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A POLAR REPRESENTATION OF A REGULARITY OF A DUAL QUATERNIONIC FUNCTION IN CLIFFORD ANALYSIS

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Abstract. The paper gives the regularity of dual quaternionic functions and the dual Cauchy-Riemann system in dual quaternions. Also, the paper researches the polar representation and properties of a dual quaternionic function and their regular quaternionic functions.

1. Introduction

A dual number z is consisted of real numbers x and y associated with a real unit 1 and the dual unit ε, where $\varepsilon^2 = 0$. A dual number is denoted in the form $z = x + \varepsilon y$. Thus, the dual numbers are elements of the two dimensional real algebra

$$D = R[\varepsilon] = \{z = x + \varepsilon y \mid x,\ y \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}$$

generated by 1 and ε (see [17]).

The algebra of dual numbers has been studied by Clifford [1] and its applications to mechanics are due to Study [20]. Dual algebra has been often used for the field of displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms. Dual numbers can be represented as follows ([3]):

1. Gaussian representation: $z = x + \varepsilon y$,
2. Polar representation: $z = r(1 + \varepsilon\phi)$,
3. Exponential representation: $z = r\exp(\varepsilon\phi)$, where $r = x$ $(x \neq 0)$, $\phi = \frac{y}{x}$ and $\exp(\varepsilon\phi) = 1 + \varepsilon\phi$.

The dual number has a geometrical property which is investigated detail in [4, 17].

Clifford [1] also has studied the following algebra

$$\mathbb{H} = \{p = z_1 + z_2 j \mid z_1 = x_0 + x_1 i,\ z_2 = x_2 + x_3 i,\ x_r \in \mathbb{R}\ (r = 0, 1, 2, 3)\}$$

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called the set of quaternions. Here imaginary basis elements i, j and k satisfy the following conditions:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

For two quaternions $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, they are given the rules of the addition and multiplication as follows, respectively,

$$p + q = (z_1 + w_1) + (z_2 + w_2)j$$

and

$$pq = (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1})j,$$

where $\overline{w_k} = y_{k0} - y_{k1} i$ for $w_k = y_{k0} + y_{k1} i$, $y_{kj} \in \mathbb{R}$, $k = 1, 2$, $j = 0, 1$. Kajiwara et al. [5, 6] applied the theory on a closed densely defined operator and a priori estimate for the adjoint operator in a Hilbert space and brconvex domains. We [9, 10, 11, 12] researched corresponding Cauchy-Riemann systems and properties of functions with values in special quaternions such as reduced quaternions, split quaternions and dual split quaternions. We [13, 14, 15] investigated a regular functions defined by the differential operators of special quaternion number systems. Porter [19] gave an explicit solution to the linear equation in the quaternions $\mathbb{H}$.

This paper gives expressions of the differential operators and the exponential functions in dual quaternions. The paper researches the polar representation of dual quaternionic functions by using a dual Cauchy-Riemann system and their regularity of that functions in dual quaternions.

2. Preliminaries

For $p = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, we denote by $Sc(p)$ the scalar part, and by $Vec(p)$ the spatial vector part:

$$p = Sc(p) + Vec(p),$$

where $Sc(p) = x_0$ and $Vec(p) = x_1 i + x_2 j + x_3 k$ with $x_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$. Then for $p,\ q \in \mathbb{H}$, we have

$$\begin{aligned} p + q &= Sc(p) + Sc(q) + Vec(p) + Vec(q), \\ pq &= Sc(p)Sc(q) - Vec(p) \cdot Vec(q) \\ &\quad + Sc(p)Vec(q) + Vec(p)Sc(q) + Vec(p) \times Vec(q), \end{aligned}$$

where $Sc(q) = y_0$, $Vec(q) = y_1 i + y_2 j + y_3 k$ with $y_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$, the symbol $\cdot$ is a usual inner product,

$$Vec(p) \cdot Vec(q) = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

and the symbol $\times$ is a usual outer product,

$$Vec(p) \times Vec(q) = (x_2 y_3 - x_3 y_2)i - (x_1 y_3 - x_3 y_1)j + (x_1 y_2 - x_2 y_1)k.$$

The norm for a quaternion is

$$|p|^2 := pp^* = Sc(p)^2 + Vec(p) \cdot Vec(p),$$

where $p^* = Sc(p) - Vec(p)$, and the inverse of p is

$$p^{-1} = \frac{p^*}{|p|^2}.$$

For a unit quaternion, $|p| = 1$, it is given by:

$$p = (\cos(\theta/2), \mathbf{n}\sin(\theta/2))$$

and

$$x_0 = \cos(\theta/2),\ x_1 = n_1 \sin(\theta/2),$$
$$x_2 = n_2 \sin(\theta/2),\ x_3 = n_3 \sin(\theta/2),$$

where an angle θ and axis $\mathbf{n} = (n_1, n_2, n_3)$ of rotation (see [7]).

We consider the following form:

$$\mathbb{D}_q = \{Z = p_1 + \varepsilon p_2 \mid p_r \in \mathbb{H},\ \varepsilon^2 = 0,\ r = 1, 2\} \cong \mathbb{H}^2 \cong \mathbb{R}^8,$$

where ε is the dual unit that commutes with i, j and k. The dual quaternion $Z = p_1 + \varepsilon p_2 \in \mathbb{D}_q$ is also written as a linear combination of a scalar, denoted by $Sc(Z)$, and a spatial vector, denoted by $Vec(Z)$ (see [7, 8]):

$$Z = Sc(Z) + Vec(Z) = Sc(p_1) + Vec(p_1) + \varepsilon\{Sc(p_2) + Vec(p_2)\},$$

where

$$Sc(Z) = Sc(p_1) + \varepsilon Sc(p_2)\ ,\ Vec(Z) = Vec(p_1) + \varepsilon Vec(p_2)$$

with $p_1, p_2 \in \mathbb{H}$.

For two elements Z and $W = Sc(W) + Vec(W) = Sc(q_1) + Vec(q_1) + \varepsilon\{Sc(q_2) + Vec(q_2)\}$ of $\mathbb{D}_q$, we give the addition and the multiplication on $\mathbb{D}_q$ as follows:

$$Z + W = Sc(Z) + Sc(W) + \varepsilon\{Vec(Z) + Vec(W)\}$$

and

$$ZW = Sc(Z)Sc(W) - Vec(Z) \cdot Vec(W) + Sc(Z)Vec(W) + Sc(W)Vec(Z) + Vec(Z) \times Vec(W),$$

where

$$Vec(Z) \cdot Vec(W) = Vec(p_1) \cdot Vec(q_1) + \varepsilon\{Vec(p_1) \cdot Vec(q_2) + Vec(p_2) \cdot Vec(q_1)\}$$

and

$$Vec(Z) \times Vec(W) = Vec(p_1) \times Vec(q_1) + \varepsilon\{Vec(p_1) \times Vec(q_2) + Vec(p_2) \times Vec(q_1)\}.$$

We give the complex conjugate element of $\mathbb{D}_q$:

$$Z^* = Sc(p_1) - Vec(p_1) + \varepsilon\{Sc(p_2) - Vec(p_2)\}.$$

It is also written as

$$Z^* = Sc(Z) - Vec(Z),$$

and the modulus of Z, denoted by $|Z|$, is described by

$$|Z|^2 := Sc(Z)Sc(Z^*) + Vec(Z) \cdot Vec(Z^*) = \{Sc(p_1)\}^2 + Vec(p_1) \cdot Vec(p_1) = |p_1|^2.$$

Since every element of the set $\{\varepsilon p \mid p \in \mathbb{H}\}$ has no inverse, the inverse of a dual quaternion is given by

$$Z^{-1} = \frac{Z^\dagger}{|p_1|^2} \in \mathbb{D}_q \quad (p_1 \neq 0),$$

where

$$Z^\dagger = (|p_1|^2 - \varepsilon p_2 p_1^*) p_1^{-1},$$

where $p_1^{-1} = \frac{p_1^*}{|p_1|^2}$, called the dual conjugate of Z with $ZZ^\dagger = Z^\dagger Z = p_1 p_1^* = |p_1|^2$.

Plucker [18] gave screw coordinates so that we can rewrite dual quaternions in a form of the spherical linear interpolation. Screw parameters have the form $(\theta, d, \mathbf{I}, \mathbf{m})$, where

$$\begin{cases} \theta & \text{is the angle of rotation,} \\ d & \text{is the translation along the axis,} \\ \mathbf{I} & \text{is the vector line direction,} \\ \mathbf{m} = \mathbf{p} \times \mathbf{I} & \text{is the line moment with } \mathbf{p} \text{ is a point on a given line.} \end{cases}$$

From the above components, Daniilidis [2] converted a unit dual quaternion to screw coordinates as follows:

$$Sc(p_1) = \cos(\theta/2),\ Vec(p_1) = \mathbf{I}\sin(\theta/2),\ Sc(p_2) = -\frac{d}{2}\sin(\theta/2), \tag{2.1}$$

$$Vec(p_2) = \mathbf{I}\frac{d}{2}\cos(\theta/2) + \mathbf{m}\sin(\theta/2). \tag{2.2}$$

Referring [2], we can write the following representation of a unit dual quaternion

$$Z = \cos\Big(\frac{\theta + \varepsilon d}{2}\Big) + (\mathbf{I} + \varepsilon\mathbf{m})\sin\Big(\frac{\theta + \varepsilon d}{2}\Big) = \cos(\phi) + \mathbf{v}\sin(\phi),$$

where $\mathbf{v} = \mathbf{I} + \varepsilon\mathbf{m}$ and $\phi = \frac{\theta + \varepsilon d}{2}$. By the properties of trigonometric functions, we have the representation of a unit dual quaternion

$$\begin{aligned} Z &= \rho\cos(\phi) + \mathbf{v}\rho\sin(\phi) \\ &= \cos\Big(\frac{\theta}{2}\Big)\cos\Big(\frac{\varepsilon d}{2}\Big) - \sin\Big(\frac{\theta}{2}\Big)\sin\Big(\frac{\varepsilon d}{2}\Big) \\ &\quad + \mathbf{v}\Big\{\sin\Big(\frac{\theta}{2}\Big)\cos\Big(\frac{\varepsilon d}{2}\Big) + \cos\Big(\frac{\theta}{2}\Big)\sin\Big(\frac{\varepsilon d}{2}\Big)\Big\}. \end{aligned}$$

From the representation of a Taylor series, since $\cos\Big(\frac{\varepsilon d}{2}\Big) = 1$ and $\sin\Big(\frac{\varepsilon d}{2}\Big) = \frac{\varepsilon d}{2}$, we have

$$Z = \cos\Big(\frac{\theta}{2}\Big) - \sin\Big(\frac{\theta}{2}\Big)\frac{\varepsilon d}{2} + \mathbf{v}\Big\{\sin\Big(\frac{\theta}{2}\Big) + \cos\Big(\frac{\theta}{2}\Big)\frac{\varepsilon d}{2}\Big\},$$

where $\rho = |Z|^2$. From the equations (2.1), we have

$$Z = Sc(p_1) + \varepsilon Sc(p_2) + \mathbf{v}\Big\{-\frac{2}{d}Sc(p_2) + \varepsilon Sc(p_1)\frac{d}{2}\Big\} := p + \mathbf{v}q,$$

where

$$p = Sc(p_1) + \varepsilon Sc(p_2)\ ,\ q = -\frac{2}{d}Sc(p_2) + \varepsilon Sc(p_1)\frac{d}{2}.$$

Since we have

$$\mathbf{v}^2 = \mathbf{I}^2 = -1,$$

we obtain a corresponding Euler's formula for a unit dual quaternion:

$$\exp(\mathbf{v}\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{v}\phi)^n = \cos(\phi) + \mathbf{v}\sin(\phi).$$

Proposition 2.1. *For any unit dual quaternion, we have*
1. $\exp(\mathbf{v}\phi_1)\exp(\mathbf{v}\phi_2) = \exp(\mathbf{v}(\phi_1 + \phi_2))$,
2. $\dfrac{\exp(\mathbf{v}\phi_1)}{\exp(\mathbf{v}\phi_2)} = \exp(\mathbf{v}(\phi_1 - \phi_2))$.

Proof. From the corresponding Euler's formula for a dual quaternion, we have

$$\begin{aligned}\exp(\mathbf{v}\phi_1)\exp(\mathbf{v}\phi_2) &= \{\cos(\phi_1) + \mathbf{v}\sin(\phi_1)\}\{\cos(\phi_2) + \mathbf{v}\sin(\phi_2)\}\\ &= \cos(\phi_1 + \phi_2) + \mathbf{v}\sin(\phi_1 + \phi_2)\\ &= \exp(\mathbf{v}(\phi_1 + \phi_2))\end{aligned}$$

and

$$\begin{aligned}\frac{\exp(\mathbf{v}\phi_1)}{\exp(\mathbf{v}\phi_2)} &= \{\cos(\phi_1) + \mathbf{v}\sin(\phi_1)\}\{\cos(\phi_2) - \mathbf{v}\sin(\phi_2)\}\\ &= \cos(\phi_1 - \phi_2) + \mathbf{v}\sin(\phi_1 - \phi_2)\\ &= \exp(\mathbf{v}(\phi_1 - \phi_2)).\end{aligned}$$

Therefore, we obtain the results. □

Proposition 2.2. *Let* $Z = \cos(\phi) + \mathbf{v}\sin(\phi)$ *be a unit dual quaternion. Then we have*

$$Z^n = (\cos(\phi) + \mathbf{v}\sin(\phi))^n = \cos(n\phi) + \mathbf{v}\sin(n\phi) \tag{2.3}$$

for all integer n.

Proof. From the induction for integers n, the equation (2.3) is obtained. □

3. Hyperholomorphic function in dual quaternions

Let Ω be a bounded open set in $\mathbb{H}^2$. A function F is given by

$$F : \Omega \ \rightarrow\ \mathbb{D}_q;\ F(Z) = f_1(p_1, p_2) + \varepsilon f_2(p_1, p_2),$$

where

$$\begin{aligned}&f_1 = g_1(z_1, z_2, w_1, w_2) + g_2(z_1, z_2, w_1, w_2)j \quad \text{and}\\ &f_2 = h_1(z_1, z_2, w_1, w_2) + h_2(z_1, z_2, w_1, w_2)j\end{aligned}$$

are quaternionic functions, g_r and h_r $(r = 1, 2)$ are complex-valued functions.

Definition. A function F is said to be hyperholomorphic on $\Omega = D \cap L$, where D is an open subset of $\mathbb{D}_q$ and $L = \mathbb{H} \setminus \{0\} + \varepsilon\mathbb{H}$, with values in $\mathbb{D}_q$ if the limit

$$\frac{dF(Z)}{dZ} := \lim_{\zeta \to 0} \{F(Z+\zeta) - F(Z)\}\zeta^{-1} = \lim_{\zeta \to 0} \frac{\{F(Z+\zeta) - F(Z)\}\zeta^{*}}{\eta_1\eta_1^{*}} \quad (\eta_1 \neq 0) \tag{3.4}$$

exists, where $\zeta = \eta_1 + \varepsilon\eta_2 \to 0$ means $\eta_1 \to 0$ and $\eta_2 \to 0$ which are referred by [16].

Theorem 3.1. *A function F is hyperholomorphic on Ω with values in $\mathbb{D}_q$ if and only if the following conditions are held*:

$$\begin{cases} \lim\limits_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \{F(Z+\zeta) - F(Z)\}\eta_1^{-1} \quad \textit{exists and} \\ \lim\limits_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1,p_2)\}\eta_2^{-1} = 0. \end{cases} \tag{3.5}$$

Proof. Since the dual part of a dual quaternion has no inverse elements, we use the dual conjugation of Z as follows:

$$\begin{aligned} \lim_{\zeta \to 0} \{F(Z+\zeta) - F(Z)\}\zeta^{-1} &= \lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \frac{\{F(Z+\zeta) - F(Z)\}(\eta_1^{*} - \varepsilon\eta_2^{\dagger})}{\eta_1\eta_1^{*}} \\ = \lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} &\{F(Z+\zeta) - F(Z)\}\eta_1^{-1} \\ &- \varepsilon \lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1,p_2)\}\eta_2^{-1}\left(\eta_2\eta_1^{-1}\right)^2. \end{aligned}$$

For the existence of the above limit, the limit

$$\lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1,p_2)\}\eta_2^{-1}$$

has to be independent to $(\eta_2\eta_1^{-1})^2$. Thus, we obtain the following equation:

$$\lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1,p_2)\}\eta_2^{-1} = 0.$$

Conversely, if the conditions (3.5) are satisfied for the function F, then the limit

$$\lim_{\zeta \to 0} \{F(Z+\zeta) - F(Z)\}\zeta^{-1}$$

exists. From the definition of a hyperholomorphic function in $\mathbb{D}_q$, the function F is hyperholomorphic. □

We give the left differential operators in $\mathbb{D}_q$.

$$D_1 := \frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}} \quad \text{and} \quad D_1^{*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}},$$

where $\frac{\partial}{\partial z_r}$ and $\frac{\partial}{\partial \overline{z_r}}$ $(r = 1, 2)$ are usual complex differential operators and j is an imaginary basis element in $\mathbb{H}$.

Remark 3.2. From the representation of differential operators in $\mathbb{D}_q$, we have

$$\begin{aligned} FD_1 &= \{g_1 + g_2 j + \varepsilon(h_1 + h_2 j)\}\Big(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}}\Big) \\ &= \Big\{\frac{\partial g_1}{\partial z_1} + \frac{\partial g_2}{\partial \overline{z_2}} + \Big(\frac{\partial g_2}{\partial \overline{z_1}} - \frac{\partial g_1}{\partial z_2}\Big)j\Big\} + \varepsilon\Big\{\frac{\partial h_1}{\partial z_1} + \frac{\partial h_2}{\partial \overline{z_2}} + \Big(\frac{\partial h_2}{\partial \overline{z_1}} - \frac{\partial h_1}{\partial z_2}\Big)j\Big\} \end{aligned}$$

and

$$\begin{aligned} FD_1^* &= \{g_1 + g_2 j + \varepsilon(h_1 + h_2 j)\}\Big(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}\Big) \\ &= \Big\{\frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial g_2}{\partial \overline{z_2}} + \Big(\frac{\partial g_2}{\partial z_1} + \frac{\partial g_1}{\partial z_2}\Big)j\Big\} + \varepsilon\Big\{-\frac{\partial h_2}{\partial \overline{z_2}} + \frac{\partial h_1}{\partial \overline{z_1}} + \Big(\frac{\partial h_1}{\partial z_2} + \frac{\partial h_2}{\partial z_1}\Big)j\Big\}. \end{aligned}$$

Definition. Let Ω be a bounded open set in $\mathbb{H}^2$. Then a function F is said to be hyperholomorphic on Ω with values in $\mathbb{D}_q$ if F satisfies the following conditions:
(i) each component of F, f_1 and f_2, is a continuously differentiable function and
(ii) F satisfies the following equations

$$\begin{cases} FD_1^* = 0 \quad \text{and} \\ \dfrac{\partial f_1}{\partial y_r} = 0 \quad (r = 0, 1, 2, 3). \end{cases} \tag{3.6}$$

In detail, the equations (3.6) is equivalent to the following system

$$\frac{\partial g_1}{\partial \overline{z_1}} = \frac{\partial g_2}{\partial \overline{z_2}},\ \frac{\partial g_2}{\partial z_1} = -\frac{\partial g_1}{\partial z_2},\ \frac{\partial h_1}{\partial \overline{z_1}} = \frac{\partial h_2}{\partial \overline{z_2}},\ \frac{\partial h_2}{\partial z_1} = -\frac{\partial h_1}{\partial z_2},$$

called the corresponding Cauchy-Riemann system on $\mathbb{D}_q$.

Let Ω be an open set in $\mathbb{H}^2$. A function can be written by

$$\begin{aligned} &F : \Omega \ \rightarrow \ \mathbb{D}_q; \\ &F(Z) = F(p(\rho,\phi), q(\rho,\phi)) = M(p(\rho,\phi), q(\rho,\phi)) + \mathbf{v}N(p(\rho,\phi), q(\rho,\phi)), \end{aligned}$$

where

$$M = Sc(f_1) + \varepsilon Sc(f_2) \ \text{ and } \ N = -\frac{2}{d}Sc(f_2) + \varepsilon Sc(f_1)\frac{d}{2}$$

are dual-valued functions and $Sc(f_1)$ and $Sc(f_2)$ are real-valued functions.

Theorem 3.3. *Let Ω be a bounded open set in $\mathbb{H}^2$. If a function $F = M + \mathbf{v}N$ is hyperholomorphic on Ω with values in $\mathbb{D}_q$, then the following equations hold*:

$$\frac{\partial Sc(f_1)}{\partial Sc(p_1)} = \frac{\partial Sc(f_2)}{\partial Sc(p_2)} \quad \text{and} \quad \frac{4}{d^2}\frac{\partial Sc(f_2)}{\partial Sc(p_1)} = \frac{\partial Sc(f_1)}{\partial Sc(p_2)}. \tag{3.7}$$

Proof. From the definition of the hyperholomorphic function F on Ω with values in $\mathbb{D}_q$ and Definition 3, if the following limit

$$\lim_{\substack{\lambda_1\to 0,\\ \lambda_2\to 0}} \frac{Sc(f_1)+\varepsilon Sc(f_2)+\mathbf{v}\{\frac{-2}{d}Sc(f_2)+\varepsilon\frac{d}{2}Sc(f_1)\}}{\lambda_1+\varepsilon\lambda_2+\mathbf{v}(\frac{-2}{d}\lambda_2+\varepsilon\frac{d}{2}\lambda_1)}$$

exists, then the function F is hyperholomorphic, where $\lambda_1 = Sc(\eta_1)$ and $\lambda_2 = Sc(\eta_2)$. By the definition of the existence of the limit and calculating of the complex conjugation of a dual quaternion, we have

$$\begin{aligned}&\frac{\partial Sc(f_1)(1+\mathbf{I}\varepsilon\frac{d}{2})+\partial Sc(f_2)(\varepsilon-\frac{2}{d}\mathbf{v})}{\partial Sc(p_1)(1+\mathbf{I}\varepsilon\frac{d}{2})}\\ &=\frac{\partial Sc(f_1)(1+\mathbf{I}\varepsilon\frac{d}{2})+\partial Sc(f_2)(\varepsilon-\frac{2}{d}\mathbf{v})}{\partial Sc(p_2)(\varepsilon-\frac{2}{d}\mathbf{v})}.\end{aligned}$$

By arranging the above equation, we have

$$\Big\{\frac{\partial Sc(f_1)}{\partial Sc(p_1)}-\frac{\partial Sc(f_2)}{\partial Sc(p_2)}\Big\}\Big(2\varepsilon-\frac{2}{d}\mathbf{v}\Big)+\Big\{\frac{\partial Sc(f_1)}{\partial Sc(p_2)}-\frac{4}{d^2}\frac{\partial Sc(f_2)}{\partial Sc(p_1)}\Big\}(1+\mathbf{I}\varepsilon d)=0.$$

Therefore, we obtain the equations (3.7). □

Theorem 3.4. *Let Ω be a bounded open set in $\mathbb{H}^2$ and a function $F = M+\mathbf{v}N$ be hyperholomorphic on Ω with values in $\mathbb{D}_q$. Then the following equations hold:*

$$\frac{\rho}{2}(1+\varepsilon)\frac{\partial M}{\partial\rho}=\frac{\partial N}{\partial\phi} \quad \textit{and} \quad \frac{\rho}{2}(1+\varepsilon)\frac{\partial N}{\partial\rho}=-\frac{\partial M}{\partial\phi}. \tag{3.8}$$

Proof. From the chain rule of multi variables calculus, we have

$$\frac{\partial M}{\partial\rho}=\frac{\partial p}{\partial\rho}\frac{\partial M}{\partial p}+\frac{\partial q}{\partial\rho}\frac{\partial M}{\partial q},\quad \frac{\partial M}{\partial\phi}=\frac{\partial p}{\partial\phi}\frac{\partial M}{\partial p}+\frac{\partial q}{\partial\phi}\frac{\partial M}{\partial q},$$

$$\frac{\partial N}{\partial\rho}=\frac{\partial p}{\partial\rho}\frac{\partial N}{\partial p}+\frac{\partial q}{\partial\rho}\frac{\partial N}{\partial q},\quad \frac{\partial N}{\partial\phi}=\frac{\partial p}{\partial\phi}\frac{\partial N}{\partial p}+\frac{\partial q}{\partial\phi}\frac{\partial N}{\partial q}.$$

Since we have the following equations:

$$\frac{\partial p}{\partial\rho}=\cos\phi,\ \frac{\partial q}{\partial\rho}=\sin\phi,$$

$$\frac{\partial p}{\partial\phi}=-\rho(1+\varepsilon)\sin\phi,\ \frac{\partial q}{\partial\phi}=\rho(1+\varepsilon)\cos\phi,$$

we have

$$\begin{aligned}\frac{\partial M}{\partial\rho}&=\cos\phi\frac{\partial M}{\partial p}+\sin\phi\frac{\partial M}{\partial q},\\ \frac{\partial M}{\partial\phi}&=-\rho\sin\phi(1+\varepsilon)\frac{\partial M}{\partial p}+\rho\cos\phi(1+\varepsilon)\frac{\partial M}{\partial q},\\ \frac{\partial N}{\partial\rho}&=\cos\phi\frac{\partial N}{\partial p}+\sin\phi\frac{\partial N}{\partial q},\end{aligned}$$

$$\frac{\partial N}{\partial \phi} = -\rho \sin\phi(1+\varepsilon)\frac{\partial N}{\partial p} + \rho\cos\phi(1+\varepsilon)\frac{\partial N}{\partial q},$$

where

$$\frac{\partial M}{\partial p} = \frac{\partial Sc(f_1)}{\partial Sc(p_1)} + \varepsilon\frac{\partial Sc(f_2)}{\partial Sc(p_1)} + \frac{\partial Sc(p_2)}{\partial p}\Big\{\frac{\partial Sc(f_1)}{\partial Sc(p_2)} + \varepsilon\frac{\partial Sc(f_2)}{\partial Sc(p_2)}\Big\},$$

$$\frac{\partial M}{\partial q} = \frac{\partial Sc(p_1)}{\partial q}\Big\{\frac{\partial Sc(f_1)}{\partial Sc(p_1)} + \varepsilon\frac{\partial Sc(f_2)}{\partial Sc(p_1)}\Big\} - \frac{d}{2}\Big\{\frac{\partial Sc(f_1)}{\partial Sc(p_2)} + \varepsilon\frac{\partial Sc(f_2)}{\partial Sc(p_2)}\Big\},$$

$$\frac{\partial N}{\partial p} = -\frac{2}{d}\frac{\partial Sc(f_2)}{\partial Sc(p_1)} + \varepsilon\frac{d}{2}\frac{\partial Sc(f_1)}{\partial Sc(p_1)} + \frac{\partial Sc(p_2)}{\partial p}\Big\{-\frac{2}{d}\frac{\partial Sc(f_2)}{\partial Sc(p_2)} + \varepsilon\frac{d}{2}\frac{\partial Sc(f_1)}{\partial Sc(p_2)}\Big\},$$

$$\frac{\partial N}{\partial q} = \frac{\partial Sc(p_1)}{\partial q}\Big\{-\frac{2}{d}\frac{\partial Sc(f_2)}{\partial Sc(p_1)} + \varepsilon\frac{d}{2}\frac{\partial Sc(f_1)}{\partial Sc(p_1)}\Big\} - \frac{d}{2}\Big\{-\frac{2}{d}\frac{\partial Sc(f_2)}{\partial Sc(p_2)} + \varepsilon\frac{d}{2}\frac{\partial Sc(f_1)}{\partial Sc(p_2)}\Big\}.$$

From Theorem 3.3, we have the following equations by comparing with the equations (3.7) and the derivative of $Sc(f_1)$ and $Sc(f_2)$ for $Sc(p_1)$ and $Sc(p_2)$:

$$\frac{\partial M}{\partial q} = \frac{\partial N}{\partial p} \quad \text{and} \quad \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}.$$

Therefore, the equations (3.8) are obtained. □

Example 3.5. Let $F(Z) = Z = \rho\cos\phi + \mathbf{v}\rho\sin\phi$ on Ω in $\mathbb{H}^2$. Then we have

$$\frac{\partial N}{\partial \phi} = \frac{\partial(\rho\sin\phi)}{\partial\phi} = \frac{1}{2}\rho\cos\phi + \frac{\varepsilon}{2}\rho\cos\phi,$$

$$\frac{\partial M}{\partial \rho} = \cos\phi, \ \frac{\partial N}{\partial \rho} = \cos\phi$$

and

$$\frac{\partial M}{\partial \phi} = \frac{\partial(\rho\cos\phi)}{\partial\phi} = -\frac{1}{2}\rho\sin\phi - \frac{\varepsilon}{2}\rho\sin\phi.$$

Therefore, the function F satisfies the equations (3.8).

References

[1] W. K. Clifford, *Preliminary sketch of bi-quaternions*, Proc. London Math. Soc. **4** (1873), 381–395.

[2] K. Daniilidis, *Hand-eye calibration using dual quaternions*, Int. J. Rob. Res. **18** (1999), no. 3, 286–298.

[3] Z. Ercan and S. Yüce, *On Properties of the Dual Quaternions*, Eur. J. Pure Appl. Math. **4** (2011), no. 2, 142–146.

[4] G. Helzer, *Special Relativity with acceleration*, Amer. Math. Monthy **107** (2000), no. 3, 215–237.

[5] J. Kajiwara, X. D. Li, and K. H. Shon, *Regeneration in complex, quaternion and Clifford analysis*, In Finite or Infinite Dimensional Complex Analysis and Applications, pp. 287–298, Springer US, New York, USA, 2004.

[6] ———, *Function spaces in complex and Clifford analysis*, In Finite or Infinite Dimensional Complex Analysis and Applications, pp. 127–155, Springer US, New York, USA, 2006.

[7] B. Kenwright, *A beginners guide to dual-quaternions: what they are, how they work, and how to use them for 3D character hierarchies*, wscg.zcu.cz/wscg2012/short/a29-full.pdf.

[8] ———, *Inverse kinematics with dual-quaternions, exponential-maps, and joint limits*, Int. J. Adv. Sys. **6** (2013), no. 1-2.
[9] J. E. Kim, S. J. Lim, and K. H. Shon, *Regular functions with values in ternary number system on the complex Clifford analysis*, Abstr. Appl. Anal. **2013** (2013), Artical ID 136120, 7 pages.
[10] ———, *Regularity of functions on the reduced quaternion field in Clifford analysis*, Abstr. Appl. Anal. **2014** (2014), Artical ID 654798, 8 pages.
[11] J. E. Kim and K. H. Shon, *The Regularity of functions on Dual split quaternions in Clifford analysis*, Abstr. Appl. Anal. **2014** (2014), Artical ID 369430, 8 pages.
[12] ———, *Polar Coordinate Expression of Hyperholomorphic Functions on Split Quaternions in Clifford Analysis*, Adv. Appl. Clifford Algebr. **25** (2015), no. 4, 915–924.
[13] ———, *Coset of a hypercomplex numbers in Clifford analysis*, Bull. Korean Math. Soc. **52** (2015), no. 5, 1721–1728.
[14] ———, *Properties of regular functions with values in bicomplex numbers*, Bull. Korean Math. Soc. **53** (2016), no. 2, 507–518.
[15] ———, *Inverse Mapping Theory on Split Quaternions in Clifford Analysis*, To appear in Filomat.
[16] K. Nono, *Hyperholomorphic functions of a quaternion variable*, Bull. Fukuoka Univ. Ed. **32** (1983), 21–37.
[17] E. Pennestrì and R. Stefanelli, *Linear algebra and numerical algorithms using dual numbers*, Multibody Syst. Dyn. **18** (2007), no. 3, 323–344.
[18] J. Plucker, *On a new geometry of space*, Phil. Trans. Roy. Soc. Lond. **155** (1865), 725–791.
[19] R. M. Porter, *Quaternionic linear and quadratic equations*, J. Nat. Geom. **11** (1997), no. 2, 101–106.
[20] E. Study, *Geometrie der Dynamen*, Leipzig, Germany, 1903.

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