gamma\sigma(\xi)\}g(Y,PZ)-\gamma B(Y,PZ)+2\gamma\theta(PZ)u(Y)\\=\ &\frac{c}{4}\{g(Y,PZ)+v(Y)u(PZ)+2u(Y)v(PZ)\}.\end{aligned}$$

Taking $Y=U$, $PZ=V$ and $Y=V$, $PZ=U$ by turns and using (5.7), we get

$$\xi\gamma-\gamma\sigma(\xi)=-2\gamma\theta(V)+\frac{3}{4}c,\qquad \xi\gamma-\gamma\sigma(\xi)=-\gamma\theta(V)+\frac{2}{4}c.$$

From these two equations, we have $c=4\gamma\theta(V)$. □

Definition. A lightlike hypersurface M is called *screen conformal* [10] if there exists a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ such that

$$C(X,PY)=\varphi B(X,Y).$$

Theorem 5.5. *Let M be a screen conformal lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-connection. Then $c=0$.*

Proof. Applying ∇_X to $C(Y,PZ)=\varphi B(Y,PZ)$, we have

$$(\nabla_X C)(Y,PZ)=(X\varphi)B(Y,PZ)+\varphi(\nabla_X B)(Y,PZ).$$

Substituting this equation into (5.4) and using (5.3), we obtain

$$\begin{aligned}&\{X\varphi-\varphi\tau(X)-\varphi\sigma(X)+av(X)\}B(Y,PZ)\\&-\ \{Y\varphi-\varphi\tau(Y)-\varphi\sigma(Y)+av(Y)\}B(X,PZ)\\=\ &\frac{c}{4}\{\eta(X)g(Y,PZ)-\eta(Y)g(X,PZ)\\&\quad+\ [v(X)-\varphi u(X)]g(FY,PZ)-[v(Y)-\varphi u(Y)]g(FX,PZ)\\&\quad+\ 2[v(PZ)-\varphi u(PZ)]\bar{g}(X,JY)\}.\end{aligned}$$

Replacing X by ξ to this equation and using $(3.6)_2$ and (3.11), we have

$$\begin{aligned}(5.8)\qquad&\{\xi\varphi-2\varphi\tau(\xi)\}B(Y,PZ)\\=\ &\frac{c}{4}\{g(Y,PZ)+[v(Y)-\varphi u(Y)]u(PZ)+2[v(PZ)-\varphi u(PZ)]u(Y)\}.\end{aligned}$$

Taking $Y=U$, $PZ=V$ and $Y=V$, $PZ=U$ to this by turns, we get

$$\{\xi\varphi-2\varphi\tau(\xi)\}B(U,V)=3c/4,\qquad \{\xi\varphi-2\varphi\tau(\xi)\}B(V,U)=2c/4.$$

As $B(U,V)=B(V,U)+\theta(V)$ by (3.3), we have

$$\{\xi\varphi-2\varphi\tau(\xi)\}\theta(V)=c/4. \tag{5.9}$$

Now we set $\mu=U-\varphi V$. From (3.13), we see that

$$B(X,\mu)=\theta(U)u(X)-\theta(V)v(X).$$

Taking $X=V$ to this equation, we obtain

$$B(V,\mu)=-\theta(V).$$

Taking $Y=V$ and $PZ=\mu$ to (5.8) and using the last equation, we have

$$\{\xi\varphi-2\varphi\tau(\xi)\}\theta(V)=-2c/4.$$

From this equation and (5.9), we obtain $c=0$. □

Theorem 5.6. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-symmetric connection. If V or U is parallel with respect to the induced connection ∇ on M, then $c=0$.*

Proof. (1) If U is parallel with respect to ∇, then, from (3.14), we have

$$F(A_{N}X)+\tau(X)U-aX+\theta(U)FX=0.$$

Taking the scalar product with N to this equation and using (3.9), we get

$$C(X,U)=0.$$

Applying ∇_X to $C(Y,U)=0$ and using the fact that $\nabla_X U=0$, we have

$$(\nabla_X C)(Y,U)=0.$$

Replacing PZ by U to (5.4) and using the last two equations, we obtain

$$a\{B(Y,U)v(X)-B(X,U)v(Y)\}=\frac{c}{2}\{v(Y)\eta(X)-v(X)\eta(Y)\}.$$

Taking $X=\xi$ and $Y=V$ and using $(3.6)_2$, we have $c=0$.

(2) If V is parallel with respect to ∇, then, from (3.15), we have

$$F(A^*_\xi X)-\sigma(X)V+bu(X)U-bX+\theta(V)FX=0.$$

Taking the scalar product with N to this and using (3.8) and (3.13), we have

$$C(X,V)=0.$$

Applying ∇_X to $C(Y,V)=0$ and using the fact that $\nabla_X V=0$, we have

$$(\nabla_X C)(Y,V)=0.$$

Replacing PZ by V to (5.4) and using the last two equations, we obtain

$$a\{B(Y,V)v(X)-B(X,V)v(Y)\}=\frac{c}{4}\{u(Y)\eta(X)-u(X)\eta(Y)+2\bar{g}(X,JY)\}.$$

Taking $X=\xi$ and $Y=U$ and using $(3.6)_2$, we have $c=0$. □

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Dae Ho Jin
Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr