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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WITH MULTIPLE VALUES

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Abstract. In this paper, we consider some normality criteria concerning multiple values. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k be a positive integer and $\psi(z) \not\equiv 0, \infty$ be a meromorphic function in D. If, for each $f \in \mathcal{F}$ and $z \in D$, (1) $f(z) \neq 0$, and all of whose poles are multiple; (2) all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $k+3$ in D; (3) all poles of $\psi(z)$ have multiplicities at most k in D, then $\mathcal{F}$ is normal in D.

1. Introduction and main results

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [3, 5]).

We shall use the basic results and standard notations of Nevanlinna theory (see [4] and [8]): $T(r, f)$, $m(r, f)$, $N(r, f), \ldots$. Let $f(z)$ be a transcendental meromorphic function in the whose complex plane and k be a positive integer. Then

(1) the Nevanlinna's First Fundamental Theorem: $T(r, f) = m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + S(r, f)$, where $T(r, f)(= m(r, f) + N(r, f))$ is Nevanlinna's characteristic function.

(2) the logarithmic derivative theorem: $m(r, \frac{f^{(k)}}{f}) = S(r, f)$.

We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\}$$

as $r \to \infty$, possibly a set of finite measure.

L. Yang [7, Theorem 2], M. Fang [2, Corollary 2] and H. Chen [1] proved independently the following result.

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Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D and let k be a positive integer. If for every $f \in \mathcal{F}$, $f(z) \neq 0$ and all the roots of $f^{(k)}(z) = 1$ are of multiplicity $> k + 4 + \left[\frac{2}{k}\right]$ in D, then $\mathcal{F}$ is normal.*

Recently, L. Zhao [10] generalized Theorem A as follows.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k, p be two positive integers and $\psi(z)(\not\equiv 0)$ be a holomorphic function in D, and all zeros of $\psi(z)$ have multiplicities at most p in D. If, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *$f(z) \neq 0$, and all poles of $f(z)$ have multiplicities at least $p + 2$ in D;*

(2) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $(k+p+2)(p+1)+1$ in D;*

(3) *$f(z)$ has at least one poles,*

then $\mathcal{F}$ is normal in D.

A natural problem arises: What can we say if the holomorphic function $\psi(z)$ is meromorphic in Theorem B, and the multiplicities of zeros of $f^{(k)}(z) - \psi(z)$ can be reduced? In this paper, we study the problem and obtain the following result.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k be a positive integer and $\psi(z)(\not\equiv 0, \infty)$ be a meromorphic function in D. If, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *$f(z) \neq 0$, and all of whose poles are multiple;*

(2) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $k + 3$ in D;*

(3) *all poles of $\psi(z)$ have multiplicities at most k in D,*

then $\mathcal{F}$ is normal in D.

As an immediate consequence of Theorem 1, we have the following result.

Corollary. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k be a positive integer, and $\psi(z)(\not\equiv 0)$ be a holomorphic function in D. If, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *$f(z) \neq 0$, and all of whose poles are multiple;*

(2) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $k + 3$ in D,*

then $\mathcal{F}$ is normal in D.

Remark 1. Clearly, from Corollary, Theorem 1 generalizes and improves Theorem B by allowing $\psi(z)$ to be meromorphic.

Example 1. Let k be a positive integer, $\Delta = \{z : |z| < 1\}$, $\psi(z) = \frac{1}{z^{k+2}}$, and

$$\mathcal{F} = \{f_n(z) = \frac{1}{nz^2} : z \in \Delta \text{ and } n \neq (-1)^k (k+1)!\}.$$

Clearly, $f_n(z) \neq 0$ and all of whose poles are multiple. We also have $f_n^{(k)}(z) - \psi(z) = (\frac{(-1)^k (k+1)!}{n} - 1)\frac{1}{z^{k+2}} \neq 0$. Thus conditions (1) and (2) in Theorem 1 are satisfied. Set $z_n = \frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i}{\sqrt{n}}$, we have $\lim\limits_{n\to\infty} z_n = 0$. Clearly, $\lim\limits_{n\to\infty} \frac{|f'(z_n)|}{1+|f(z_n)|^2} = \infty$, by Marty's Theorem [5], we have that $\mathcal{F}$ is not normal at $z_0 = 0$.

Remark 2. The above example shows that the restriction on the multiplicities of the poles of $\psi(z)$ in Theorem 1 is indispensable comparing to the holomorphic function $\psi(z)$ in Corollary.

2. Some lemmas

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version for $f(z) \neq 0$, which is due to Xue and Pang [6] and Zalcman [9].

Lemma 1. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D such that $f(z) \neq 0$ and all poles of functions in f have multiplicity greater than or equal to j. Let α be a real number satisfying $-\infty < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in D$, if and only if there exist*

(a) *points z_n, $z_n \to z_0$;*

(b) *functions $f_n \in \mathcal{F}$; and*

(c) *positive numbers $\rho_n \to 0^+$ such that $\rho_n^\alpha f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$. Moreover, the order of $g(\xi)$ is less than 2 and the poles of $g(\xi)$ are of multiplicity $\geq j$.*

Here, as usual, $g^{\#}(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

Lemma 2 (See [10, Lemma 2.2]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D and k be a positive integer, and let $b(z)(\neq 0)$, $a_0(z)$, $a_1(z), \ldots, a_{k-1}(z)$ be analytic functions in D. If, for every function $f \in \mathcal{F}, f \neq 0$ and all poles of $f(z)$ are multiple, and all zeros of $f^{(k)}(z)+a_{k-1}(z)f^{(k-1)}(z)+\cdots+a_1(z)f'(z)+a_0(z)f(z)-b(z)$ have multiplicity at least $k+3$, then $\mathcal{F}$ is normal in D.*

Lemma 3. *Let $k > 0, l \geq 0$ be two integers, and let $f(z)$ be a non-constant rational function. If $f(z) \neq 0$, and all poles of $f(z)$ are multiple in $\mathbb{C}$, then $f^{(k)}(z) - z^l$ has at least one zero which has multiplicity $\leq k+2$ in $\mathbb{C}$.*

Proof. We may assume that all zeros of $f^{(k)}(z) - z^l$ have multiplicities at least $k+3$. Since $f(z) \neq 0$, we can deduce that $f(z)$ is a non-polynomial rational and has the following form

$$f(z) = \frac{A}{(z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2}\cdots(z-\alpha_t)^{n_t}}, \tag{2.3.1}$$

where A is a non-zero constant and $n_j \geqslant 2$ $(j = 1, 2, \ldots, t)$ are integers.

By mathematical induction, from (2.3.1), we have

$$f^{(k)}(z)=\frac{g_{k(t-1)}(z)}{(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}, \tag{2.3.2}$$

where $g_{k(t-1)}(z)$ is a polynomial.

We use $\deg(g(z))$ to denote the degree of a polynomial and easily obtain that

$$\deg(g_{k(t-1)}(z))=k(t-1). \tag{2.3.3}$$

Because all zeros of $f^{(k)}(z)-z^l$ are of multiplicity $\geq k+3$ in $\mathbb{C}$, so we can get

$$f^{(k)}(z)-z^l=B\frac{(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_s)^{m_s}}{(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}, \tag{2.3.4}$$

where B is a non-zero constant and $m_i\geqslant k+3$ $(i=1,2,\ldots,s)$ are integers.

For simplicity, we denote

$$m_1+m_2+\cdots+m_s=M\geq(k+3)s, \tag{2.3.5}$$

$$n_1+n_2+\cdots+n_t=N\geq 2t. \tag{2.3.6}$$

From (2.3.4), we have $s\geq 1$ and $M=N+kt+l\geq 2t+kt+l$, thus we can get

$$t\leq\frac{M-l}{2+k}. \tag{2.3.7}$$

From (2.3.2) and (2.3.4) we have

$$\frac{f^{(k)}(z)}{z^l}=\frac{g_{k(t-1)}(z)}{z^l(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}, \tag{2.3.8}$$

$$\frac{f^{(k)}(z)}{z^l}-1=B\frac{(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_s)^{m_s}}{z^l(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}. \tag{2.3.9}$$

We distinguish the following two cases.

Case 1. Assume that $\alpha_1\alpha_2\cdots\alpha_t\neq 0$.

From (2.3.8) and (2.3.9), by taking derivative once, we derive

$$\left(\frac{f^{(k)}(z)}{z^l}\right)'=\frac{g_{k(t-1)+t}(z)}{z^{l+1}(z-\alpha_1)^{n_1+k+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.3.10}$$

where $g_{k(t-1)+t}(z)$ is a polynomial and easily obtained that $\deg(g_{k(t-1)+t}(z))=k(t-1)+t$.

$$\left(\frac{f^{(k)}(z)}{z^l}\right)'=\frac{(z-\beta_1)^{m_1-1}(z-\beta_2)^{m_2-1}\cdots(z-\beta_s)^{m_s-1}g_{s+t}(z)}{z^{l+1}(z-\alpha_1)^{n_1+k+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.3.11}$$

where $g_{s+t}(z)$ is a polynomial.

By comparing the above equations, we deduce that

$$M-s\leq\deg(g_{k(t-1)+t}(z))=k(t-1)+t=(k+1)t-k$$

i.e.,

$$t \geq \frac{M+k-s}{k+1}. \tag{2.3.12}$$

By (2.3.7) and (2.3.12), we get $\frac{M+k-s}{k+1} \leq \frac{M-l}{2+k}$. Through a simple calculation, we have

$$M \leq (2+k)(s-k) - (k+1)l < (k+2)s.$$

Note that $s \geq 1$, so the above inequality contradicts $M \geq (k+3)s$.

Case 2. Assume that $\alpha_1\alpha_2 \cdots \alpha_t = 0$.

Without loss of generality, we may assume $\alpha_1 = 0$. From (2.3.8) and (2.3.9), we have

$$(\frac{f^{(k)}(z)}{z^l})' = \frac{g_{(k+1)(t-1)}(z)}{z^{n_1+k+l+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.3.10$'$}$$

where $g_{(k+1)(t-1)}(z)$ is a polynomial and easily obtained that $\deg(g_{(k+1)(t-1)}(z)) = (k+1)(t-1)$.

$$(\frac{f^{(k)}(z)}{z^l})' = \frac{(z-\beta_1)^{m_1-1}(z-\beta_2)^{m_2-1}\cdots(z-\beta_s)^{m_s-1}g_{s+t-1}(z)}{z^{n_1+k+l+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.3.11$'$}$$

where $g_{s+t-1}(z)$ is a polynomial.

Proceeding as in the proof for Case 1, we have a contradiction.

This completes the proof of Lemma 3. □

Lemma 4. *Let $k > 0, 0 \leq l \leq k$ be two integers, and let $f(z)$ be a non-constant rational function. If $f(z) \neq 0$, and all poles of $f(z)$ are multiple in $\mathbb{C}$, then $f^{(k)}(z) - \frac{1}{z^l}$ has at least one zero which has multiplicity $\leq k+2$ in $\mathbb{C}$.*

Proof. We may assume that all zeros of $f^{(k)}(z) - \frac{1}{z^l}$ have multiplicities at least $k+3$. Since $f(z) \neq 0$, we can deduce that $f(z)$ is a non-polynomial rational function and has the following form

$$f(z) = \frac{A}{(z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2}\cdots(z-\alpha_t)^{n_t}}, \tag{2.4.1}$$

where A is a non-zero constant and $n_j \geqslant 2$ $(j = 1, 2, \ldots, t)$ are integers.

By mathematical induction, from (2.4.1), we have

$$f^{(k)}(z) = \frac{g_{k(t-1)}(z)}{(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}, \tag{2.4.2}$$

where $g_{k(t-1)}(z)$ is a polynomial.

We use $\deg(g(z))$ to denote the degree of a polynomial and easily obtain that

$$\deg(g_{k(t-1)}(z)) = k(t-1). \tag{2.4.3}$$

We distinguish the following two cases.

Case 1. Assume that $\alpha_1\alpha_2 \cdots \alpha_t \neq 0$.

From (2.4.2), we have

$$f^{(k)}(z)-\frac{1}{z^l}=\frac{g_{k(t-1)}(z)z^l-(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}{z^l(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}. \tag{2.4.4}$$

Since all zeros of $f^{(k)}(z)-\frac{1}{z^l}$ are of multiplicity $\geq k+3$ in $\mathbb{C}$, so we can get

$$f^{(k)}(z)-\frac{1}{z^l}=B\frac{(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_s)^{m_s}}{z^l(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}, \tag{2.4.5}$$

where B is a non-zero constant and $m_i\geqslant k+3$ $(i=1,2,\ldots,s)$ are integers.

For simplicity, we denote

$$m_1+m_2+\cdots+m_s=M\geq(k+3)s, \tag{2.4.6}$$

$$n_1+n_2+\cdots+n_t=N\geq 2t. \tag{2.4.7}$$

From (2.4.2) and (2.4.5) we have

$$z^lf^{(k)}(z)=\frac{z^lg_{k(t-1)}(z)}{(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}=\frac{p(z)}{q(z)}, \tag{2.4.8}$$

$$z^lf^{(k)}(z)-1=B\frac{(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_s)^{m_s}}{(z-\alpha_1)^{n_1+k}(z-\alpha_2)^{n_2+k}\cdots(z-\alpha_t)^{n_t+k}}. \tag{2.4.9}$$

We know $l\leq k$ and $\deg(g_{k(t-1)}(z))=k(t-1)$, we get $\deg(p(z))\leq kt$ and $\deg(q(z))\geq kt+2t$, so $\deg(q(z))>\deg(p(z))$. Combining this with (2.4.9), we have $s\geq 1$. It follows from (2.4.9) that

$$M=N+kt\geq 2t+kt$$

i.e.,

$$t\leq\frac{M}{2+k}. \tag{2.4.10}$$

We derive from (2.4.8) and (2.4.9)

$$(z^lf^{(k)}(z))'=\frac{g_{(k+1)(t-1)+l}(z)}{(z-\alpha_1)^{n_1+k+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.4.11}$$

where $g_{(k+1)(t-1)+l}(z)$ is a polynomial and easily obtained that

$$\deg(g_{(k+1)(t-1)+l}(z))=(k+1)(t-1)+l.$$

$$(z^lf^{(k)}(z))'=\frac{(z-\beta_1)^{m_1-1}(z-\beta_2)^{m_2-1}\cdots(z-\beta_s)^{m_s-1}g_{s+t-1}(z)}{(z-\alpha_1)^{n_1+k+1}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}}, \tag{2.4.12}$$

where $g_{s+t-1}(z)$ is a polynomial.

We obtain from (2.4.11) and (2.4.12) that

$$M-s\leq(k+1)(t-1)+l.$$

So

$$t \geq \frac{M + (k+1) - s - l}{k+1}. \tag{2.4.13}$$

The inequality (2.4.10) and (2.4.13) imply $\frac{M+(k+1)-s-l}{k+1} \leq \frac{M}{2+k}$. Through a simple calculation, note that $l \leq k$ and $s \geq 1$, we have

$$M \leq (2+k)(s+l-k-1) \leq (2+k)(s-1)$$

which is a contradiction.

Case 2. Assume that $\alpha_1\alpha_2\cdots\alpha_t = 0$.

Without loss of generality, we may assume $\alpha_1 = 0$. We derive from (2.4.8) and (2.4.9)

$$(2.4.10)' \qquad (z^l f^{(k)}(z))' = \frac{g_{(k+1)(t-1)}(z)}{z^{n_1+k+1-l}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}},$$

where $g_{(k+1)(t-1)}(z)$ is a polynomial and easily obtained that $\deg(g_{(k+1)(t-1)}(z)) = (k+1)(t-1)$.

$$(2.4.11)' \quad (z^l f^{(k)}(z))' = \frac{(z-\beta_1)^{m_1-1}(z-\beta_2)^{m_2-1}\cdots(z-\beta_s)^{m_s-1}g_{s+t-1}(z)}{z^{n_1+k+1-l}(z-\alpha_2)^{n_2+k+1}\cdots(z-\alpha_t)^{n_t+k+1}},$$

where $g_{s+t-1}(z)$ is a polynomial.

We can arrive at a contradiction by using the same argument as in the proof for Case 1.

The proof is complete. □

Lemma 5. *Let $k > 0$, $l \geq -k$ be two integers, and let $f(z)$ be a non-constant function. If $f(z) \neq 0$, and all poles of $f(z)$ are multiple in $\mathbb{C}$, then $f^{(k)}(z) - z^l$ has at least one zero which has multiplicity $\leq k+2$ in $\mathbb{C}$.*

Proof. We may assume that all zeros of $f(z)$ have multiplicities at least $k+3$. The Lemma 3 and Lemma 4 imply that $f(z)$ is a transcendental function. We know

$$\frac{1}{f} = \frac{f^{(k)}}{z^l f} - \frac{(z^{-l}f^{(k)})'}{f}\frac{z^{-l}f^{(k)}-1}{(z^{-l}f^{(k)})'}.$$

Therefore

$$\begin{aligned} m(r,\frac{1}{f}) &\leq m(r,\frac{f^{(k)}}{f}) + m(r,z^{-l}) + m(r,\frac{(z^{-l}f^{(k)})'}{z^{-l}f}z^{-l}) \\ &\quad + m(r,\frac{z^{-l}f^{(k)}-1}{(z^{-l}f^{(k)})'}) + \log 2 \\ &\leq m(r,\frac{z^{-l}f^{(k)}-1}{(z^{-l}f^{(k)})'}) + S(r,f). \end{aligned}$$

Combining

$$m(r,\frac{z^{-l}f^{(k)}-1}{(z^{-l}f^{(k)})'}) = m(r,\frac{(z^{-l}f^{(k)})'}{z^{-l}f^{(k)}-1}) + N(r,\frac{(z^{-l}f^{(k)})'}{z^{-l}f^{(k)}-1})$$

$$
\begin{aligned}
&- N(r, \frac{z^{-l}f^{(k)}-1}{(z^{-l}f^{(k)})'}) + O(1)\\
&\leq S(r,f) + N(r,(z^{-l}f^{(k)})') + N(r, \frac{1}{z^{-l}f^{(k)}-1})\\
&- N(r, \frac{1}{(z^{-l}f^{(k)})'}) - N(r, z^{-l}f^{(k)}-1)\\
&\leq \overline{N}(r, z^{-l}f^{(k)}) + \overline{N}(r, \frac{1}{z^{-l}f^{(k)}-1}) + S(r,f)\\
&\leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{f^{(k)}-z^{l}}) + S(r,f) \qquad (2.5.1)
\end{aligned}
$$

with

$$N(r, \frac{1}{f}) = 0.$$

We get

$$
\begin{aligned}
T(r,f) &= m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + S(r,f)\\
&\leq N(r,\frac{1}{f}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f^{(k)}-z^{l}}) + S(r,f)\\
&\leq \overline{N}(r,f) + \frac{1}{k+3}N(r,\frac{1}{f^{(k)}-z^{l}}) + S(r,f)\\
&\leq \overline{N}(r,f) + \frac{1}{k+3}T(r,f^{(k)}) + S(r,f)\\
&\leq \overline{N}(r,f) + \frac{1}{k+3}T(r,f) + \frac{k}{k+3}\overline{N}(r,f) + S(r,f)\\
&\leq (1+\frac{k}{k+3})\frac{1}{2}N(r,f) + \frac{1}{k+3}T(r,f) + S(r,f)\\
&\leq \frac{2k+3}{2(k+3)}N(r,f) + \frac{1}{k+3}T(r,f) + S(r,f)\\
&\leq \frac{2k++5}{2(k+3)}T(r,f) + S(r,f)\\
&\leq \frac{2k+5}{2k+6}T(r,f) + S(r,f)\\
&< T(r,f) + S(r,f)
\end{aligned}
$$

which is impossible. This completes the proof of Lemma 5. □

Lemma 6. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k be a integer, and let $\psi_n(z)$ be a sequence of holomorphic functions on D such that $\psi_n(z) \to \psi(z)$ locally uniformly on D, where $\psi(z)(\neq 0)$ is a holomorphic on D. If, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *$f(z) \neq 0$, and all poles of $f(z)$ are multiple in D;*

(2) *all zeros of $f^{(k)}(z) - \psi_n(z)$ have multiplicities at least $k+3$ in D,*

then $\mathcal{F}$ is normal in D.

Proof. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 1, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0^+$ such that

$$g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi) \to g(\xi) \tag{2.6.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. Hurwitz's theorem implies that $g(\xi) \neq 0$ and all poles of $g(\xi)$ are multiple.

From (2.6.1), we deduce that

$$g_n^{(k)}(\xi) = f_n^{(k)}(z_n + \rho_n \xi) \to g^{(k)}(\xi)$$

spherically uniformly on every compact subset of $\mathbb{C}$ which contains no pole of $g(\xi)$.

Since $g_n^{(k)}(\xi) - \psi_n(z_n + \rho_n \xi) = f_n^{(k)}(\xi) - \psi_n(z_n + \rho_n \xi) \to g^{(k)}(\xi) - \psi(z_0)$. Hurwitz's theorem implies that all zeros of $g^{(k)}(\xi) - \psi(z_0)$ have multiplicities at least $k+3$. It follows from Lemma 5 (for $l = 0$) that $g(\xi)$ is a constant, which contradicts the fact that $g(\xi)$ is a non-constant meromorphic function. Lemma 6 is proved. □

3. Proof of Theorem 1

Without loss of generality, we may assume that $D = \Delta = \{z : |z| < 1\}$, and

$$\psi(z) = z^m \phi(z) \ (z \in \Delta),$$

where m is a integer with $m \geq -k$, $\phi(0) = 1$, $\phi(z) \neq 0, \infty$ on $\Delta' = \{z : 0 < |z| < 1\}$.

If $m = 0$, from Theorem 1, we have that $\psi(z) (\neq 0)$ is a holomorphic function. By Lemma 2, Theorem 1 is proved. Since normality is local property, we only need to prove that $\mathcal{F}$ is normal at $z = 0$ for $m \neq 0$.

We distinguish two cases:

Case 1. $m < 0$.

By Lemma 1, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to 0$, and a sequence of positive numbers $\rho_n \to 0^+$, such that

$$W_n(\xi) = \rho_n^{-k-m} f_n(z_n + \rho_n \xi) \to W(\xi) \tag{3.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $W(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$. Hurwitz's theorem implies that $W(\xi) \neq 0$.

We now consider two subcases:

Case 1.1. $z_n/\rho_n \to \infty$.

Set $\omega_n(\xi) = z_n^{-m-k} f_n(z_n + z_n \xi) = z_n^{-m-k} f_n(z_n(1+\xi))$. Clearly, $\omega_n(\xi) \neq 0$ and all poles of $\omega_n(\xi)$ are multiple. From (3.1), we get

$$\omega_n^{(k)}(\xi) - (1+\xi)^m \phi(z_n(1+\xi))$$

$$= z_n^{-m}\left[f_n^{(k)}(z_n(1+\xi)) - (z_n(1+\xi))^m\phi(z_n(1+\xi))\right]$$
$$= z_n^{-m}\left[f_n^{(k)}(z_n(1+\xi)) - \psi(z_n(1+\xi))\right].$$

Since $z_n \to 0$, then there exists a natural number N such that, for every natural number $n > N$, we have $|z_n| < \frac{1}{2}$. By the assumption of the theorem, we have that all zeros of $\omega_n^{(k)}(\xi) - (1+\xi)^m\phi(z_n(1+\xi))$ have multiplicities at least $k+3$ in Δ for $n > N$. On the other hand, $(1+\xi)^m\phi(z_n(1+\xi))$ is holomorphic in Δ for $n > N$, and

$$(1+\xi)^m\phi(z_n(1+\xi)) \to (1+\xi)^m(\neq 0)$$

for $\xi \in \Delta$. Then, by Lemma 6, $\{\omega_n(\xi) : n > N\}$ is normal in Δ.

Hence, we can find a subsequence $\{\omega_{n_j}(\xi)\} \subset \{\omega_n(\xi) : n > N\}$, and a function $\omega(z)$ such that

$$\omega_{n_j}(\xi) = z_{n_j}^{-m-k} f_{n_j}(z_{n_j}(1+\xi)) \to \omega(z) \tag{3.2}$$

spherically locally uniformly on Δ.

If $\omega(0) \neq \infty$, from (3.1) and (3.2), and noting that $z_n/\rho_n \to \infty$, we have

$$\begin{aligned} W^{(m+k)}(\xi) &= \lim_{j\to\infty} f_{n_j}^{(m+k)}(z_{n_j} + \rho_{n_j}\xi)) \\ &= \lim_{j\to\infty} f_{n_j}^{(m+k)}(z_{n_j}(1 + \frac{\rho_{n_j}}{z_{n_j}}\xi)) \\ &= \lim_{j\to\infty} \omega_{n_j}^{(m+k)}(\frac{\rho_{n_j}}{z_{n_j}}\xi) = \omega^{(m+k)}(0). \end{aligned}$$

This implies that $W^{(m+k)}(\xi)$ is a finite constant, and then $W(\xi)$ is a polynomial. But this is impossible since $W(\xi)$ is a non-constant meromorphic function and $W(\xi) \neq 0$.

If $\omega(0) = \infty$. From (3.3), we have

$$z_{n_j}^{-m-k} f_{n_j}(z_{n_j} + \rho_{n_j}\xi) = z_{n_j}^{-m-k} f_{n_j}(z_{n_j}(1 + \frac{\rho_{n_j}}{z_{n_j}}\xi))$$

$$= \omega_{n_j}(\frac{\rho_{n_j}}{z_{n_j}}\xi) \to \omega(0) = \infty$$

and hence

$$W(\xi) = \lim_{j\to\infty} \rho_{n_j}^{-k-m} f_{n_j}(z_{n_j}+\rho_{n_j}\xi) = \lim_{j\to\infty} (\frac{z_{n_j}}{\rho_{n_j}})^{k+m} z_{n_j}^{-m-k} f_{n_j}(z_{n_j}+\rho_{n_j}\xi) = \infty$$

that is, $W(\xi) \equiv \infty$, a contradiction.

Case 1.2. $z_n/\rho_n \to \alpha$, a finite complex number. We have

$$W_n^{(k)}(\xi) - \rho_n^{-m}(z_n + \rho_n\xi)^m\phi(z_n + \rho_n\xi) \to W^{(k)}(\xi) - (\alpha + \xi)^m$$

on $\mathbb{C}\{-\alpha\}$, and

$$W_n^{(k)}(\xi) - \rho_n^{-m}(z_n + \rho_n\xi)^m\phi(z_n + \rho_n\xi) = \rho_n^{-m}(f_n^{(k)}(z_n + \rho_n\xi) - \psi(z_n + \rho_n\xi)).$$

Hurwitz's theorem implies that all zeros of $W^{(k)}(\xi)-(\alpha+\xi)^m$ have multiplicities at least $k+3$. It follows from Lemma 5 that $W(\xi)$ must be a constant, a contradiction.

Case 2. $m>0$.

Consider the family $\mathcal{G}=\{g(z)=\frac{f(z)}{\psi(z)}:f\in\mathcal{F},z\in\Delta\}$. Since $f\neq 0$ for $f\in\mathcal{F}$, we have that $g(0)=\infty$ for each $g\in\mathcal{G}$.

We first prove that $\mathcal{G}$ is normal in Δ. Suppose, on the contrary, that $\mathcal{G}$ is not normal at $z_0=0$. By Lemma 1, there exist a sequence of functions $g_n\in\mathcal{G}$, a sequence of complex numbers $z_n\to 0$, and a sequence of positive numbers $\rho_n\to 0^+$, such that

$$G_n(\xi)=\rho_n^{-k}g_n(z_n+\rho_n\xi)\to G(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$. Hurwitz's theorem implies that $G(\xi)\neq 0$.

We now consider two subcases:

Case 2.1. $z_n/\rho_n\to\infty$.

By simple calculations, we have

$$\begin{aligned} g_n^{(k)}(z)&=\frac{f_n^{(k)}(z)}{\psi(z)}-\sum_{j=1}^{k}\binom{k}{j}g_n^{(k-j)}(z)\frac{\psi^{(j)}(z)}{\psi(z)}\\ &=\frac{f_n^{(k)}(z)}{\psi(z)}-\sum_{j=1}^{k}\left[\binom{k}{j}g_n^{(k-j)}(z)\sum_{i=0}^{j}A_{ji}\frac{1}{z^{j-i}}\frac{\phi^{(i)}(z)}{\phi(z)}\right], \end{aligned}\tag{3.3}$$

where $A_{jj}=1$, $A_{ji}=m(m-1)\cdots(m-j+i+1)\binom{j}{i}$ if $m\geq j$, and $A_{ji}=0$ if $1\leq m<j$ for $i=0,1,\ldots,j-1$.

Thus, from (3.3), we have

$$\begin{aligned} G_n^{(k)}(\xi)&=g_n^{(k)}(z_n+\rho_n\xi)\\ &=\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)}\\ &\quad-\sum_{j=1}^{k}\left[\binom{k}{j}g_n^{(k-j)}(z_n+\rho_n\xi)\sum_{i=0}^{j}A_{ji}\frac{1}{(z_n+\rho_n\xi)^{j-i}}\frac{\phi^{(i)}(z_n+\rho_n\xi)}{\phi(z_n+\rho_n\xi)}\right]\\ &=\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)}\\ &\quad-\sum_{j=1}^{k}\left[\binom{k}{j}\frac{g_n^{(k-j)}(z_n+\rho_n\xi)}{\rho_n^j}\sum_{i=0}^{j}A_{ji}\frac{1}{(z_n/\rho_n+\xi)^{j-i}}\frac{\rho_n^i\phi^{(i)}(z_n+\rho_n\xi)}{\phi(z_n+\rho_n\xi)}\right]. \end{aligned}$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{1}{(z_n/\rho_n+\xi)}=0$$

and

$$\lim_{n\to\infty}\frac{\rho_n^i\phi^{(i)}(z_n+\rho_n\xi)}{\phi(z_n+\rho_n\xi)}=0$$

for $i\geq 1$. Noting that $g_n^{(k-j)}(z_n+\rho_n\xi)/\rho_n^j$ is locally bounded on $\mathbb{C}$ minus the set of poles of $G(\xi)$ since $g_n(z_n+\rho_n\xi)/\rho_n^k\to G(\xi)$. Therefore, on every compact subset of $\mathbb{C}$ which contains no poles of $G(\xi)$, we have

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)}\to G^{(k)}(\xi)$$

thus

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)-\psi(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)}\to G^{(k)}(\xi)-1.$$

Noting that $\psi(z_n+\rho_n\xi)$ has only one zero $\xi=-\frac{z_n}{\rho_n}\to\infty$, by the assumption of theorem, we have that all poles of $G(\xi)$ are multiple. Since all zeros of $f_n^{(k)}(z_n+\rho_n\xi)-\psi(z_n+\rho_n\xi)$ have multiplicities at least $k+3$, and It follows from Lemma 5 (for $l=0$) that $G(\xi)$ must be a constant, which contradicts the fact that $G(\xi)$ is a non-constant meromorphic function.

Case 2.2. $z_n/\rho_n\to\alpha$, a finite complex number. We have

$$\rho_n^{-k}g_n(\rho_n\xi)=\rho_n^{-k}g_n(z_n+\rho_n(\xi-z_n/\rho_n))=G_n(\xi-z_n/\rho_n)\to G(\xi-\alpha)$$

spherically uniformly on compact subsets of $\mathbb{C}$. Clearly, $G(\xi-\alpha)\neq 0$, and the pole of $G(\xi-\alpha)$ at $\xi=0$ has multiplicity at least m. Now

$$F_n(\xi)=\frac{f_n(\rho_n\xi)}{\rho_n^{k+m}}=\frac{f_n(\rho_n\xi)}{\rho_n^k\psi(\rho_n\xi)}\frac{\psi(\rho_n\xi)}{\rho_n^m}=\frac{g_n(\rho_n\xi)}{\rho_n^k}\frac{\psi(\rho_n\xi)}{\rho_n^m}. \tag{3.4}$$

Noting that $\frac{\psi(\rho_n\xi)}{\rho_n^m}\to\xi^m$, we get

$$F_n(\xi)\to\xi^m G(\xi-\alpha)=F(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$. Since the pole of $G(\xi-\alpha)$ at $\xi=0$ has multiplicity at least m, we have $F(0)\neq 0$, hence $F(\xi)\neq 0$.

From (3.4), we have

$$\frac{f_n^{(k)}(\rho_n\xi)-\psi(\rho_n\xi)}{\rho_n^m}\to F^{(k)}(\xi)-\xi^m. \tag{3.5}$$

By the assumption of Theorem and (3.5), Hurwitz's theorem implies that all zeros of $F^{(k)}(\xi)-\xi^m$ have multiplicities at least $k+3$. It follows from Lemma 5 that $F(\xi)$ must be a constant, a contradiction.

We thus have proved that $\mathcal{G}$ is normal in Δ. Thus the family $\mathcal{G}$ is equicontinuous on Δ with respect to the spherical distance. We see that $f_n(z)$ and $\psi(z)$ have no common zeros. On the other hand, $g(0)=\infty$ for each $g\in\mathcal{G}$, so there exists $\delta>0$ such that $|g(z)|\geq 1$ for all $g\in\mathcal{G}$ and each $z\in\Delta_\delta=\{z:|z|<\delta\}$. Suppose that $\mathcal{F}$ is not normal at $z=0$. Since $\mathcal{F}$ is normal in Δ'_δ, the family $\mathcal{F}_1=\{\frac{1}{f}:f\in\mathcal{F}\}$ is normal in Δ'_δ, but it is not normal at $z=0$. Then there exists a sequence $\{\frac{1}{f_n}\} \subset \mathcal{F}_1$ which converges locally uniformly in Δ'_δ, but not in Δ_δ. Noting that $f_n \neq 0$ in Δ, it follows that $\frac{1}{f_n}$ is holomorphic in Δ for each n. If $\frac{1}{f_n}$ converges a analytic function locally uniformly in Δ', by maximum modulus principle, we have that $\frac{1}{f_n}$ is locally bounded uniformly on Δ, which contradicts the assumption that $\mathcal{F}_1$ is not normal at $z = 0$. So we have $\frac{1}{f_n} \to \infty$ in Δ'. Thus $f_n \to 0$ converges locally uniformly in Δ', and hence so does $\{g_n\} \subset \mathcal{G}$, where $g_n = f_n/\psi$, which contradicts $|g(z)| \geq 1$ for $z \in \Delta_\delta = \{z : |z| < \delta\}$. Thus $\mathcal{F}$ is normal in D.

This proves the theorem.

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