Bull. Korean Math. Soc. **54** (2017), No. 2, pp. 573–581
https://doi.org/10.4134/BKMS.b160198
pISSN: 1015-8634 / eISSN: 2234-3016

SOME LIMITING RESULTS OF REFLECTED ORNSTEIN-UHLENBECK PROCESSES WITH TWO-SIDED BARRIERS

Chenglian Zhu

Abstract. Reflected Ornstein-Uhlenbeck process is a process that returns continuously and immediately to the interior of the state space when it attains a certain boundary. In this work, we are concerned with the study of asymptotic behaviours of parametric estimation for ergodic reflected Ornstein-Uhlenbeck processes with two-sided barriers. Moreover, we also focus on the relations between regulators and the local time process.

1. Introduction

It is well-known that the parameter estimation for stochastic processes attracts more and more attention. Especially for Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dW_t,\ t \in [0,\ T], \tag{1.1}$$

involved the unknown parameter $\theta \in \mathcal{R}$, $W = \{W_t,\ t \in [0,\ \infty)\}$. The maximum likelihood estimation of $\theta \in \mathcal{R}$, from the observation of a sample path of the process along the finite interval $[0,\ T]$ as $T \to \infty$, is as follows

$$\hat{\theta}_T = \frac{\int_0^T X_s dX_s}{\int_0^T X_s^2 ds},$$

and its behaviour as $T \to \infty$ is well-known (see e.g. [3], [7], [15]).

(i) If the unknown parameter $\theta < 0$, the process X of (1.1) is positive recurrent, ergodic with invariant distribution $\mathcal{N}(0,\ \frac{1}{-2\theta})$, and for $T \to \infty$ it holds

$$\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\ -2\theta).$$

Here and in the sequel, $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and $\mathcal{N}$ is the normal random variable.

Received March 7, 2016; Revised August 16, 2016.
2010 *Mathematics Subject Classification.* Primary 60F15; Secondary 60F05.
Key words and phrases. reflected Ornstein-Uhlenbeck processes, maximum likelihood estimation, ergodic.

(ii) If $\theta = 0$, the process X of (1.1) is null recurrent with limiting distribution

$$T(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W_t dW_t}{\int_0^1 W_t^2 dt} = \frac{W_1^2 - 1}{2\int_0^1 W_t^2 dt}$$

as $T \to \infty$. Observe that this limiting distribution is neither normal nor a mixture of normals.

(iii) If $\theta > 0$, the process X of (1.1) is not recurrent or transient; it holds $|X_t| \to \infty$ as $t \to \infty$ with probability one and

$$\frac{1}{\sqrt{2\theta}} e^{\theta T}(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{D}} \frac{v}{X_0 + \xi^\theta}$$

on $\{X_0 + \xi^\theta \neq 0\}$, as $T \to \infty$, where $v \sim \mathcal{N}(0,\ 1)$ and $\xi^\theta \sim \mathcal{N}(0,\ \frac{1}{2\theta})$ are two independent Gaussian random variables.

Furthermore, Jiang and Dong [13] studied the asymptotics behaviors for estimators of the parameters in the non-stationary Ornstein-Uhlenbeck process with linear drift, and some more complicate results were derived by them.

In the present work, our goal is to investigate Ornstein-Uhlenbeck process with the reflected barrier. The reflected Ornstein-Uhlenbeck (abbr. ROU) process is a modification of the Ornstein-Uhlenbeck process with an additional regulator, which keeps the ROU process nonnegative.

Given a filtered probability space $\Lambda := (\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. The ROU processes $\{X_t, t \geq 0\}$ reflected at the boundary $b \in \mathcal{R}_+$ on Λ is defined as follows. Let $\{X_t, t \geq 0\}$ be the strong solution whose existence is guaranteed by an extension of the results of Lions and Sznitman [17] to the stochastic differential equation

$$\begin{cases} dX_t = (\beta - \alpha X_t)dt + \sigma dW_t + dL_t - dU_t, & \\ X_t \in [0,\ b] & \text{for all } t \geq 0, \\ X_0 = x, & \end{cases} \tag{1.2}$$

where $b,\ \alpha \in \mathcal{R}_+$, $\sigma \in (0, +\infty)$, $\beta \in \mathcal{R}$ and $\{W_t, t \geq 0\}$ is a one-dimensional standard Wiener process. $L = (L_t)_{t\geq 0}$ and $U = (U_t)_{t\geq 0}$ are uniquely determined by (see, [10]).

• For $t \in [0,\ \infty)$, the sample paths $t \to L_t$ and $t \to U_t$ are continuous non-decreasing and $L_0 = U_0 = 0$.

• The processes L and U increase only on the respective time sets $\{t \in [0,\ \infty);\ X_t = 0\}$ and $\{t \in [0,\ \infty);\ X_t = b\}$. This is equivalent to

$$\int_0^t I(X_t > 0) dL_t = 0, \tag{1.3}$$

and

$$\int_0^t I(X_t < b) dU_t = 0 \tag{1.4}$$

for all $t > 0$, where $I(\cdot)$ denotes the indicator function. Sometimes L and U are called the regulators of the point 0 and b (see, [10]) and by virtue of Ata et al. [1], the paths of the regulator are nondecreasing, right continuous with left limits and possess the support property

$$\int_0^t I(X_s = 0)dL_s = L_t, \tag{1.5}$$

and

$$\int_0^t I(X_s = b)dU_s = U_t. \tag{1.6}$$

In many cases, the stochastic processes are not allowed to cross a certain boundary, or are even supposed to remain within two boundaries. The stochastic processes with the reflection behave like the standard Ornstein-Uhlenbeck processes in the interior of their domain. However, when they reaches the boundary, the sample path returns to the interior in a manner that the "pushing" force is minimal. This kind of processes, which can be applied into the field of queueing system, financial engineering, mathematical biology, has attracted the attention of scholars around the world.

Many attempts have been made to research the ROU processes in the aspects of theory and application, see for example, Ricciardi and Sacerdote [21] applied the ROU processes into the field of mathematical biology. Krugman [14] limited the currency exchange rate dynamics in a target zone by two reflecting barriers. Goldstein and Keirstead [8] explored the term structure of interest rates for the short rate processes with reflecting boundaries. In Hanson et al. [9]), the asset pricing models with truncated price distributions had been investigated. Linetsky [16] studied the analytical representation of transition density for reflected diffusions in terms of their Sturm-Liouville spectral expansions. Bo et al. [4, 5] applied the ROU processes to model the dynamics of asset prices in a regulated market, and the conditional default probability with incomplete (or partial) market information was calculated. Ward and Glynn [22, 23, 24] shown that the ROU processes serve as a good approximation for a Markovian queue with reneging when the arrival rate is either close to or exceeds the processing rate and the reneging rate is small and the ROU processes also well approximate queues having renewal arrival and service processes in which customers have deadlines constraining total sojourn time. Customers either renege from the queue when their deadline expires or balk if the conditional expected waiting time given the queue-length exceeds their deadline.

In practice, some important aspects of performance of a queueing system (e.g. customers' waiting times, traffic intensities) may not be directly observable and therefore such performance measures and their related model parameters need to be statistically inferred from the available observed data. In the case of Ornstein-Uhlenbeck processes driven by Wiener processes, the statistical inference for these processes has been studied and a comprehensive survey of various methods was given in Prakasa Rao [18] and Bishwal [3].

Recently, based on continuous observations, Bo et al. [4] first presented the maximum likelihood estimator (MLE) for the ergodic ROU processes and derived the MLE of the unknown parameter α, on the basis of the process $\{X_t\}$ up to a previously determined fixed time T. Zang and Zhang [25] derived parameter estimation for generalized diffusion processes with reflected boundary.

In this paper, our interest lies in the asymptotic behaviors of maximum likelihood estimator of the reflected Ornstein-Uhlenbeck processes with two barriers and the relation between regulators and local time of the observable process.

Noting that σ in our model is an unknown constant which is independent of the parameter α and the quadratic variation process $[X]_t$ equals to $\sigma^2 t$, $t \geq 0$, we assume that σ is known and set it equal to one in the situation of continuous observations.

2. Main results

Theorem 2.1. *In our model* (1.2)*, we have*

$$L_t = \frac{1}{2}\ell_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I(0 < X_s < \varepsilon) ds, \tag{2.1}$$

and

$$U_t = \frac{1}{2}\ell_t^b = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I(0 < X_s - b < \varepsilon) ds, \tag{2.2}$$

where $\ell = \{\ell_t^a;\ a \geq 0\}$*, denotes the local time process of ROU process* X *at point* a*.*

Theorem 2.2. *In our model* (1.2)*, if* $\alpha > 0$*, we have*

$$\sqrt{T}(\hat{\alpha}_T - \alpha) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\ \frac{1 - \Phi(-\sqrt{\frac{2\beta^2}{\alpha}})}{\frac{1}{4\alpha} + \beta\sqrt{\frac{2\pi}{\alpha}} + \frac{\beta^2}{2}}\right) \ a.s., \tag{2.3}$$

where $\Phi(\cdot)$ *denotes the standard normal distribution function.*

Remark 2.3. If $\beta = 0$ in (1.2), our model turns to be

$$\begin{cases} dX_t = -\alpha X_t dt + \sigma dW_t + dL_t - dU_t, \\ X_t \in [0,\ b] & \text{for all } t \geq 0, \\ X_0 = x, \end{cases}$$

we have

$$\sqrt{T}(\hat{\alpha}_T - \alpha) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\ 2\alpha), \tag{2.4}$$

which is the same as the ergodic Ornstein-Uhlenbeck processes without reflection.

3. Proofs

Proof of Theorem 2.1. In view of Tanaka formula (Protter [19], Huang [12], or Revuz and Yor [20]) for ROU process X, we have

$$\begin{aligned}
X_t &= X_0 + \int_0^t I(X_s > 0)dX_s + \frac{1}{2}\ell_t^0 \\
&= X_t - \int_0^t I(X_s = 0)dX_s + \frac{1}{2}\ell_t^0 \\
&= X_t - \beta \int_0^t I(X_s = 0)ds + \alpha \int_0^t X_s I(X_s = 0)ds - \int_0^t I(X_s = 0)dW_s \\
&\quad - \int_0^t I(X_s = 0)dL_s + \int_0^t I(X_s = 0)dU_s + \frac{1}{2}\ell_t^0 \\
&= X_t - \beta \int_0^t I(X_s = 0)ds - \int_0^t I(X_s = 0)dW_s - L_t + \frac{1}{2}\ell_t^0,
\end{aligned}$$

where we used the equation (1.5), $\int_0^t I(X_s = 0)dU_s = 0$ and $X_t \geq 0$. Then

$$L_t = -\beta \int_0^t I(X_s = 0)ds - \int_0^t I(X_s = 0)dW_s + \frac{1}{2}\ell_t^0.$$

Since $\int_0^t I(X_s = 0)dW_s$ is a continuous local martingale and has finite variation from the above equation, it equals to the initial value 0. Hence, by $\int_0^t I(X_s = 0)ds = 0$, we have

$$L_t = \frac{1}{2}\ell_t^0.$$

The local time of X at b equals to the local time of $-X$ at $-b$. Thus

$$\begin{aligned}
-X_t + b = &\; - X_0 + b - \int_0^t I(-X_s > -b)dX_s + \frac{1}{2}\ell_t^b \\
= &\; - X_t + b + \int_0^t I(X_s = b)dX_s + \frac{1}{2}\ell_t^b \\
= &\; - X_t + b + \beta \int_0^t I(X_s = b)ds - \alpha \int_0^t X_s I(X_s = b)ds \\
&\; + \int_0^t I(X_s = b)dW_s \\
&\; + \int_0^t I(X_s = b)dL_s - \int_0^t I(X_s = b)dU_s + \frac{1}{2}\ell_t^b \\
= &\; - X_t + b + \beta \int_0^t I(X_s = b)ds + \int_0^t I(X_s = b)dW_s - U_t + \frac{1}{2}\ell_t^b,
\end{aligned}$$

where we used the equation (1.6), $\int_0^t I(X_s = b)dL_s = 0$ and $X_t \leq b$. Then

$$U_t = \beta \int_0^t I(X_s = b)ds + \int_0^t I(X_s = b)dW_s + \frac{1}{2}\ell_t^b.$$

Since $\int_0^t I(X_s = b)dW_s$ is a continuous local martingale and has finite variation from the above equation, it equals to the initial value 0. Hence, by $\int_0^t I(X_s = b)ds = 0$, we have

$$U_t = \frac{1}{2}\ell_t^b.$$

The proof is desired. □

Proof of Theorem 2.2. From Theorem 4.1 of Bo et al. [6], we have

$$\begin{aligned} l_T(\alpha) &= \log \frac{dP_\alpha^T}{dP_W^T} \\ &= \beta T - \alpha \int_0^T X_t dX_t - \frac{\alpha^2}{2} \int_0^T X_t^2 dt + \alpha \int_0^T X_t dL_t - \alpha \int_0^T X_t dU_t. \end{aligned}$$

The estimator $\hat{\alpha}_T$ of α is naturally defined as

$$\hat{\alpha}_T := \arg \sup_{\alpha \in \Theta} l_T(\alpha).$$

Then, we can derive the maximum likelihood estimator $\hat{\alpha}_T$ of the parameter α by solving the likelihood equation $l'_T(\alpha) = 0$. On the other hand, it follows that

$$l'_T(\alpha) = -\int_0^T X_t dW_t \tag{3.1}$$

and

$$l''_T(\alpha) = -\int_0^T X_t^2 dt. \tag{3.2}$$

Thus, in view of Taylor's expansion, it follows that

$$l'_T(\alpha) = l'_T(\hat{\alpha}_T) + (\alpha - \hat{\alpha}_T) l''_T(\xi_T),$$

where $|\xi_T| \leq 1$. Then

$$\hat{\alpha}_T - \alpha = \frac{l'_T(\alpha)}{\int_0^T X_t^2 dt}.$$

Furthermore, by (3.1) and local martingale central limit theorem, we get

$$(\hat{\alpha}_T - \alpha)\sqrt{\int_0^T X_t^2 dt} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\ 1). \tag{3.3}$$

It can be proved that the process $\{X(t)\}_{t\geq 0}$ in the model is ergodic and the unique invariant density of $\{X(t)\}_{t\geq 0}$ is given (Hu et al. [11]) by

$$p(x) = \frac{\sqrt{2\alpha}\phi(\sqrt{2\alpha}(x-\beta))}{1-\Phi(-\sqrt{\frac{2\beta^2}{\alpha}})},\ x \in [0,\ \infty), \tag{3.4}$$

where $\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\Phi(u) = \int_{-\infty}^{y}\phi(u)du$ are the Gaussian density and the error function, respectively. Therefore, the mean ergodic theorem holds (Hu et al. [11]), i.e.,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))ds = \int_0^\infty f(x)p(x)dx \quad \text{a.s. } [P_\alpha^T] \tag{3.5}$$

for any $x \in S := [0,\ \infty)$ and any $f \in L_1(S,\ \mathcal{B}(s))$. Let $f(x) = x^2$, we have

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)ds &= \int_0^\infty x^2p(x)dx \\
&= \frac{\sqrt{2\alpha}}{1-\Phi(-\sqrt{\frac{2\beta^2}{\alpha}})}\int_0^\infty x^2\phi(\sqrt{2\alpha}(x-\beta))dx \\
&= \frac{1}{1-\Phi(-\sqrt{\frac{2\beta^2}{\alpha}})}\int_0^\infty (\frac{t}{\sqrt{2\alpha}}+\beta)^2\phi(t)dt \\
&= \frac{\frac{1}{4\alpha}+\beta\sqrt{\frac{2\pi}{\alpha}}+\frac{\beta^2}{2}}{1-\Phi(-\sqrt{\frac{2\beta^2}{\alpha}})} \quad \text{a.s. } [P_\alpha^T].
\end{aligned} \tag{3.6}$$

Coupled with (3.3), we have

$$\sqrt{T}(\hat{\alpha}_T-\alpha) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\ \frac{1-\Phi(-\sqrt{\frac{2\beta^2}{\alpha}})}{\frac{1}{4\alpha}+\beta\sqrt{\frac{2\pi}{\alpha}}+\frac{\beta^2}{2}}\right) \quad \text{a.s.} \tag{3.7}$$

The proof is completed. □

Acknowledgements. The author acknowledges partial research support from National Natural Science Foundation of China (Grant No. 11326174 and 11401245), Natural Science Foundation of Jiangsu Province (Grant No. BK20130412), Natural Science Research Project of Ordinary Universities in Jiangsu Province (Grant No. 12KJB110003), China Postdoctoral Science Foundation (Grant No. 2014M551720).

References

[1] B. Ata, J. M. Harrison, and L. A. Shepp, *Drift rate control of a Brownian processing system*, Ann. Appl. Probab. **15** (2005), no. 2, 1145–1160.

[2] J. P. N. Bishwal, *Large deviations inequalities for the maximum likelihood estimator and the Bayes estimators in nonlinear stochastic differential equations*, Statist. Probab. Lett. **43** (1999), no. 2, 207–215.

[3] ———, *Parameter estimation in stochastic differential equations*, Lecture Notes in Mathematics, vol. 1923, Springer, Berlin Heidelberg, New York, 2008.

[4] L. Bo, D. Tang, Y. Wang, and X. Yang, *On the conditional default probability in a regulated market: a structural approach*, Quant. Finance **11** (2011), no. 12, 1695–1702.

[5] L. Bo, Y. Wang, and X. Yang, *Some integral functionals of reflected SDEs and their applications in finance*, Quant. Finance **11** (2011), no. 3, 343–348.

[6] L. Bo, Y. Wang, X. Yang, and G. Zhang, *Maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes*, J. Statist. Plann. Inference **141** (2011), no. 1, 588–596.

[7] P. D. Feigin, *Some comments concerning a curious singularity*, J. Appl. Prob. **16** (1979), 440–444.

[8] R. S. Goldstein and W. P. Keirstead, *On the term structure of interest rates in the presence of reflecting and absorbing boundaries*, DOI:10.2139/ssrn.10.2139/ssrn.19840 (1997).

[9] S. D. Hanson, R. J. Myers, and J. H. Hilker, *Hedging with futures and options under truncated cash price distribution*, J. Agricultural Appl. Economics **31** (1999), 449–459.

[10] M. Harrison, *Brownian Motion and Stochastic Flow Systems*, John Wiley and Sons, New York, 1986.

[11] Y. Hu, C. Lee, M. H. Lee, and J. Song, *Parameter estimation for reflected Ornstein-Uhlenbeck processes with discrete observations*, preprint, 2013.

[12] Z. Y. Huang, *Foundation in Stochastic Calculus*, Science Press, Beijing (in Chinese), 2001.

[13] H. Jiang and X. Dong, *Parameter estimation for the non-stationary Ornstein-Uhlenbeck process with linear drift*, Statist. Papers **56** (2015), no. 1, 257–268.

[14] P. R. Krugman, *Target zones and exchange rate dynamics*, Quarterly J. Economics **106** (1991), 669–682.

[15] Y. A. Kutoyants, *Statiatical Inference for Ergodic Diffusion Processes*, Springer, London, 2004.

[16] V. Linetsky, *On the transition densities for reflected diffusions*, Adv. in Appl. Probab. **37** (2005), no. 2, 435–460.

[17] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math. **37** (1984), no. 4, 511–537.

[18] B. L. S. Prakasa Rao, *Statistical Inference for Diffusion Type Processes*, Oxford University Press, New York, 1999.

[19] P. E. Protter, *Stochastic Integration and Differential Equations*, Springer Verlag, Berlin, 2004.

[20] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer Verlag, Berlin, 1998.

[21] L. M. Ricciardi and L. Sacerdote, *On the probability densities of an Ornstein-Uhlenbeck process with a reflecting boundary*, J. Appl. Probab. **24** (1987), no. 2, 355–369.

[22] A. Ward and P. Glynn, *A diffusion approximation for a Markovian queue with reneging*, Queueing Syst. **43** (2003), no. 1-2, 103–128.

[23] ———, *Properties of the reflected Ornstein-Uhlenbeck process*, Queueing Syst. **44** (2003), no. 2, 109–123.

[24] ———, *A diffusion approximation for a GI/GI/1 queue with balking or reneging*, Queueing Syst. **50** (2005), no. 4, 371–400.

[25] Q. Zang and L. Zhang, *Parameter estimation for generalized diffusion processes with reflected boundary*, Sci. China Math. **59** (2016), no. 6, 1163–1174.

Chenglian Zhu
School of Mathematical Science
Huaiyin Normal University
Huaian, Jiangsu 223300, P. R. China
E-mail address: zqphunhu@hytc.edu.cn