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ON LEBESGUE NONLINEAR TRANSFORMATIONS

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Abstract. In this paper, we introduce a quadratic stochastic operators on the set of all probability measures of a measurable space. We study the dynamics of the Lebesgue quadratic stochastic operator on the set of all Lebesgue measures of the set $[0, 1]$. Namely, we prove the regularity of the Lebesgue quadratic stochastic operators.

1. Introduction

Quadratic stochastic operator (in short QSO) was first introduced in Bernstein's work [5]. The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modeling in various fields such as biology [18, 19], physics [31], game theory [8], control system [27, 28, 29]. Such operators frequently arise in many models of mathematical genetics [10, 11]. The analytic theory of stochastic processes generated by quadratic operators was established in [7, 30]. A fixed point set and an omega limiting set of quadratic stochastic operators defined on the finite dimensional simplex were deeply studied in [13, 14, 15, 16, 20, 21, 22]. Ergodicity and chaotic dynamics of quadratic stochastic operators on the finite dimensional simplex were studied in the papers [8, 11, 12, 23, 24, 25, 26, 32]. In the paper [9], the nonlinear Poisson quadratic stochastic operators over the countable state space was studied. Ergodic theory of quadratic stochastic operators on the infinite space was established in [1, 2, 3, 4]. In [17], it was given a self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators. In this paper, we shall study the dynamics of Lebesgue quadratic stochastic operators on the set of all Lebesgue measures of the set $[0, 1]$. Let us first recall some notions and notations (see [7, 9, 30]).

Let $(X, \mathbb{F})$ be a measurable space and $S(X, \mathbb{F})$ be the set of all probability measures on $(X, \mathbb{F})$, where X is a state space and $\mathbb{F}$ is σ-algebra of subsets of X. It is evident that the set $S(X, \mathbb{F})$ is a convex space and a form of Dirac

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measure δ_x which defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for any $A \in \mathbb{F}$ is an extremal element of $S(X, \mathbb{F})$.

Let $\{P(x, y, A) : x, y \in X, A \in \mathbb{F}\}$ be a family of functions on $X \times X \times \mathbb{F}$ that satisfy the following conditions:

(i) $P(x, y, \cdot) \in S(X, \mathbb{F})$, for any fixed $x, y \in X$, that is, $P(x, y, \cdot) : \mathbb{F} \to [0, 1]$ is the probability measure on $\mathbb{F}$;

(ii) $P(x, y, A)$ regarded as a function of two variables x and y with fixed $A \in \mathbb{F}$ is measurable function on $(X \times X, \mathbb{F} \otimes \mathbb{F})$;

(iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in X, A \in \mathbb{F}$.

We consider a nonlinear transformation (quadratic stochastic operator) $V : S(X, \mathbb{F}) \to S(X, \mathbb{F})$ defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A)d\lambda(x)d\lambda(x), \tag{1.1}$$

where $\lambda \in S(X, \mathbb{F})$ is an arbitrary initial probability measure and $A \in \mathbb{F}$ is an arbitrary measurable set.

Definition 1.1. A probability measure μ on $(X, \mathbb{F})$ is said to be discrete, if there exists a countable set of elements $\{x_1, x_2, \dots\} \subset X$, such that $\mu(\{x_i\}) = p_i$ for $i = 1, 2, \dots$, with $\sum_i p_i = 1$. Then $\mu(X \setminus \{x_1, x_2, \dots\}) = 0$ and for any $A \in \mathbb{F}, \mu(A) = \sum_{x_i \in A} \mu(\{x_i\})$.

A family $\{P(x, y, A) : x, y \in X, A \in \mathbb{F}\}$ on an arbitrary state space X, such that for any $x, y \in X$ a measure $P(x, y, \cdot)$ is a discrete measure, is shown in the following example.

Example 1. Let $(X, \mathbb{F})$ be a measurable space. For any $x, y \in X$ and $A \in \mathbb{F}$, assume

$$P(x, y, A) = \begin{cases} 0 & \text{if } x \notin A \text{ and } y \notin A, \\ \frac{1}{2} & \text{if } x \in A, y \notin A \text{ or } x \notin A, y \in A, \\ 1 & \text{if } x \in A \text{ and } y \in A. \end{cases}$$

It is easy to verify that the quadratic stochastic operator V generated by this family is identity transformation, that is for any measure $\lambda \in S(X, \mathbb{F})$ we have $V\lambda = \lambda$. In fact, for any $A \in \mathbb{F}$,

$$\begin{aligned} V\lambda(A) = & \int_X \int_X P(x, y, A)d\lambda(x)d\lambda(y) \\ = & \int_A \int_A 1 \cdot d\lambda(x)d\lambda(y) + \int_A \int_{A^c} \frac{1}{2} \cdot d\lambda(x)d\lambda(y) \\ & + \int_{A^c} \int_A \frac{1}{2} \cdot d\lambda(x)d\lambda(y) + \int_{A^c} \int_{A^c} 0 \cdot d\lambda(x)d\lambda(y) \\ = & \lambda^2(A) + \frac{1}{2}\lambda(A)(1 - \lambda(A)) + \frac{1}{2}(1 - \lambda(A))\lambda(A) \end{aligned}$$

$$= \lambda(A),$$

where $A^c = X \setminus A$.

Assume $\{V^n\lambda : n = 0, 1, 2, \ldots\}$ is the trajectory of QSO (1.1) starting from an initial point $\lambda \in S(X, \mathbb{F})$, i.e., a sequence of probability measures, where $V^{n+1}\lambda = V(V^n\lambda)$ for all $n = 0, 1, 2, \ldots$, with $V^0\lambda = \lambda$. In measure theory, there are various notions of the convergence of measures. In what follows, we assume that X is a compact metric space. We say that a sequence of probability measures μ_n *weak* converges* to μ, as $n \to \infty$ if for every continuous function f,

$$\int_X f(x)d\mu_n \to \int_X f(x)d\mu \quad \text{as} \quad n \to \infty.$$

It is known that if X be a compact metric space, then $S(X, \mathbb{F})$ is weak* compact [6]. For $(X, \mathbb{F})$ a measurable space, a sequence μ_n is said to *converge strongly* to a limit μ if

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)$$

for every set $A \in \mathbb{F}$.

Definition 1.2. A quadratic stochastic operator V is called a regular (weak regular), if for any initial measure $\lambda \in S(X, \mathbb{F})$, the strong limit (respectively weak limit) $\lim_{n\to\infty} V^n(\lambda) = \mu$ exists.

If a state space $X = \{1, 2, \ldots, m\}$ is a finite set and the corresponding σ-algebra is a power set $\mathcal{P}(X)$, i.e., the set of all subsets of X, then the set of all probability measures on $(X, \mathcal{F})$ has the following form:

(1.2)

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ for any } i, \text{ and } \sum_{i=1}^{m} x_i = 1\}$$

that is called a $(m-1)$-dimensional simplex. In this case for any $i, j \in X$ a probabilistic measure $P(i, j, \cdot)$ is a discrete measure with $\sum_{k=1}^{m} P(ij, \{k\}) = 1$, where $P(ij, \{k\}) \equiv P_{ij,k}$ and the corresponding qso V has the following form

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j \tag{1.3}$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \ldots, m$, where

$$\text{a) } P_{ij,k} \geq 0, \quad \text{b) } P_{ij,k} = P_{ji,k} \text{ for all } i, j, k; \quad \text{c) } \sum_{k=1}^{m} P_{ij,k} = 1.$$

Such operator can be reinterpreted in terms of evolutionary operator of free population and in this form it has a long history. Note that the theory of quadratic stochastic operators on the finite state space was well developed and the most substantial works were referred to such operators (see [17] for a survey). In the papers [1, 2, 3, 4, 9], the authors studied QSO on the infinite state space. In this paper, we construct the family of quadratic stochastic operators defined on the continual compact state space $X = [0,1]$ and investigate their asymptotic behavior.

Definition 1.3. A transformation V given by (1.1) is called a Lebesgue QSO, if $X = [0,1]$ and $\mathbb{F}$ is a Borel σ-algebra on $[0,1]$.

In the next sections, we present a family of Lebesgue QSO with measurable partitions.

2. A construction of Lebesgue qso

Let $X = [0,1]$ and $\mathbb{F}$ be a Borel σ-algebra on $[0,1]$. For any element $(x,y) \in X \times X$, we define a discrete probability $P(x,y,\cdot)$ as follows:

(2.1) (i) for $x < y$ assume $P(x,y,\{x\}) = p$ and $P(x,y,\{y\}) = q$,

(2.2) (ii) for $x = y$ assume $P(x,x,\{x\}) = 1$,

(2.3) (iii) for $x > y$ assume $P(y,x,\cdot) = P(x,y,\cdot)$,

where $p+q = 1$, with $p \geq 0$ and $q \geq 0$. Let V be a quadratic stochastic operator

$$(V\lambda)(A) = \int_X \int_X P(x,y,A)d\lambda(x)d\lambda(x), \tag{2.4}$$

generated by family of functions (2.1)-(2.3), where $\lambda \in S(X,\mathbb{F})$ is an arbitrary initial probability measure and $A \in \mathbb{F}$ is an arbitrary measurable set. This operator is a generalization of Volterra QSO (see [17]). Note that if $p = q = 0.5$, then the corresponding operator is the identity operator. We show that for any initial measure $\lambda \in S(X,\mathbb{F})$, there exists a limit of the sequence $\{V^n\lambda : n = 0,1,2,\ldots\}$.

3. A limit behaviour of the trajectories

In this section we study the limit behaviour of the trajectory $\{V^n\lambda : n = 0,1,2,\ldots\}$ for any initial measure $\lambda \in S(X,\mathbb{F})$.

3.1. A discrete initial measure λ

It is easy to verify that for any $a \in [0,1]$ an extremal Dirac measure δ_a is a fixed point of the operator V. Since $\delta_a(\{a\}) = 1$, then from (2.2) we have

$$(V\delta_a)(\{a\}) = \int_X \int_X P(x,y,\{a\})d\delta_a(x)d\delta_a(y) = P(a,a,\{a\}) = 1 = \delta_a(\{a\}), \tag{3.1}$$

that is the Dirac measure δ_a for any $a \in [0,1]$ is a fixed point.

Let a measure λ be a convex combination of two Dirac measures δ_a and δ_b, i.e., $\lambda = \alpha\delta_a + (1-\alpha)\delta_b$, where $\alpha \in [0,1]$ and $a, b \in [0,1]$ with $a < b$. After simple algebra, we have that

$$(V\lambda)(\{a\}) = \int_X \int_X P(x,y,\{a\})d\lambda(x)d\lambda(y) = \lambda(a)[\lambda(a) + 2p\lambda(b)], \tag{3.2}$$

and

$$(V\lambda)(\{b\}) = \int_X \int_X P(x, y, \{b\}) d\lambda(x) d\lambda(y) = \lambda(b)[\lambda(b) + 2q\lambda(a)], \tag{3.3}$$

i.e., $V\lambda$ is the convex combination of the same two Dirac measures δ_a and δ_b with

$$V\lambda = \alpha_1 \delta_a + (1 - \alpha_1)\delta_b,$$

where $\alpha_1 = \alpha[\alpha + 2p(1 - \alpha)]$ and $1 - \alpha_1 = (1 - \alpha)[1 - \alpha + 2q\alpha]$. Then it is evident that $V^2\lambda$ is the convex combination of the same two Dirac measures δ_a and δ_b with

$$V^2\lambda = \alpha_2 \delta_a + (1 - \alpha_2)\delta_b,$$

where $\alpha_2 = \alpha_1[\alpha_1 + 2p(1 - \alpha_1)]$ and $1 - \alpha_2 = (1 - \alpha_1)[1 - \alpha_1 + 2q\alpha_1]$. Thus, one can show that $V^n\lambda$ is the convex combination of the same two Dirac measures δ_a and δ_b with

$$V^n\lambda = \alpha_n \delta_a + (1 - \alpha_n)\delta_b,$$

where $\alpha_n = \alpha_{n-1}[\alpha_{n-1} + 2p(1 - \alpha_{n-1})]$ and $1 - \alpha_n = (1 - \alpha_{n-1})[1 - \alpha_{n-1} + 2q\alpha_{n-1}]$. After simple algebra, we have that

$$\lim_{n\to\infty} \alpha_n = \begin{cases} 1 & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2} \end{cases}$$

that is

$$\lim_{n\to\infty} V^n\lambda = \begin{cases} \delta_a & \text{if } p > \frac{1}{2} \\ \delta_b & \text{if } p < \frac{1}{2}. \end{cases}$$

Let a measure λ be a convex combination of n Dirac measures $\{\delta_{a_i}, i = 1, \dots, n\}$ i.e., $\lambda = \sum_{i=1}^n \alpha_i \delta_{a_i}$, where $\alpha_i \in [0, 1], i = 1, \dots, n$ and $a_i \in [0, 1], i = 1, \dots, n$ with $\sum_{i=1}^n \alpha = 1$ and $a_1 < a_2 < \cdots < a_n$. After simple algebra, we have that

$$\begin{aligned} (V\lambda)(\{a_1\}) &= \int_X \int_X P(x, y, \{a_1\}) d\lambda(x) d\lambda(y) \\ &= \lambda(a_1)[\lambda(a_1) + 2p(1 - \lambda(a_1))], \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} (V\lambda)(\{a_n\}) &= \int_X \int_X P(x, y, \{a_n\}) d\lambda(x) d\lambda(y) \\ &= \lambda(a_n)[\lambda(a_n) + 2q(1 - \lambda(a_n))]. \end{aligned} \tag{3.5}$$

As shown above, we have

$$\lim_{n\to\infty} V^n\lambda = \begin{cases} \delta_{a_1} & \text{if } p > \frac{1}{2}, \\ \delta_{a_n} & \text{if } p < \frac{1}{2}. \end{cases}$$

3.2. A continuous initial measure λ

Let $\lambda \in S(X, \mathbb{B})$ be a continuous probability measure and $A = [a, b] \in \mathbb{B}$ be a segment in X with $A^c = [0, a) \cup (b, 1]$. Then,

$$\begin{aligned} V\lambda(A) = & \int_a^b \int_a^b 1 \cdot d\lambda(x)d\lambda(y) + \int_a^b \int_0^a q \cdot d\lambda(x)d\lambda(y) \\ & + \int_a^b \int_b^1 p \cdot d\lambda(x)d\lambda(y) + \int_0^a \int_a^b q \cdot d\lambda(x)d\lambda(y) \\ & + \int_b^1 \int_a^b p \cdot d\lambda(x)d\lambda(y) + \int_0^a \int_0^a 0 \cdot d\lambda(x)d\lambda(y) \\ & + \int_0^a \int_b^1 0 \cdot d\lambda(x)d\lambda(y) + \int_b^1 \int_0^a 0 \cdot d\lambda(x)d\lambda(y) \\ & + \int_b^1 \int_b^1 0 \cdot d\lambda(x)d\lambda(y) \\ = & \lambda([a,b]) \left[\lambda([a,b]) + 2q\lambda([0,a)) + 2p\lambda((b,1])\right]. \end{aligned}$$

It is evident that the measure $V\lambda$ is absolutely continuous with respect to λ. Then according to the Radon-Nikodym Theorem, there exists non-negative measurable function $f_\lambda^{(1)} : X \to R$, so-called a density, such that

$$V\lambda(A) = \int_A f_\lambda^{(1)}(x)d\lambda(x). \tag{3.6}$$

The density functions are obtained as follows. For rather small segment $[x, x+\Delta x]$ we have

$$\begin{aligned} & V\lambda([x, x+\Delta x]) \\ & = \lambda([x, x+\Delta x])[\lambda([x, x+\Delta x]) + 2q\lambda([0, x)) + 2p\lambda((x+\Delta x, 1])] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} f_\lambda^{(1)}(x) &= \lim_{\Delta x \to 0} \frac{V\lambda([x, x+\Delta x])}{\lambda([x, x+\Delta x])} \\ &= \lim_{\Delta x \to 0} [\lambda([x, x+\Delta x]) + 2q\lambda([0, x)) + 2p\lambda((x+\Delta x, 1])] \\ &= 2q\lambda([0, x)) + 2p\lambda((x, 1]). \end{aligned}$$

Now consider a measure $V^2\lambda = V(V\lambda)$. It is evident that

$$V^2\lambda(A) = \int_A f_{V\lambda}^{(1)}(x)dV\lambda(x), \tag{3.8}$$

and since $V^2\lambda$ is absolutely continuous with respect to measure λ, we have

$$V^2\lambda(A) = \int_A f_\lambda^{(2)}(x)d\lambda(x). \tag{3.9}$$

According (3.7), we have

$$
\begin{aligned}
&V^2\lambda([x,x+\Delta x])\\
&= V\lambda([x,x+\Delta x])[V\lambda([x,x+\Delta x]) + 2qV\lambda([0,x)) + 2pV\lambda((x+\Delta x,1])]\\
&= \lambda([x,x+\Delta x])[\lambda([x,x+\Delta x]) + 2q\lambda([0,x)) + 2p\lambda((x+\Delta x,1])]\\
&\quad \cdot\{\lambda([x,x+\Delta x])[\lambda([x,x+\Delta x]) + 2q\lambda([0,x)) + 2p\lambda((x+\Delta x,1])]\\
&\quad + 2q\lambda([0,x])[\lambda([0,x]) + 2p\lambda((x,1])]\\
&\quad + 2p\lambda([x+\Delta x,1])[\lambda([x+\Delta x,1)) + 2q\lambda([0,x+\Delta x))\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
f_\lambda^{(2)}(x) = {}&[2q\lambda([0,x)) + 2p\lambda([x,1])]\{2q\lambda([0,x))[\lambda([0,x)) + 2p\lambda([x,1])]\\
&+ 2p\lambda([x,1))[\lambda([x,1)) + 2q\lambda([0,x])\}.
\end{aligned}
$$

Similarly, one can show that a measure $V^n\lambda$ is absolutely continuous with respect to λ for any n and

$$V^n\lambda(A) = \int_A f_\lambda^{(n)}(x)d\lambda(x). \tag{3.10}$$

Let $g_\lambda(x) = \lambda([0,x))$ and $g_\lambda^{(n)}(x) = g_\lambda^{(n-1)}(x)(g_\lambda^{(n-1)}(x)+2p(1-g_\lambda^{(n-1)}(x))$, for $n = 1,2,3,\ldots$, where $g_\lambda^{(0)}(x) = x$ and $g_\lambda^{(1)}(x) = g_\lambda(x)$. It is evident that

$$1 - g_\lambda^{(n)}(x) = (1-g_\lambda^{(n-1)}(x))(1-g_\lambda^{(n-1)}(x) + 2qg_\lambda^{(n-1)}(x)).$$

Then, since $\lambda([x,1)) = 1-\lambda([0,x))$, we have

$$f(x) = 2qx + 2p(1-x), \tag{3.11}$$

$$f_\lambda^{(1)}(x) = f(g_\lambda^{(1)}(x)), \tag{3.12}$$

and

$$f_\lambda^{(2)}(x) = f(g_\lambda^{(1)}(x))\cdot f(g_\lambda^{(2)}(x)). \tag{3.13}$$

Using induction, one can prove that for any n we have

$$f_\lambda^{(n)}(x) = \prod_{i=1}^n f(g_\lambda^{(i)}(x)). \tag{3.14}$$

It is easy to see that $f_\lambda^{(n)}(0) = (2p)^n$ and $f_\lambda^{(n)}(1) = (2q)^n$. Clearly, we have that

$$\int_0^1 f_\lambda^{(n)}(x)d\lambda(x) = 1,$$

$$f_\lambda^{(n)}(0)\to\infty \quad\text{and}\quad f_\lambda^{(n)}(1)\to 0 \quad \text{if } p > 1/2, \tag{3.15}$$

and

$$f_\lambda^{(n)}(0)\to 0 \quad\text{and}\quad f_\lambda^{(n)}(1)\to \infty \quad \text{if } p < 1/2. \tag{3.16}$$

If $\lambda = m$ is a usual Lebesgue measure on $[0,1]$, then $m([0,x)) = x$ and $g_m^{(1)}(x) = x$. In this case one can explicitly find the functions $f_m^{(n)}(x)$ for any n.

4. Regularity of Lebesgue qso

Now, we are aiming to study the limit behavior of the Radon-Nikodym derivatives $f_\lambda^{(n)}(\cdot)$ for $n \to \infty$. Let $f(x) = 2qx + 2p(1-x)$ and $G(x) = x(x + 2p(1-x))$ for $p, q \geq 0$ and $p + q = 1$. We always assume $p, q \neq \frac{1}{2}$. One can easily check that $G'(x) = f(x)$ and $G''(x) = f'(x) = 2(q-p)$. Since $f(x) \geq 0$ for any $x \in [0,1]$, the function $G : [0,1] \to [0,1]$ is increasing. Moreover, the function $f : [0,1] \to \mathbb{R}_+$ is increasing whenever $q > p$ (or equivalently $q > \frac{1}{2}$) and decreasing whenever $q < p$ (or equivalently $q < \frac{1}{2}$).

It is easy to check that $g_\lambda^{(n)}(x) = G\left(g_\lambda^{(n-1)}(x)\right)$. We know that $g_\lambda : [0,1] \to [0,1]$, $g_\lambda(x) = \lambda([0,x))$ is increasing. Without loss of generality, we can assume that the function $g_\lambda : [0,1] \to [0,1]$ is strictly increasing. It is clear that

$$(g_\lambda^{(n)}(x))' = G'\left(g_\lambda^{(n-1)}(x)\right) \cdot (g_\lambda^{(n-1)}(x))'.$$

Consequently, $g_\lambda^{(n)} : [0,1] \to [0,1]$, for all $n \in \mathbb{N}$, are strictly increasing functions.

Proposition 4.1. *Let $f_\lambda^{(n)} : [0,1] \to \mathbb{R}_+$, $n \in \mathbb{N}$ be functions given by* (3.14). *Then, the function $f_\lambda^{(n)}$ is increasing whenever $q > \frac{1}{2}$ and decreasing whenever $q < \frac{1}{2}$.*

Proof. We know that

$$(f_\lambda^{(n)}(x))' = \sum_{k=1}^{n} \left(\prod_{i=1,\ i\neq k}^{n} f(g_\lambda^{(i)}(x)) \right) \cdot f'(g_\lambda^{(k)}(x)) \cdot \left(g_\lambda^{(k)}(x)\right)'.$$

Since $\prod\limits_{i=1,\ i\neq k}^{n} f(g_\lambda^{(i)}(x)) \geq 0$ and $\left(g_\lambda^{(k)}(x)\right)' \geq 0$, the function $f_\lambda^{(n)}$ is increasing whenever $q > \frac{1}{2}$ and decreasing whenever $q < \frac{1}{2}$. This completes the proof. $\square$

Let $q > \frac{1}{2}$ (or equivalently $p < \frac{1}{2}$) and $\alpha_0 \in (2p, 1)$. Let $\beta_n \in (2p, 1)$ be a sequence with $\lim\limits_{n\to\infty} \beta_n = 1$ such that for any given n there exists the smallest $N_0(n) < n$ such that

$$\beta_n^{N_0(n)} < \frac{\alpha_0 - 2p}{2\,(1-2p)}.$$

Let $M_0(n) = \prod\limits_{i=1}^{N_0(n)} \left[2\,(1-2p)\,\beta_n^{i-1} + 2p\right]$ and $B_n = \dfrac{\beta_n - 2p}{1-2p}$ for any $n \in \mathbb{N}$. It is clear that $0 < B_n < 1$ and $\lim\limits_{n\to\infty} B_n = 1$.

Proposition 4.2. *Let $q > \frac{1}{2}$ and β_n, B_n be given as above. The following statements hold*:

(i) *One has that* $G(x) \le \beta_n x$ *for any* $x \in [0, B_n]$;
(ii) *One has that* $G[0, B_n] = [0, \beta_n B_n] \subset [0, B_n]$.

Proof. Since $0 \le x \le B_n = \frac{\beta_n - 2p}{1-2p}$, we have that $x + 2p(1-x) \le \beta_n$ or equivalently $G(x) \le \beta_n x$. On the other hand, since $G(x)$ is increasing, we get that $0 = G(0) \le G(x) \le G(B_n) = \beta_n B_n < B_n$. □

Corollary 4.3. *Let* $q > \frac{1}{2}$. *Then* $0 \le g_\lambda^{(i)}(x) \le \beta_n^{i-1} g_\lambda(x) < \beta_n^{i-1}$ *for all* $x \in [0, B_n]$ *and* $i = 2, \dots, n$.

Theorem 4.4. *Let* $q > \frac{1}{2}$ *and* $\alpha_0, N_0(n), M_0(n)$ *be defined as above. Then* $0 \le f_\lambda^{(n)}(x) \le M_0(n) \alpha_0^{n - N_0(n)}$ *for any* $x \in [0, B_n]$.

Proof. We know that $f_\lambda^{(n)}(x) = \prod_{i=1}^{n} f(g_\lambda^{(i)}(x))$. Due to Corollary 4.3, since f is the increasing function, we then obtain for any $x \in [0, B_n]$ that

$$f(g_\lambda^{(i)}(x)) < f\left(\beta_n^{i-1}\right) = 2\left(1-2p\right)\beta_n^{i-1} + 2p.$$

Since $2\left(1-2p\right)\beta_n^{i-1} + 2p \le \alpha_0$ for any $i \ge N_0$, we then have that

$$\begin{aligned} f_\lambda^{(n)}(x) &= \prod_{i=1}^{n} f(g_\lambda^{(i)}(x)) \\ &\le \prod_{i=1}^{n} f\left(\beta_n^{i-1}\right) \\ &\le \prod_{i=1}^{N_0(n)} \left[2\left(1-2p\right)\beta_n^{i-1} + 2p\right] \cdot \prod_{i=N_0(n)}^{n} \left[2\left(1-2p\right)\beta_n^{i-1} + 2p\right] \\ &\le M_0(n) \alpha_0^{n-N_0(n)} \end{aligned}$$

for any $x \in [0, B_n]$. This completes the proof. □

Similarly, one can prove the following result.

Theorem 4.5. *Let* $q < \frac{1}{2}$ *and* $\alpha_0 \in (2q, 1)$. *For any given* n, *there exist numbers* $N_0(n) \in \mathbb{N}$, $K_0(n) \in \mathbb{R}_+$, *and* $A_n \in (0,1)$ *with* $\lim_{n\to\infty} A_n = 0$ *such that* $0 \le f_\lambda^{(n)}(x) \le K_0(n) \alpha_0^{n-N_0(n)}$ *for any* $x \in [A_n, 1]$.

Hence, the sequence of functions $f_\lambda^{(n)}(x)$ has a tall spike at the end points of the segment $[0,1]$ whenever $p, q \neq \frac{1}{2}$. Consequently, the limit of this sequence of the functions $f_\lambda^{(n)}(x)$ is the Dirac delta function concentrated at the end points of the segment $[0,1]$.

Theorem 4.6. *Let* V *be the Lebesgue quadratic stochastic operator generated by a family of functions* (2.1)-(2.3). *Let* $\lambda \in S(X, \mathbb{F})$ *be an initial continuous measure such that* $g_\lambda : [0,1] \to [0,1]$, $g_\lambda(x) = \lambda([0,x))$ *is a strictly increasing* *function. Then, there exists a strong limit of the sequence of measures* $\{V^k\lambda\}$ *where*

$$\lim_{n\to\infty} V^n\lambda = \begin{cases} \delta_0 & \textit{if } q < \frac{1}{2} \\ \delta_1 & \textit{if } q > \frac{1}{2}. \end{cases}$$

Recall that if $p = q = \frac{1}{2}$, then the corresponding QSO is the identity transformation.

Corollary 4.7. *The Lebesgue quadratic stochastic operator V generated by family of functions* (2.1)-(2.3) *is a regular transformation.*

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