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INJECTIVE PROPERTY RELATIVE TO NONSINGULAR EXACT SEQUENCES

Marzieh Arabi-Kakavand, Shadi Asgari, and Yaser Tolooei

Abstract. We investigate modules M having the injective property relative to nonsingular modules. Such modules are called "$\mathcal{N}$-injective modules". It is shown that M is an $\mathcal{N}$-injective R-module if and only if the annihilator of $Z_2(R_R)$ in M is equal to the annihilator of $Z_2(R_R)$ in $E(M)$. Every $\mathcal{N}$-injective R-module is injective precisely when R is a right nonsingular ring. We prove that the endomorphism ring of an $\mathcal{N}$-injective module has a von Neumann regular factor ring. Every (finitely generated, cyclic, free) R-module is $\mathcal{N}$-injective, if and only if $R^{(\mathbb{N})}$ is $\mathcal{N}$-injective, if and only if R is right t-semisimple. The $\mathcal{N}$-injective property is characterized for right extending rings, semilocal rings and rings of finite reduced rank. Using the $\mathcal{N}$-injective property, we determine the rings whose all nonsingular cyclic modules are injective.

1. Introduction

To describe the content of the paper we first state some notations and recall a few relevant results. Throughout, all rings are associative with unity and all modules are unitary right modules. For a subset K of an R-module M, we denote $r_R(K) = \{r \in R : Kr = 0\}$, and for a subset I of R we denote $l_M(I) = \{m \in M : mI = 0\}$. Recall that the singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that $mI = 0$ for some essential right ideal I of R, or equivalently, $r_R(m) \leq_e R_R$ (the notation $\leq_e$ denotes an essential submodule). The Goldie torsion (or second singular) submodule $Z_2(M)$ of M is defined by $Z_2(M)/Z(M) = Z(M/Z(M))$. The following facts are well known: $Z_2(M/Z_2(M)) = 0$. If $f : M \to N$ is a homomorphism, then $f(Z_2(M)) \leq Z_2(N)$. Moreover, $Z_2(M) \cap A = Z_2(A)$ for every submodule A of M, and $Z_2(\bigoplus_\lambda M_\lambda) = \bigoplus_\lambda Z_2(M_\lambda)$ for every class of R-modules M_λ. A module M is called singular if $Z(M) = M$ and nonsingular if $Z(M) = 0$, or equivalently, $Z_2(M) = 0$. The module M is called Z_2-torsion if $Z_2(M) = M$.

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Clearly, a submodule A of M is Z_2-torsion if and only if $A \leq Z_2(M)$. The class of Z_2-torsion modules is closed under submodules, factor modules, direct sums, and extensions. In [2], a submodule A of M is called t-essential in M (written by $A \leq_{tes} M$) if for every submodule B of M, $A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. Using this notion, it is easy to see that $Z_2(M)$ is the set of $m \in M$ such that $mI = 0$ for some t-essential right ideal I of R, or equivalently, $r_R(m) \leq_{tes} R_R$. Following [2], a submodule C of M is said to be t-closed in M if $C \leq_{tes} C' \leq M$ implies that $C = C'$; and a module M is called t-extending if every t-closed submodule of M is a direct summand. In fact, t-extending modules are precisely the modules M for which every closed submodule of M containing $Z_2(M)$ is a direct summand of M.

Over the last 50 years numerous mathematicians have investigated rings over which certain cyclic modules have a homological property. Among these, determining the rings whose certain cyclic modules are injective has been of interest. Osofsky [12] proved that every cyclic R-module is injective, if and only if every R-module is injective, if and only if R is semisimple. A cyclic R-module is called proper cyclic if it is not isomorphic to R. A ring R is called a right PCI-ring if every proper cyclic R-module is injective. Faith [5] proved that a right PCI-ring is either a semisimple ring or a simple right semihereditary right Ore domain. An excellent reference for a thorough study of these rings is [8]. The rings for which every singular module is injective were studied by Goodearl [6]. He called them right SI-rings and characterized such rings as those nonsingular ones for which R/I is semisimple for every essential right ideal I of R. Osofsky and Smith [13] showed that every singular cyclic R-module is injective if and only if R is a right SI-ring. More results on such rings can be found in [4] and [14]. Motivated by these, a natural question is: "What are the rings whose all nonsingular cyclic modules are injective?" In [3] the rings whose all nonsingular modules are injective were studied. Such rings are called right t-semisimple rings. It was shown that R is right t-semisimple, if and only if every nonsingular R-module is semisimple, if and only if $R/Z_2(R_R)$ is a semisimple ring, if and only if R is a direct product of two rings, one is semisimple and the other is right Z_2-torsion. By [3, Example 4.15], the class of right t-semisimple rings is properly contained in that of rings R for which every nonsingular cyclic R-module is injective. This raises another question: "Under which condition(s) the class of rings R for which every nonsingular cyclic R-module is injective coincides with that of right t-semisimple rings?" But, it is a fact, obtained by Baer's criterion, that a nonsingular R-module M is injective precisely when M is injective relative to the nonsingular R-module $R/Z_2(R_R)$. This leads us to investigate the modules M which are injective relative to nonsingular modules for finding the answers of the above questions.

Let M and L be R-modules. Recall that M is said to be L-injective (or, injective relative to L) if for every monomorphism $f : K \to L$ and every homomorphism $g : K \to M$, there is a homomorphism $h : L \to M$ such that $hf = g$. We say that an R-module M is $\mathcal{N}$-injective if M is injective relative to every nonsingular R-module; in other words, M is injective relative to every nonsingular exact sequence $0 \to K \to L$. (Note that every submodule of a nonsingular module is nonsingular.) Section 2 is devoted to study $\mathcal{N}$-injective modules. Every injective module and every module over a right t-semisimple ring are $\mathcal{N}$-injective. It is proved that M is $\mathcal{N}$-injective, if and only if M is injective relative to $R/Z_2(R_R)$, if and only if $l_M(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$, if and only if $M = Z_2(M) \oplus M'$, where $Z_2(M)$ is $\mathcal{N}$-injective and M' is injective (Theorem 2.2). A nonsingular module is $\mathcal{N}$-injective if and only if it is injective (Corollary 2.3(i)). For a module M,

$$\text{injective} \Rightarrow \mathcal{N}\text{-injective} \Rightarrow t\text{-extending},$$

but none of these implications is reversible (Corollary 2.3(ii)). The classes of injective R-modules and $\mathcal{N}$-injective R-modules coincide if and only if R is a right nonsingular ring (Proposition 2.7). We prove that if M is an $\mathcal{N}$-injective module, then S/T is a von Neumann regular ring, where $S = \mathrm{End}(M)$ and $T = \{\varphi \in S : \varphi M \leq Z_2(M)\}$ (Theorem 2.9). This implies that $R/Z_2(R_R)$ is a von Neumann regular ring whenever R is $\mathcal{N}$-injective (Corollary 2.10).

In Section 3, we give several characterizations obtained by the $\mathcal{N}$-injective property. It is proved that R is a right t-semisimple ring, if and only if every (finitely generated, cyclic, free) R-module is $\mathcal{N}$-injective, if and only if $R^{(\mathbb{N})}$ is $\mathcal{N}$-injective (Theorem 3.1). This, in particular, implies that a semilocal ring is $\mathcal{N}$-injective precisely when R is right t-semisimple (Corollary 3.2). In the sequel, it is shown that R is $\mathcal{N}$-injective if and only if $Z_2(R_R)$ is $R/Z_2(R_R)$-injective and every nonsingular cyclic R-module is injective and projective (Proposition 3.6). A right extending ring R is $\mathcal{N}$-injective if and only if $R/Z_2(R_R)$ is a right self-injective ring (Theorem 3.7). Moreover, if R is a ring of finite reduced rank, then R is $\mathcal{N}$-injective if and only if R is right t-semisimple (Proposition 3.8).

By the obtained results, we find some answers to the above mentioned questions: i) The rings whose every nonsingular cyclic module is injective are characterized. In fact, R is such a ring if and only if $R/Z_2(R_R)$ is a right self-injective ring, and if R is right extending, these are equivalent to R being right $\mathcal{N}$-injective (Theorem 3.7). ii) The class of rings R for which every nonsingular cyclic R-module is injective coincides with that of right t-semisimple rings whenever R is either semilocal or of finite reduced rank (Corollary 3.10).

2. $\mathcal{N}$-injective modules

We say that an R-module M is $\mathcal{N}$-injective if M is injective relative to every nonsingular R-module. Clearly, every injective R-module is $\mathcal{N}$-injective. The following example shows that the class of $\mathcal{N}$-injective R-modules properly contains that of injective R-modules. More examples of $\mathcal{N}$-injective modules will be given in Examples 2.6.

Example 2.1. Let R_1 be a right Z_2-torsion ring (e.g., $R_1 = \mathbb{Z}/p^2\mathbb{Z}$, where p is a prime number), R_2 be a semisimple ring (e.g., $R_2 = D$ is a division ring), and $R = R_1 \times R_2$. Assume that M is an R-module, $f : A \to B$ is an R-monomorphism where B is a nonsingular R-module, and $g : A \to M$ is an R-homomorphism. By [3, Theorems 3.2(4) and 3.8(3)], A is a direct summand of B, and hence g can be extended to an R-homomorphism $h : B \to M$. This shows that M is $\mathcal{N}$-injective.

The next result gives several equivalent conditions for an $\mathcal{N}$-injective module.

Theorem 2.2. *The following statements are equivalent for an R-module M.*

(1) *M is $\mathcal{N}$-injective.*

(2) *M is $R/Z_2(R_R)$-injective.*

(3) *$l_M(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$.*

(4) *$l_M(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module.*

(5) *$M = Z_2(M) \oplus M'$, where $Z_2(M)$ is $\mathcal{N}$-injective and M' is injective.*

(6) *For every monomorphism $f : A \to B$ of R-modules where A is nonsingular, and every R-homomorphism $g : A \to M$, there exists an R-homomorphism $h : B \to M$ such that $hf = g$.*

Proof. (1) $\Rightarrow$ (6). Let $f : A \to B$ be a monomorphism of R-modules where A is nonsingular, and $g : A \to M$ be a homomorphism. Assume that $\pi : B \to B/Z_2(B)$ is the natural epimorphism. Since A is nonsingular, $\pi f : A \to B/Z_2(B)$ is a monomorphism. So by hypothesis, there exists a homomorphism $\theta : B/Z_2(B) \to M$ such that $\theta\pi f = g$. Set $h = \theta\pi$.

(6) $\Rightarrow$ (5). Let C be a complement of $Z_2(M)$ in M, and $f : C \to E(C)$ be the inclusion map, where $E(C)$ is the injective hull of C. Moreover, assume that $g : C \to M$ is the inclusion map. By hypothesis, there exists a homomorphism $h : E(C) \to M$ such that $hf = g$. Since g is a monomorphism and $C \leq_e E(C)$, we conclude that h is a monomorphism. Thus $h(E(C)) \cong E(C)$ is injective, and so $h(E(C))$ is a direct summand of M, say $M = K \oplus h(E(C))$. Since C is nonsingular we conclude that $E(C)$ is nonsingular, and so $h(E(C))$ is nonsingular. Thus $Z_2(M) \leq K$. On the other hand, $c = g(c) = hf(c) = h(c)$, for every $c \in C$. Thus $C \leq h(E(C))$. Hence $Z_2(M) \oplus C \leq_e M$ implies that $Z_2(M) \leq_e K$. But $Z_2(M)$ is closed, and so $Z_2(M) = K$. Since M satisfies (6) and $Z_2(M)$ is a direct summand of M, it is easy to see that $Z_2(M)$ also satisfies (6). Thus $Z_2(M)$ is $\mathcal{N}$-injective. Now by setting $M' = h(E(C))$, the desired decomposition is obtained.

(5) $\Rightarrow$ (2). Since $Z_2(M)$ and M' are $R/Z_2(R_R)$-injective, so is M.

(2) $\Rightarrow$ (4). Let $\overline{R} = R/Z_2(R_R)$, and $\overline{I}$ be a right ideal of $\overline{R}$. Moreover, assume that $g : \overline{I} \to l_M(Z_2(R_R))$ is an $\overline{R}$-homomorphism. By hypothesis g can be extended to an R-homomorphism $h : \overline{R} \to M$. But clearly, $h(\overline{R}) \leq l_M(Z_2(R_R))$, and so g can be extended to the $\overline{R}$-homomorphism $h : \overline{R} \to l_M(Z_2(R_R))$. Thus by Baer's criterion, $l_M(Z_2(R_R))$ is an injective $\overline{R}$-module.

(4) $\Rightarrow$ (3). Set $\overline{R} = R/Z_2(R_R)$, and $K = l_M(Z_2(R_R))$. By [7, Exercise 5J], $l_{E(K)}(Z_2(R_R)) = E(K_{\overline{R}})$. Now we show that $l_{E(K)}(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$. Clearly, $E(K)$ is a direct summand of $E(M)$, say $E(K) \oplus D = E(M)$. Let

$x \in l_{E(M)}(Z_2(R_R))$ and $x = e + d$, where $e \in E(K)$ and $d \in D$. Obviously, $e \in l_{E(K)}(Z_2(R_R))$ and $d \in l_D(Z_2(R_R))$. If $d \neq 0$, then there exists $r \in R$ such that $0 \neq dr \in M$. Thus $drZ_2(R_R) \leq dZ_2(R_R) = 0$, and so $dr \in K \cap D = 0$ which is impossible. Hence $d = 0$ and $x = e \in l_{E(K)}(Z_2(R_R))$. This shows that $l_{E(K)}(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$, as desired. Therefore $E(K_{\overline{R}}) = l_{E(M)}(Z_2(R_R))$. Since $K_{\overline{R}}$ is injective we conclude that $l_M(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$.

(3) ⇒ (1). First note that $l_{E(M)}(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module. In fact, let $\overline{R} = R/Z_2(R_R)$, and $\overline{I}$ be a right ideal of $\overline{R}$. Moreover, let $\varphi : \overline{I} \to l_{E(M)}(Z_2(R_R))$ be an $\overline{R}$-homomorphism. Then φ can be extended to an R-homomorphism $\psi : \overline{R} \to E(M)$. But clearly, $\psi(\overline{R}) \leq l_{E(M)}(Z_2(R_R))$, and so φ is extended to the $\overline{R}$-homomorphism $\psi : \overline{R} \to l_{E(M)}(Z_2(R_R))$. Thus by Baer's criterion we conclude that $l_{E(M)}(Z_2(R_R))$ is an injective $\overline{R}$-module, as desired.

Now let N be a nonsingular R-module, $f : A \to N$ be an R-monomorphism and $g : A \to M$ be an R-homomorphism. Since A is nonsingular, $AZ_2(R_R) = 0$, and hence $g(A) \leq l_M(Z_2(R_R))$. But, by hypothesis and what we have shown above $l_M(Z_2(R_R))$ is an injective $\overline{R}$-module. So there exists an $\overline{R}$-homomorphism $h : N \to l_M(Z_2(R_R))$ such that $hf = g$. Clearly, $h : N \to M$ is an R-homomorphism. This shows that M is $\mathcal{N}$-injective. □

Corollary 2.3. (i) *A nonsingular module M is $\mathcal{N}$-injective if and only if M is injective.*

(ii) *If M is an $\mathcal{N}$-injective module, then M is t-extending.*

Proof. (i) This follows from Theorem 2.2(5).

(ii) This is obtained by Theorem 2.2(5) and [2, Theorem 2.11(3)]. □

The converse implication of Corollary 2.3(ii) is not always true. For example, $\mathbb{Z}$ is an extending module which is not injective, hence it is not $\mathcal{N}$-injective by Corollary 2.3(i).

Corollary 2.4. *The following statements are equivalent for a ring R.*

(1) *$R/Z_2(R_R)$ is a right Noetherian ring.*

(2) *$M^{(\mathbb{N})}$ is $\mathcal{N}$-injective, for every $\mathcal{N}$-injective module M.*

(3) *Every direct sum of $\mathcal{N}$-injective modules is $\mathcal{N}$-injective.*

Proof. (1) ⇒ (3). Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where each M_λ is $\mathcal{N}$-injective. By Theorem 2.2(4), $l_{M_\lambda}(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module. Hence $l_M(Z_2(R_R)) = \bigoplus_{\lambda\in\Lambda} l_{M_\lambda}(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module since $R/Z_2(R_R)$ is right Noetherian. Thus by Theorem 2.2(4), M is $\mathcal{N}$-injective.

(3) ⇒ (2). This implication is clear.

(2) ⇒ (1). By [11, Theorem 7.48(4)], it suffices to show that $M^{(\mathbb{N})}$ is an injective $R/Z_2(R_R)$-module, for every injective $R/Z_2(R_R)$-module M. Since M

is $R/Z_2(R_R)$-injective as an R-module, Theorem 2.2(2) implies that M is $\mathcal{N}$-injective. Thus by hypothesis, $M^{(\mathbb{N})}$ is $\mathcal{N}$-injective, hence $R/Z_2(R_R)$-injective. So $M^{(\mathbb{N})}$ is an injective $R/Z_2(R_R)$-module. □

A ring R is called a right V-ring (or right co-semisimple) if every simple R-module is injective.

Corollary 2.5. *The following statements are equivalent for a ring R.*
(1) *Every simple R-module is $\mathcal{N}$-injective.*
(2) *$R/Z_2(R_R)$ is a right V-ring.*

Proof. (1) ⇒ (2). Let S be a simple $R/Z_2(R_R)$-module. Clearly, S is a simple R-module, and so as an R-module, S is $\mathcal{N}$-injective, hence $R/Z_2(R_R)$-injective. Thus S is an injective $R/Z_2(R_R)$-module.

(2) ⇒ (1). Let S be a simple R-module. Clearly, $l_S(Z_2(R_R))$ is S or 0. So by hypothesis, $l_S(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module. Hence S is $\mathcal{N}$-injective by Theorem 2.2(4). □

In the following we give more examples of $\mathcal{N}$-injective modules.

Examples 2.6. (i) Let U be a right Z_2-torsion ring (e.g., $U = \mathbb{Z}/p^2\mathbb{Z}$ for a prime number p). Then $T = \left(\begin{smallmatrix} U & U \\ 0 & U \end{smallmatrix}\right)$ is a right Z_2-torsion ring; see [3, Proposition 3.11]. Set $R = T \times \mathbb{Z}$, and $M = T \times \mathbb{Q}$. Since T is right Z_2-torsion, every T-module X is Z_2-torsion (note that $XZ_2(T_T) \leq Z_2(X)$), and hence every T-module is $\mathcal{N}$-injective. On the other hand, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Therefore T is an $\mathcal{N}$-injective R-module and $\mathbb{Q}$ is an injective R-module. But, $Z_2(M) = T$, and so by Theorem 2.2(5), M is an $\mathcal{N}$-injective R-module.

(ii) Let R_1 be a right Z_2-torsion ring (e.g., $R_1 = \prod_p \mathbb{Z}/p^2\mathbb{Z}$, where p runs through the set of prime numbers), R_2 a right nonsingular right Noetherian ring (e.g., $R_2 = \left(\begin{smallmatrix} D & D \\ 0 & D \end{smallmatrix}\right)$, where D is a division ring), and $R = R_1 \times R_2$. By [3, Lemma 3.10], $Z_2(R_R) = R_1$, and so $R/Z_2(R_R) \cong R_2$ is right Noetherian. Now let M be an R-module and Λ be a set. By Corollary 2.4, $E(M)^{(\Lambda)}$ is an $\mathcal{N}$-injective R-module.

(iii) Let R_1 be a right Z_2-torsion ring (e.g., $R_1 = \prod_\Lambda \mathbb{Z}/p^2\mathbb{Z}$, where p is a prime number and Λ is a set), R_2 a right nonsingular right V-ring (e.g., R_2 is a field), and $R = R_1 \times R_2$. Then $Z_2(R_R) = R_1$, and so $R/Z_2(R_R) \cong R_2$ is a right V-ring. Thus by Corollary 2.5, R/L is an $\mathcal{N}$-injective R-module, for every maximal right ideal L of R.

The following result shows that the classes of $\mathcal{N}$-injective R-modules and injective R-modules coincide if and only if R is a right nonsingular ring.

Proposition 2.7. *The following statements are equivalent for a ring R.*
(1) *Every $\mathcal{N}$-injective R-module is injective.*
(2) *R is right nonsingular.*

Proof. The implication (2) ⇒ (1) follows from Theorem 2.2. For (1) ⇒ (2), set $A = l_R(Z_2(R_R))$. We show that A is an essential right ideal of R. Let I be a right ideal of R such that $A \cap I = 0$. So $l_K(Z_2(R_R)) = 0$ for every R-submodule K of I. Thus by Theorem 2.2(4), K is $\mathcal{N}$-injective, and so by hypothesis it is injective. This implies that I is a semisimple direct summand of R. On the other hand, if J is a nonsingular right ideal of R, then $JZ_2(R_R) \leq Z_2(J) = 0$, and so $J \leq A$. Hence by the semisimple property of I we conclude that I is singular. But R cannot contain a nonzero singular direct summand, and so $I = 0$. This shows that A is an essential right ideal of R. Thus $E(A) = E(R_R)$. By Theorem 2.2(4), $l_{E(A)}(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module, and so it is $\mathcal{N}$-injective as an R-module. Thus by hypothesis, $l_{E(A)}(Z_2(R_R))$ is an injective R-module. But $A \leq l_{E(A)}(Z_2(R_R))$, and so $l_{E(A)}(Z_2(R_R)) = E(A)$. Thus $Z_2(R_R) = RZ_2(R_R) \leq E(R_R)Z_2(R_R) = E(A)Z_2(R_R) = 0$. Hence R is right nonsingular. □

Corollary 2.8. *The following statements are equivalent for a ring R.*
(1) *Every $\mathcal{N}$-injective R-module is projective.*
(2) *R is semisimple.*

Proof. It suffices to show that (1) ⇒ (2). By hypothesis, every injective R-module is projective. So R is quasi-Frobenius, and hence every projective R-module is injective; see [11, Theorems 7.55 and 7.56(2)]. Thus hypothesis implies that every $\mathcal{N}$-injective R-module is injective. Hence R is right nonsingular by Proposition 2.7. So by [3, Corollary 4.6], R is semisimple. □

We end this section by proving that the endomorphism ring of an $\mathcal{N}$-injective module has a von Neumann regular factor ring. It will be observed that the endomorphism ring of an $\mathcal{N}$-injective module is not necessarily von Neumann regular; see Remark 3.5.

Theorem 2.9. *Let M be a module, $S = \mathrm{End}(M)$, and $T = \{\varphi \in S : \varphi M \leq Z_2(M)\}$. If M is $\mathcal{N}$-injective, then S/T is a von Neumann regular ring.*

Proof. First we show that T is a two-sided ideal of S. Let $\varphi \in T$ and $\psi \in S$. Since $\varphi \in T$ we conclude that $\varphi^{-1}(Z_2(M)) = M$. But clearly, $\varphi^{-1}(Z_2(M)) \leq (\psi\varphi)^{-1}(Z_2(M))$, hence $(\psi\varphi)^{-1}(Z_2(M)) = M$. So $\psi\varphi \in T$. On the other hand, $(\varphi\psi)^{-1}(Z_2(M)) = \psi^{-1}(\varphi^{-1}(Z_2(M))) = \psi^{-1}(M) = M$. Hence $\varphi\psi \in T$. This shows that T is a two-sided ideal of S.

Now we show that S/T is von Neumann regular. Let $\psi \in S$. By Corollary 2.3(ii), M is t-extending. So by [2, Theorem 2.11(5)], there exists a direct summand D of M, say $M = D \oplus E$, such that $\psi^{-1}(Z_2(M)) \leq_{tes} D$. Assume that 'bar' denotes the image in $M/Z_2(M)$. Since $Z_2(M) \leq \psi^{-1}(Z_2(M))$ we conclude that $\overline{M} = \overline{D} \oplus \overline{E}$. Moreover, $\overline{\psi} : \overline{E} \to \overline{\psi E}$ defined by $\overline{\psi}\,\overline{x} = \overline{\psi x}$ is an isomorphism ($\overline{\psi}$ is one-to-one, since $\psi x \in Z_2(M)$ implies that $x \in \psi^{-1}(Z_2(M)) \cap E \leq D \cap E = 0$). But $\overline{M}$ is injective by Theorem 2.2(5), and so $\overline{M}$ has C_2 condition. Thus $\overline{\psi E}$ is a direct summand of $\overline{M}$, say $\overline{M} = \overline{\psi E} \oplus \overline{K}$. This implies that $M = \psi E \oplus (K + Z_2(M))$; in fact, it is enough to show that $\psi E \cap (K + Z_2(M)) = 0$. Let $\psi x = k + z$, where $x \in E$, $k \in K$ and $z \in Z_2(M)$. Then $\psi x + Z_2(M) =$

$k + Z_2(M) \in \overline{\psi E} \cap \overline{K} = 0$. Thus $x \in \psi^{-1}(Z_2(M)) \cap E = 0$, and hence $\psi E \cap (K+Z_2(M)) = 0$, as desired. On the other hand, $\psi^{-1}(Z_2(M)) \cap E = 0$ implies that $\psi|_E : E \to \psi E$ is an isomorphism. Set $\theta = (\psi|_E)^{-1} \oplus 1_{K+Z_2(M)} \in S$. Clearly, $\psi^{-1}(Z_2(M)) \oplus E \leq (\psi - \psi\theta\psi)^{-1}(Z_2(M))$. But $\psi^{-1}(Z_2(M)) \leq_{tes} D$ implies that $\psi^{-1}(Z_2(M)) \oplus E \leq_{tes} D \oplus E = M$ by [2, Proposition 2.2(4)]. Thus $(\psi - \psi\theta\psi)^{-1}(Z_2(M)) \leq_{tes} M$. Moreover, $(\psi - \psi\theta\psi)^{-1}(Z_2(M))$ is t-closed in M by [2, Corollary 2.7]. Thus $(\psi - \psi\theta\psi)^{-1}(Z_2(M)) = M$. Hence $\psi - \psi\theta\psi \in T$, and so S/T is von Neumann regular. □

Corollary 2.10. *Let a ring R be $\mathcal{N}$-injective.*

(i) *$R/Z_2(R_R)$ is a von Neumann regular ring.*

(ii) *$\mathrm{Rad}(P) \leq Z_2(P)$ for every projective R-module P.*

Proof. (i) Let $r \in R$, and f_r be the endomorphism of R defined by $f_r(x) = rx$. If $r \in Z_2(R_R)$, then $f_r(R) \leq Z_2(R_R)$. If $f_r(R) \leq Z_2(R_R)$, then $f_r(1) = r \in Z_2(R_R)$. Therefore under the ring isomorphism $\Phi : R \to S = \mathrm{End}(R_R)$ defined by $\Phi(r) = f_r$, the ideal $Z_2(R_R)$ is isomorphic to $T = \{\varphi \in S : \varphi R \leq Z_2(R_R)\}$. Hence $R/Z_2(R_R) \cong S/T$, and so by Theorem 2.9, $R/Z_2(R_R)$ is a von Neumann regular ring.

(ii) Since the Jacobson radical of a von Neumann regular ring is zero, (i) implies that $\mathrm{Rad}(R) \leq Z_2(R_R)$. Hence $\mathrm{Rad}(P) = P\mathrm{Rad}(R) \leq PZ_2(R_R) \leq Z_2(P)$. □

3. More characterizations

In this section we give several characterizations obtained by the $\mathcal{N}$-injective property. For right extending rings, semilocal rings and rings of finite reduced rank, the $\mathcal{N}$-injective property is characterized. Moreover, we determine the rings R for which every nonsingular cyclic R-module is injective. Recall that a ring R is right t-semisimple if and only if $R/Z_2(R_R)$ is a semisimple ring.

Theorem 3.1. *The following statements are equivalent for a ring R.*

(1) *Every free (projective) R-module is $\mathcal{N}$-injective.*

(2) *Every cyclic R-module is $\mathcal{N}$-injective.*

(3) *Every R-module is $\mathcal{N}$-injective.*

(4) *R is right t-semisimple.*

(5) *$R^{(\mathbb{N})}$ is $\mathcal{N}$-injective.*

(6) *$[l_R(Z_2(R_R))]^{(\mathbb{N})}$ is an injective $R/Z_2(R_R)$-module.*

Proof. (1) ⇒ (4). Let $[R/Z_2(R_R)]^{(\Lambda)}$ be a free $R/Z_2(R_R)$-module. Since $Z_2(R^{(\Lambda)}) = Z_2(R_R)^{(\Lambda)}$ we conclude that $[R/Z_2(R_R)]^{(\Lambda)} \cong R^{(\Lambda)}/Z_2(R^{(\Lambda)})$. Hence by hypothesis and Theorem 2.2(5), the module $[R/Z_2(R_R)]^{(\Lambda)}$ is an injective R-module, and so it is an injective $R/Z_2(R_R)$-module. Thus $R/Z_2(R_R)$ is a right Σ-injective ring, and so it is quasi-Frobenius by [4, 18.1]. On the other hand, $R/Z_2(R_R)$ is a right nonsingular ring. Thus by [3, Corollary 4.6], $R/Z_2(R_R)$ is a semisimple ring.

(2) ⇒ (4). Let M be a cyclic $R/Z_2(R_R)$-module. Then M is a cyclic R-module, and so by hypothesis, M is $R/Z_2(R_R)$-injective. Hence M is an injective $R/Z_2(R_R)$-module. Thus $R/Z_2(R_R)$ is a semisimple ring.

(4) ⇒ (3). Assume that B and M are R-modules, and A is a nonsingular submodule of B. By [3, Theorem 3.2(4)], A is a direct summand of B. So clearly, every R-homomorphism $g : A \to M$ can be extended to an R-homomorphism $h : B \to M$. Thus by Theorem 2.2(6), M is $\mathcal{N}$-injective.

(3) ⇒ (1), (3) ⇒ (2) and (1) ⇒ (5). These implications are obvious.

(5) ⇒ (6). Clearly, $l_{R^{(\mathbb{N})}}(Z_2(R_R)) = [l_R(Z_2(R_R))]^{(\mathbb{N})}$. Thus by Theorem 2.2(4), $[l_R(Z_2(R_R))]^{(\mathbb{N})}$ is an injective $R/Z_2(R_R)$-module.

(6) ⇒ (1). Let $R^{(\Lambda)}$ be a free R-module. By hypothesis, $[l_R(Z_2(R_R))]^{(\mathbb{N})}$ is an injective $R/Z_2(R_R)$-module. Thus by [1, Theorem 25.1], $[l_R(Z_2(R_R))]^{(\Lambda)}$ is an injective $R/Z_2(R_R)$-module. So by Theorem 2.2(4), $R^{(\Lambda)}$ is $\mathcal{N}$-injective. □

A ring R is called semilocal if $R/\mathrm{Rad}(R)$ is semisimple. Semiperfect rings (hence right and left perfect rings, semiprimary rings, right and left Artinian rings, and local rings) are semilocal. The next result determines the $\mathcal{N}$-injective semilocal rings. Moreover, by Corollary 2.10, if R is $\mathcal{N}$-injective, then $\mathrm{Rad}(R) \leq Z_2(R_R)$. The converse implication is not necessarily true even though R is right Noetherian; e.g., $R = \mathbb{Z}$. The next result shows that the converse implication holds for semilocal rings.

Corollary 3.2. *Let R be a semilocal ring. The following statements are equivalent.*

(1) *R is $\mathcal{N}$-injective.*

(2) *R is right t-semisimple.*

(3) $\mathrm{Rad}(R) \leq Z_2(R_R)$.

If R is local, the above statements are equivalent to

(4) *R is right Z_2-torsion.*

Proof. (3) ⇒ (2). If R is semilocal, then $R/\mathrm{Rad}(R)$ is semisimple. Thus by hypothesis, $R/Z_2(R_R)$ is semisimple, and so R is right t-semisimple.

(2) ⇒ (1). This follows from Theorem 3.1.

(4) ⇒ (2). This is clear by [3, Theorem 2.3].

Now assume that R is a local ring. We show that (3) ⇒ (4). Since R is local, $\mathrm{Rad}(R)$ is essential in R. So by [2, Proposition 2.2(4)], $R/\mathrm{Rad}(R)$ is Z_2-torsion. Moreover, by hypothesis, $\mathrm{Rad}(R)$ is Z_2-torsion. Therefore R is right Z_2-torsion. □

Recall that a ring R is called quasi-Frobenius if R is right (or left) Artinian and right (or left) self-injective.

Corollary 3.3. *A ring R is quasi-Frobenius if and only if R is right t-semisimple and $R^{(\mathbb{N})}$ is $Z_2(R_R)$-injective.*

Proof. (⇒) Since $R^{(\mathbb{N})}$ is injective, it is $Z_2(R_R)$-injective. Moreover, by [3, Proposition 4.5], R is right t-semisimple.

($\Leftarrow$) By Theorems 3.1(3) and 2.2(5), $Z_2(R_R)$ is a direct summand of R. Moreover, by Theorem 3.1(5), $R^{(\mathbb{N})}$ is $R/Z_2(R_R)$-injective. Thus by hypothesis, $R^{(\mathbb{N})}$ is R-injective, so $R^{(\mathbb{N})}$ is injective. Hence R is quasi-Frobenius by [4, 18.1(b)] and [1, Theorem 25.1]. □

Recall that R is called a right pseudo-Frobenius ring if R is an injective cogenerator in Mod-R. Every quasi-Frobenius ring is right pseudo-Frobenius; see [9, Theorem 19.25]. The next result shows that a right pseudo-Frobenius ring for which the second singular ideal is Noetherian is quasi-Frobenius.

Corollary 3.4. *Let R be a ring.*

(1) *If R is right pseudo-Frobenius, then R is right t-semisimple.*

(2) *R is quasi-Frobenius if and only if R is right pseudo-Frobenius and $Z_2(R_R)$ is Noetherian (Artinian).*

(3) *R is quasi-Frobenius if and only if R is right Kasch and $Z_2(R_R)$ is injective and Noetherian (Artinian).*

Proof. (1) Since R is right pseudo-Frobenius, R is right self-injective and semi-perfect. Hence Corollary 3.2 implies that R is right t-semisimple.

(2) Let R be right pseudo-Frobenius and $Z_2(R_R)$ be Noetherian (Artinian). By (1), R is right t-semisimple, and so $R/Z_2(R_R)$ is Noetherian (Artinian). Thus R is Noetherian (Artinian), and hence R is quasi-Frobenius. The converse is clear.

(3) Let R be quasi-Frobenius. Then $Z_2(R_R)$ is injective and Noetherian (Artinian). Moreover, R is right pseudo-Frobenius, and so by [9, Theorem 19.25], R is right Kasch. The converse implication follows from [15, Theorem 5] and (2). □

Remark 3.5. (i) The endomorphism ring of an $\mathcal{N}$-injective module has a von Neumann regular factor ring (Theorem 2.9), but itself is not necessarily von Neumann regular. In fact, by Theorem 3.1(5) and [10, Proposition 2.17], if R is a right t-semisimple ring which is not semisimple, then $R^{(\mathbb{N})}$ is $\mathcal{N}$-injective and $\mathrm{End}(R^{(\mathbb{N})})$ is not von Neumann regular.

(ii) Recall that every injective R-module is projective if and only if every projective R-module is injective (and these are equivalent to R being quasi-Frobenius). However, Corollary 2.8 and Theorem 3.1 show that this equivalence does not hold if we replace injective by $\mathcal{N}$-injective.

Proposition 3.6. *The following statements are equivalent for a ring R.*

(1) *R is $\mathcal{N}$-injective.*

(2) *$Z_2(R_R)$ is $R/Z_2(R_R)$-injective and every finitely generated (cyclic) nonsingular R-module is injective and projective.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.2(5), $Z_2(R_R)$ is $R/Z_2(R_R)$-injective. Let M be a finitely generated nonsingular R-module. There exists a finitely generated free R-module F such that $M \cong F/C$ for some submodule C of F. By [2, Proposition 2.6(6)], C is a t-closed submodule of F. On the other hand, F is

$\mathcal{N}$-injective, and so by Corollary 2.3(ii), F is t-extending. Thus C is a direct summand of F, and so M is isomorphic to a direct summand of F. This implies that M is projective and $\mathcal{N}$-injective which implies that M is injective by Corollary 2.3(i).

(2) $\Rightarrow$ (1). By Theorem 2.2(2), $Z_2(R_R)$ is $\mathcal{N}$-injective. Since $R/Z_2(R_R)$ is projective by hypothesis, $Z_2(R_R)$ is a direct summand of R, say $R = Z_2(R_R) \oplus R'$. But, $R' \cong R/Z_2(R_R)$ is injective by hypothesis, and so by Theorem 2.2(5), R is $\mathcal{N}$-injective. □

The following result characterizes the rings over which every cyclic (finitely generated) nonsingular module is injective. Moreover, this result determines that when a right extending ring is $\mathcal{N}$-injective.

Theorem 3.7. *The following statements are equivalent for a ring R.*

(1) *Every cyclic (finitely generated) nonsingular R-module is injective.*

(2) *$R/Z_2(R_R)$ is a right self-injective ring.*

If R is right extending, then the above statements are equivalent to

(3) *R is $\mathcal{N}$-injective.*

Proof. (1) $\Rightarrow$ (2). By hypothesis, $R/Z_2(R_R)$ is an injective R-module, and hence, a right self-injective ring.

(2) $\Rightarrow$ (1). Let M be a finitely generated nonsingular R-module. Then M is a finitely generated nonsingular $R/Z_2(R_R)$-module. But, $R/Z_2(R_R)$ is a right self-injective ring, and by Proposition 3.6, every finitely generated nonsingular module over a right self-injective ring is injective. So M is an injective $R/Z_2(R_R)$-module. Therefore Baer's criterion implies that M is an injective R-module.

(3) $\Rightarrow$ (1). This follows from Proposition 3.6.

Now assume that R is right extending. We show that (1) $\Rightarrow$ (3). Since R is right extending, $Z_2(R_R)$ is a direct summand of R, say $R = Z_2(R_R) \oplus R'$. By [4, 7.11], $Z_2(R_R)$ is R'-injective. Hence $Z_2(R_R)$ is $R/Z_2(R_R)$-injective. On the other hand, R' is injective since R' is a cyclic nonsingular R-module. Thus by [2, Theorem 2.11(3)], $R^{(n)} = Z_2(R_R)^{(n)} \oplus R'^{(n)}$ is t-extending. So by hypothesis and [2, Remark 3.14], every finitely generated nonsingular R-module is injective and projective. Thus by Proposition 3.6, R is $\mathcal{N}$-injective. □

A ring R is called of finite (Goldie) reduced rank if the uniform dimension of $R/Z_2(R_R)$ is finite. Every ring of finite uniform dimension is of finite reduced rank; see [9, (7.35)].

Proposition 3.8. *The following statements are equivalent for a ring R of finite reduced rank.*

(1) *R is $\mathcal{N}$-injective.*

(2) *R is right t-semisimple.*

(3) *Every nonsingular principal right ideal of R is injective.*

(4) *Every nonsingular principal right ideal of R is a direct summand.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Theorem 3.1, the implication (1) $\Rightarrow$ (3) follows from Proposition 3.6, and the implication (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2). By [3, Theorem 2.3(4)], it suffices to show that a nonsingular right ideal K of R is a direct summand. Since R is of finite reduced rank, so is K. Hence K is of finite uniform dimension as it is nonsingular. Thus by [9, Proposition (6.30)'] and [1, Proposition 10.14], K is a finite direct sum of indecomposable right ideals. So by hypothesis, K is a finite direct sum of minimal right ideals, say $K = a_1R \oplus a_2R \oplus \cdots \oplus a_nR$. If $n = 1$, then K is a direct summand of R. Let $n > 1$. By induction, assume that $a_2R \oplus \cdots \oplus a_nR = eR$ for some idempotent $e \in R$. Since $(1-e)a_1R$ is a submodule of K, it is nonsingular. Hence by hypothesis, $(1-e)a_1R = e'R$ for some idempotent $e' \in R$. However, $K = eR + e'R$ and $ee' = 0$. Therefore $e'' = e + e' - e'e$ is an idempotent and $K = e''R$ is a direct summand of R, as desired. □

Following [2], a ring R is called right Σ-t-extending if every free R-module is t-extending.

Corollary 3.9. *A ring R is right t-semisimple if and only if R is $\mathcal{N}$-injective and right Σ-t-extending.*

Proof. ($\Rightarrow$) This follows from Theorem 3.1 and [3, Corollary 3.6].

($\Leftarrow$) Let $R^{(\Lambda)}$ be a free R-module. By [2, Theorem 2.11(3)], $[R/Z_2(R_R)]^{(\Lambda)} \cong R^{(\Lambda)}/Z_2(R^{(\Lambda)})$ is an extending R-module. Thus $[R/Z_2(R_R)]^{(\Lambda)}$ is an extending $R/Z_2(R_R)$-module. So $R/Z_2(R_R)$ is a right Σ-extending ring. Thus by [4, 12.21((d) $\Leftrightarrow$ (e))], $R/Z_2(R_R)$ is an Artinian ring. So R is of finite reduced rank. Thus by Proposition 3.8, R is right t-semisimple. □

Our last result shows that a ring R for which every nonsingular cyclic R-module is injective is precisely a right t-semisimple ring, whenever R is either semilocal or of finite reduced rank; see [3, Example 4.15].

Corollary 3.10. *Let R be a ring which is either semilocal or of finite reduced rank. Then every cyclic (finitely generated) nonsingular R-module is injective if and only if R is right t-semisimple.*

Proof. The implication ($\Leftarrow$) is obtained by [3, Theorem 3.2(4)]. For ($\Rightarrow$), set $\overline{R} = R/Z_2(R_R)$. By Theorem 3.7, $\overline{R}$ is right self-injective. So $\mathrm{Rad}(\overline{R}) \leq Z_2(\overline{R}_{\overline{R}})$ by Corollary 2.10(ii). But $Z_2(\overline{R}_{\overline{R}}) = 0$, hence $\mathrm{Rad}(R) \leq Z_2(R_R)$. Moreover, $\overline{R}$ is von Neumann regular by Corollary 2.10(i). So by [3, Lemma 4.12], every nonsingular cyclic right ideal of R is a direct summand. Thus Corollary 3.2(3) and Proposition 3.8(4) imply that R is right t-semisimple. □

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Marzieh Arabi-Kakavand
Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, Iran
E-mail address: m.arabikakavand@math.iut.ac.ir

Shadi Asgari
Department of Mathematical Sciences
University of Isfahan
Isfahan, Iran
and
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
Tehran, Iran
E-mail address: sh_asgari@ipm.ir

Yaser Tolooei
Department of Mathematics
Faculty of Science
Razi University
Kermanshah, Iran
E-mail address: y.toloei@razi.ac.ir