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GEOMETRIC CHARACTERIZATION OF q-PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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Abstract. In this paper, we investigate the notion of q-pseudoconvexity to discuss and describe some geometric characterizations of q-pseudoconvex domains $\Omega \subset \mathbb{C}^n$. In particular, we establish that Ω is q-pseudoconvex, if and only if, for every boundary point, the Levi form of the boundary is semipositive on the intersection of the holomorphic tangent space to the boundary with any $(n-q+1)$-dimensional subspace $E \subset \mathbb{C}^n$. Furthermore, we prove that the Kiselman's minimum principal holds true for all q-pseudoconvex domains in $\mathbb{C}^p \times \mathbb{C}^n$ such that each slice is a convex tube in $\mathbb{C}^n$.

1. Introduction

We study in this paper the notion of q-pseudoconvexity from a geometric point of view. We consider smoothly q-pseudoconvex domains Ω in $\mathbb{C}^n$ which are defined by smooth q-subharmonic function ρ such that $d\rho \neq 0$ on $\partial\Omega$. We will prove in Section 2 that, for q-pseudoconvex domains $\Omega \subset \mathbb{C}^n$ such that $2 \leq q \leq n$, the function $-\log d(z, \complement\Omega)$ is q-subharmonic. Note that by taking in account our convention about this notion, according to [5], this isn't generally the case for 1-pseudoconvex domains. By considering the function

$$\delta_\Omega(z, E) = \sup\{r > 0,\ z + B_E(r) \subset \Omega\}. \tag{1.1}$$

which is the distance from z to $\partial\Omega$ in the multi-complex direction supported by a q-dimensional subspace E of $\mathbb{C}^n$, we obtain a new understanding of the concept of q-pseudoconvexity. We will see here that the function $-\log d(z, \complement\Omega)$ is one of the most important tools in studying q-pseudoconvexity. In addition, We will show that the concepts of the weak q-pseudoconvexity and the strong q-pseudoconvexity are equivalent and we say simply q-pseudoconvexity. By using the function $\delta_\Omega(z, E)$, we legitimate the q-pseudoconvexity of the Hartogs domains when $n \geq 2q + 2$.

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In Section 3, we shall give a full rigorous proof of the local property of q-pseudoconvexity, that differs of such given in [5]. Furthermore, we characterize the q-pseudoconvexity by the Levy form of the defining function and we prove that Ω is q-pseudoconvex, if and only if, for every boundary point, the Levi form of the boundary is semi-positive on the intersection of the holomorphic tangent space to the boundary with any $(n-q+1)$-dimensional subspace $E \subset \mathbb{C}^n$.

In Section 4, we attempt to show that the Kiselman's minimum principal holds true for all q-pseudoconvex domains in $\mathbb{C}^p \times \mathbb{C}^n$ such that each slice is a convex tube in $\mathbb{C}^n$.

Now, let's give the definition of a q-subharmonic function.

Definition 1.1. A function $u : \Omega \to [-\infty, +\infty[$, $u \not\equiv -\infty$, is called q-subharmonic if for every $(n-q+1)$-dimensional complex subspace $E \subset \mathbb{C}^n$, the restriction $u_{|E\cap\Omega}$ is subharmonic. This means that for all compact set $K \subset E \cap \Omega$ and for every continuous harmonic function h on K such that $u \leq h$ on ∂K, we have $u \leq h$ on K.

Observe here that n-subharmonic functions are usual plurisubharmonic functions and 1-subharmonic functions are usual subharmonic functions. Further details about the notion of q-subharmonic functions and their properties can be obtained from [5] or [8].

The set of q-subharmonic functions on Ω will be denoted $q\text{-}Sh(\Omega)$.

Example 1.1. Consider in $\mathbb{C}^n$ the Riez kernel [6], $K(\alpha, z)$ defined by the expression

(1.2)
$$K(\alpha, z) = -\frac{|z|^{2(\alpha-q)}}{H_q(\alpha)} \quad \text{where} \quad H_q(\alpha) = \frac{\pi^{2n} 2^{2\alpha}\Gamma(\alpha)}{\Gamma(q-\alpha)} \quad \text{and} \quad 1 \leq \alpha < q \leq n.$$

For every q-dimensional subspace $E \subset \mathbb{C}^n$, an easy computation far from the origin of the Laplacian $\Delta K_{|E}$ of the restriction on E of the function $K(\alpha, .)$ defined by (1.2), yields up to a positive constant

$$\Delta K_{|E}(\alpha, z) = -K_{|E}(\alpha - 1, z). \tag{1.3}$$

Then (1.3) implies that K is $(n-q+1)$-subharmonic on $\mathbb{C}^n$. In case $q = n$ and $\alpha = 1$, K is the Newton kernel.

We may introduce the notion of a q-pseudoconvex domain in $\mathbb{C}^n$ where $n \geq 2$, by considering an integer $1 \leq q \leq n$ and a smoothly domain $\Omega \subset \mathbb{C}^n$ with a defining function ρ such that $d\rho \neq 0$ on $\partial\Omega$ and we define this notion as the following:

Definition 1.2. We say that Ω is q-pseudoconvex if there is a neighborhood U of $\overline{\Omega}$ and a q-subharmonic function $\rho : U \to \mathbb{R} \cup \{-\infty\}$ such that $d\rho \neq 0$ on $\partial\Omega$ and $\Omega = \{z \in \mathbb{C}^n \ / \ \rho(z) < 0\}$.

Example 1.2. Consider an example of 3-pseudoconvex domain in $\mathbb{C}^5 = \mathbb{C}^3 \times \mathbb{C}^2$, which is a variant of the Kohn-Nirenberg example [4] of a pseudoconvex domain in $\mathbb{C}^2$:

$\Omega = \{(z', z, w) \in \mathbb{C}^5 \ ; \ 3|z_1|^2 - |z_2|^4 - |z_3|^4 + \Re(w) + |z|^{2k} + t|z|^2\Re(z^{2k-2}) < 0\}$

where $t \in \mathbb{R}$ and $k \in \mathbb{N},\ k \geq 2$, are fixed parameters. We can easily check that if $|t| \leq \frac{k^2-2}{2k-1}$, then the restriction on every 3-complex subspace $E \subset \mathbb{C}^5$, of the defining function of Ω given by $\rho(z_1, z_2, z_3, z, w) = 3|z_1|^2 - |z_2|^4 - |z_3|^4 + \Re(w) + |z|^{2k} + t|z|^2\Re(z^{2k-2})$ is subharmonic. Which means that Ω is a 3-pseudoconvex domain in $\mathbb{C}^5$.

In [2], Dinh introduced the notion of p-pseudoconcavity of a closed subset X of a complex manifold V of dimension $n \geq 2$ as follows:

We say that X is p-pseudoconcave if for every open set $U \Subset V$ and every holomorphic map f from a neighborhood of $\overline{U}$ into C^p, we have $f(X \cap U) \subset \mathbb{C}^p \setminus \Omega$ where Ω is the unbounded component of $\mathbb{C}^p \setminus f(X \cap \partial\Omega)$.

As it is mentioned above, n-pseudoconvex domains are just the usual pseudoconvex domains which are domains of holomorphy with smooth boundary. In addition, strictly q-pseudoconvex domains are defined at the boundary by smooth strictly q-subharmonic functions.

Definition 1.3. A function $u \in q$-Sh(Ω) is said to be strictly q-subharmonic if $u \in L^1_{loc}(\Omega)$ and if for every point $x_0 \in \Omega$ there exist a neighborhood ω of x_0 and $c > 0$ such that $u - c|z|^2$ is q-subharmonic in ω.

Remark 1.1. By induction on $1 \leq k \leq q$, we can show that a function u is strictly q-subharmonic on Ω means that for every point $x_0 \in \Omega$, there exist $c > 0$ and a neighborhood ω of x_0 such that

$$(dd^c u)^k \wedge \beta^{n-k} \geq c\beta^n \quad \text{on} \quad \omega \quad \forall\, k = 1, \ldots, q,$$

where β is the Kahler form on $\mathbb{C}^n$.

Definition 1.4. Let $\Omega \subset \mathbb{C}^n$ be an open subset and a function $\psi : \Omega \to [-\infty, +\infty[$. Then ψ is said to be an exhaustion, if all sub-level sets $\Omega_c = \{z \in \Omega \ / \ \psi(z) < c\},\ c \in \mathbb{R}$, are relatively compact. Furthermore, we say that

(1) Ω is weakly q-pseudoconvex, if there exists a smooth q-subharmonic exhaustion function $\psi \in q\text{-Sh}(\Omega) \cap \mathscr{C}^\infty(\overline{\Omega})$;
(2) Ω is strongly q-pseudoconvex, if there exists a smooth strictly q-subharmonic exhaustion function $\psi \in q\text{-}Sh(\Omega) \cap \mathscr{C}^\infty(\overline{\Omega})$.

The main results of this paper are the followings:

Theorem 2.2. *Let $2 \leq q \leq n$ be a nonnegative integer, Ω be an open subset in $\mathbb{C}^n$ and E be a $(n-q+1)$-dimensional complex subspace. Then the following properties are equivalent*:

(1) Ω *is strongly q-pseudoconvex*;
(2) Ω *is weakly q-pseudoconvex*;
(3) Ω *has a q-subharmonic exhaustion function*;

(4) *the function* $(z, \xi_1, \ldots, \xi_{n-q+1}) \mapsto -\log \delta_\Omega(z, \xi_1, \ldots \xi_{n-q+1})$ *is* q*-subharmonic on* $\Omega \times E^{n-q+1}$;

(5) *the function* $z \mapsto -\log d(z, \complement\Omega)$ *is* q*-subharmonic on* Ω.

Theorem 3.2. *Let* $2 \leq q \leq n$ *be a nonnegative integer. An open subset* $\Omega \subset \mathbb{C}^n$ *with smooth boundary is* q*-pseudoconvex, if and only if, for every* $(n-q+1)$*-dimensional complex subspace* $E \subset \mathbb{C}^n$*, the Levi form* $L_{\partial\Omega,z_{|E\cap hT_{\partial\Omega,z}}}$ *is semi-positive at every point of* $\partial\Omega$.

In case $q = n$, Theorem 2.2 and Theorem 3.2 were proved in [1].

Theorem 4.1. *Let* $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^p \times \mathbb{C}^n$ *be a* q*-pseudoconvex domain such that each slice*

$$\Omega_\zeta = \{z \in \mathbb{C}^n;\ (\zeta, z) \in \Omega\}, \quad \zeta \in \mathbb{C}^p,$$

is a convex tube $\omega_\zeta + i\mathbb{R}^n$, $\omega_\zeta \subset \mathbb{C}^p$. *Then, for every* q*-subharmonic function* $v(\zeta, z)$ *on* Ω *that does not depend on* $\Im(z)$*, the function* $u(\zeta) = \inf_{z\in\Omega_1} v(\zeta, z)$ *is* q*-subharmonic or locally* $\equiv -\infty$ *on* $\Omega_2 = pr_{\mathbb{C}^n}(\Omega)$.

In case $q = n$, Theorem 4.1 was proved in [3].

2. Geometric characterizations of q-pseudoconvex domains

In this section, we will discuss some characterizations of q-pseudoconvex domains in $\mathbb{C}^n$.

Let $E \subset \mathbb{C}^n$ be a q-dimensional subspace. We denote by $B_E(r)$ the ball in E of center 0 and radius r, when $E = \mathbb{C}^q$, $B_{\mathbb{C}^q}(r)$ will be simply denoted $B(r)$. For $r_0 > 0$ and $z_0 \in \Omega$, we denote by $z_0 + B_E(r_0)$ the set of points of the form $z_0 + t_1\xi_1 + \cdots + t_q\xi_q$, where $(t_1, \ldots, t_q) \in B_E(1)$ and $\{\xi_1, \ldots, \xi_q\}$ is any orthonormal basis of E. We also denote $S_E(r)$ the sphere of center 0 and of radius r in E. For any $z \in \Omega$, we put

$$\delta_\Omega(z, E) = \sup\{r > 0,\ z + B_E(r) \subset \Omega\}. \tag{2.4}$$

The expression (2.4) is the distance from z to $\partial\Omega$ in the multi-complex direction supported by E.

If $\{\xi_1, \ldots, \xi_q\}$ is an orthonormal basis of E, then we will sometimes denote the distance from z to $\partial\Omega$ by $\delta_\Omega(z, \xi_1, \ldots, \xi_q)$. So we have

$$\delta_\Omega(z, \xi_1, \ldots, \xi_q) = \sup\{r > 0 \ /\ z + t_1\xi + \cdots + t_q\xi_q \in \Omega,\ (t_1, \ldots, t_q) \in B(r)\}. \tag{2.5}$$

We will need the following elementary proposition to characterize q-subharmonic functions.

Proposition 2.1. *Let* $v : \Omega \to [-\infty, +\infty[$ *be an upper semi continuous function and suppose that* $1 \leq q \leq n$*. Then* v *is* q*-subharmonic, if and only if, for every* $(n - q + 1)$*-dimensional complex subspace* $E \subset \mathbb{C}^n$*, for any closed ball* $\bar{B} = z_0 + \bar{B}_E(1) \subset \Omega$ *and any polynomial* $P \in \mathbb{C}[t_1, \ldots, t_{n-q+1}]$ *such that*

$$v(z_0 + t_1\eta_1 + \cdots + t_{n-q+1}\eta_{n-q+1}) \leq \Re P(t_1, \ldots, t_{n-q+1})$$

$$\textit{whenever } |t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$$

then $v(z_0) \leq \Re P(0)$, *where* $\{\eta_1, \ldots, \eta_{n-q+1}\}$ *is any orthonormal basis of* E.

Proof. It is clear that the condition is necessary. Indeed, the function

$$(t_1, \ldots, t_{n-q+1}) \mapsto \Re P(t_1, \ldots, t_{n-q+1})$$

is pluriharmonic and hence the function $(t_1, \ldots, t_{n-q+1}) \mapsto v(z_0 + t_1\eta_1 + \ldots t_{n-q+1}\eta_{n-q+1}) - \Re P(t_1, \ldots, t_{n-q+1})$ is subharmonic in a neighborhood of $\bar{B}_E(1)$, so it satisfies the maximum principal on $B_E(1)$. To prove the sufficiency, let $v = \lim v_\mu$ be a strictly decreasing sequence of continuous functions on ∂B such that $v = \lim v_\mu$ on ∂B.

Without loss of generalities, we may assume that v_μ is smooth on a small neighborhood of S_E and

$$\begin{aligned} &v_\mu(z_0 + t_1\eta_1 + \cdots + t_{n-q+1}\eta_{n-q+1}) = \Re P_\mu(t_1, \ldots, t_{n-q+1}) \\ &\text{whenever } |t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1 \end{aligned} \tag{2.6}$$

where $P_\mu \in \mathbb{C}[t_1, \ldots, t_{n-q+1}]$. Then, we have

$$\begin{aligned} &v(z_0 + t_1\eta_1 + \cdots + t_{n-q+1}\eta_{n-q+1}) \leq \Re P_\mu(t_1, \ldots, t_{n-q+1}) \\ &\text{whenever } |t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1, \end{aligned}$$

and thanks to (2.6), we get

$$\begin{aligned} v(z_0) &\leq \Re P_\mu(0) \\ &\leq \frac{1}{area(S_E)} \int_{S_E} \Re P_\mu(\xi) d\sigma(\xi) \\ &= \frac{1}{area(S_E)} \int_{S_E} v_\mu(z_0 + t_1\eta_1 + \ldots t_{n-q+1}\eta_{n-q+1}) d\sigma(t). \end{aligned} \tag{2.7}$$

If we take the limit of (2.7) when $\mu \to +\infty$, then we find that v satisfies the mean value inequality. □

In the following theorem, we give some characterizations of q-pseudoconvex domains.

Theorem 2.2. *Let* $2 \leq q \leq n$ *be a nonnegative integer,* Ω *be an open subset in* $\mathbb{C}^n$ *and* E *be a* $(n - q + 1)$*-dimensional complex subspace. Then, the following properties are equivalent*:

(1) Ω *is strongly* q*-pseudoconvex*;
(2) Ω *is weakly* q*-pseudoconvex*;
(3) Ω *has a* q*-subharmonic exhaustion function*;
(4) *the function* $(z, \xi_1, \ldots, \xi_{n-q+1}) \mapsto -\log \delta_\Omega(z, \xi_1, \ldots, \xi_{n-q+1})$ *is* q*-subharmonic on* $\Omega \times E^{n-q+1}$;
(5) *the function* $z \mapsto -\log d(z, \complement\Omega)$ *is* q*-subharmonic on* Ω.

We say that Ω is a q-pseudoconvex domain, when one of these properties holds.

Proof. We have to prove the following sequence of implications:

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (1)$$

• It is clear by definitions, that implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious.
• For the implication (3) $\Longrightarrow$ (4), we use Proposition 2.1. Consider in $\Omega \times E^{n-q+1}$ a ball of the form

$$B = (z_0, \xi^1, \dots, \xi^{n-q+1}) + B_E(1)(\eta^1, \dots, \eta^{n-q+1}, \alpha^1, \dots, \alpha^{n-q+1})$$

where, for all $j = 1, \dots, n-q+1$, $\xi^j = (\xi^j_1, \dots, \xi^j_{n-q+1})$, $\eta^j = (\eta^j_1, \dots, \eta^j_{n-q+1})$, $\alpha^j = (\alpha^j_1, \dots, \alpha^j_{n-q+1})$ are vectors in E and $B_E(1)(\eta^1, \dots, \eta^{n-q+1}, \alpha^1, \dots, \alpha^{n-q+1})$ is defined by the set

$$\{(t_1\eta^1, \dots, t_{n-q+1}\eta^{n-q+1}, t_1\alpha^1, \dots, t_{n-q+1}\alpha^{n-q+1}),\ (t_1, \dots, t_{n-q+1}) \in B(1)\}.$$

Consider also a polynomial $P \in \mathbb{C}[t_1, \dots, t_{n-q+1}]$ such that

$$\begin{aligned} &-\log \delta(z_0 + t_1\eta^1 + \cdots + t_{n-q+1}\eta^{n-q+1}, \\ &\xi^1 + t_1\alpha^1, \dots, \xi^{n-q+1} + t_{n-q+1}\alpha^{n-q+1}) \le \Re P(t_1, \dots, t_{n-q+1}) \end{aligned} \tag{2.8}$$

for $|t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$.

We have to show that the inequality (2.8) holds for $|t_1|^2 + \cdots + |t_{n-q+1}|^2 < 1$. Consider the holomorphic function $h : E \times E \to \mathbb{C}^n$ defined by

$$h(t, w) = z_0 + \sum_{j=1}^{n-q+1} t_j\eta^j + w_j \exp(-P(t_1, \dots, t_{n-q+1}))(\xi^j + t_j\alpha^j). \tag{2.9}$$

By (2.9), we have for all $t \in \bar{B}$, $f(t, 0) = z_0 + \sum_{j=1}^{n-q+1} t_j\eta^j \in pr_1(\bar{B})$, where $pr_1 : E \times \mathbb{C}^n \to \mathbb{C}^n$ is the first projection. Hence we may deduce

$$h(\bar{B}_E \times \{0\}) = pr_1(\bar{B}) \subset \Omega. \tag{2.10}$$

Equation (2.8) implies that $|\exp(-P)| \le \delta$ on ∂B, which leads to deduce that the following assertion holds

$$h(\partial(B_E) \times B_E) \subset \Omega. \tag{2.11}$$

We want to conclude that $h(\bar{B}_E \times B_E) \subset \Omega$. Let I be the set of radii $r \ge 0$ such that $h(\bar{B}_E \times rB_E) \subset \Omega$. Then, I is an open interval $]0, R[$, $R > 0$. Suppose that $R < 1$, let $\psi \in q\text{-}Sh(\Omega)$ be an exhaustion function and $K = h(\partial\bar{B}_E \times R\bar{B}_E) \Subset \Omega$, $c = \sup_K \psi$. Since any q-dimensional complex subspace of $E \times E$ is isomorphic to $\{0\} \times E$ or $E \times \{0\}$, we may deduce that $\psi \circ h$ is a q-subharmonic function on a neighborhood of $\bar{B}_E \times RB_E$. The maximum principle applied with respect to $t = (t_1, \dots, t_{n-q+1})$ implies that $\psi \circ h(t, w) \le c$ on $\bar{B}_E \times RB_E$. Hence $h(\bar{B}_E \times RB_E) \subset \Omega_c \Subset \Omega$ and $h(\bar{B}_E \times (R + \varepsilon)B_E) \subset \Omega$ for some $\varepsilon > 0$, a contradiction.
• The implication (4) $\Longrightarrow$ (5): we have

$$-\log d(z, \complement\Omega) = \sup_{\xi_1, \dots, \xi_{n-q+1} \in \bar{B}_E, E \subset \mathbb{C}^n} (-\log \delta(z, \xi_1, \dots, \xi_{n-q+1})). \tag{2.12}$$

Assertion (2.12) implies that $-\log d(z, \complement\Omega)$ is a continuous function on Ω and satisfies the mean value inequality.

• The implication (5) $\Longrightarrow$ (1). It is clear that

$$u(z) = |z|^2 + \max(\log d(z, \complement\Omega)^{-1}, 0))$$

is a strictly q-subharmonic continuous exhaustion function. Replace $|z|^2$ by $M|z|^2$, if necessary, where $M > 0$ is sufficiently big we get

$$u(z) = M|z|^2 + \max(\log d(z, \complement\Omega)^{-1}, 0). \tag{2.13}$$

Applying the Richberg's theorem for the function defined by (2.13), we may conclude the existence of $\Psi \in \mathscr{C}^\infty(\Omega)$ strictly q-subharmonic such that $u \leq \Psi \leq u + 1$. Then Ψ is the required exhaustion function. □

Example 2.1. Consider in $\mathbb{C}^4$

$$\Omega = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \,;\, 3|z_1 + z_2 + z_3 + z_4|^2 - 2|z_3 + z_4|^2 - 2|z_4|^2 < 0\}.$$

A direct calculation shows that the complex Hessian of the defining function of Ω, given by $\rho(z) = 3|z_1 + z_2 + z_3 + z_4|^2 - 2|z_3 + z_4|^2 - 2|z_4|^2$, is not positive. Hence ρ is not plurisubharmonic and so Ω is not pseudoconvex. However, we can easily check that the restriction of ρ, on each complex subspace $\{z_j = z_k = 0\}$, $1 \leq j \neq k \leq 4$, is subharmonic. So Ω has a 3-subharmonic exhaustion function, which leads to conclude by Theorem 2.2 that Ω is 3-pseudoconvex.

Proposition 2.3. (1) *Let $\Omega \Subset \mathbb{C}^n_z = \mathbb{C}^{n-p}_{z'} \times \mathbb{C}^p_w$ and $\Omega' \Subset \mathbb{C}^p_w$ be q-pseudoconvex domains ($p \leq n$). Then, $\Omega \times \Omega'$ is a q-pseudoconvex domain of $\mathbb{C}^n \times \mathbb{C}^p$. Furthermore, if $F : \mathbb{C}^n \to \mathbb{C}^p$ is a map defined by $F(z) = F(z', w) = f(w)$ where $f : \mathbb{C}^p \to \mathbb{C}^p$ is a unitary transformation, then the inverse image $F^{-1}(\Omega')$ is q-pseudoconvex.*

(2) *If $(\Omega_s)_{s\in I}$ is a family of q-pseudoconvex open subsets of $\mathbb{C}^n$, the interior of the intersection $\Omega = (\cap_{s\in I}\Omega_s)^\circ$ is q-pseudoconvex.*

(3) *If $(\Omega_j)_{j\in\mathbb{N}}$ is a non decreasing sequence of q-pseudoconvex open subsets of $\mathbb{C}^n$, then $\Omega = \cup_{j\in\mathbb{N}}\Omega_j$ is q-pseudoconvex.*

Proof. (1) If we have for all $c \in \mathbb{R}$ and for all $c' \in \mathbb{R}$, $\Omega_c = \{z \in \mathbb{C}^n \,/\, \psi_1(z) < c\} \Subset \Omega$ and $\Omega'_{c'} = \{w \in \mathbb{C}^p \,/\, \psi_2(w) < c'\} \Subset \Omega'$ where ψ_1 and ψ_2 are smooth q-subharmonic exhaustion functions, then we can write $(\Omega \times \Omega')_{c+c'} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p \,/\, \psi_1(z) + \psi_2(w) < c + c'\} \Subset \Omega \times \Omega'$ and $\left(F^{-1}(\Omega')\right)_{c+c'} = \{z = (z', w) \in \mathbb{C}^{n-p} \times \mathbb{C}^p \,/\, \psi_1(z) + \psi_2(f(w)) < c + c'\} \Subset F^{-1}(\Omega')$. The second assertion holds since $\psi_2 \circ f$ is q-subharmonic because f is a unitary transformation. So $(z, w) \mapsto \psi_1(z) + \psi_2(w)$ and $z \mapsto \psi_1(z) + \psi_2(F(z))$ are exhaustion functions of $\Omega \times \Omega'$ and $F^{-1}(\Omega')$ respectively.

(2) We have $-\log d(z, \complement\Omega) = \sup_{s\in I} -\log d(z, \complement\Omega_s)$, so the function $z \mapsto -\log d(z, \complement\Omega)$ is q-subharmonic.

(3) We have $-\log d(z, \complement\Omega) = \lim\downarrow_{j\to+\infty} - \log d(z, \complement\Omega_j)$ and this limit is q-subharmonic. □

2.1. Further examples

Example 2.2. Let $(f_{i,j})_{1\le i\le N, 1\le j\le N'}$ be a finite family of analytic functions on $\mathbb{C}^n$ such that for all $i = 1,\dots,N$, $\dim Vect\{f_{i,j}, j = 1,\dots N'\} \ge n-q+1$. Recall here that for all $i = 1,\dots,N$, the dimension of each subspace $V_i = Vect\{f_{i,j}, j = 1,\dots N'\}$, depends on the functions $f_{i,j}, j = 1,\dots N'$. For all $1 \le j \le N$, let

$$P_j = \{z \in \mathbb{C}^n \ ; \ |f_{i_1,j}(z)|^2 + \cdots + |f_{i_q,j}(z)|^2 - |f_{i_q,j}(z)|^2 - \cdots - |f_{i_N,j}(z)|^2 < 1\}$$

where $(f_{i_s,j})_{s=1,\dots,n-q+1}$ is an independent subfamily of $(f_{i,j})_{1\le i\le N,1\le j\le N'}$. Put $P = \cup_{j=1}^{N'} P_j$, then P is a q-pseudoconvex domain. In case $\dim Vect\{f_{i,j}, i = 1,\dots,N\} = n-q+1 = N = 1$ (which means that $q = n$) then P is a polyhedron and it is pseudoconvex.

Example 2.3. Consider n and q such that $n \ge 2q+2$ and $\omega \subset \mathbb{C}^{n-q}$ be a q-pseudoconvex domain. Let $u : \omega \to [-\infty, +\infty[$ be an upper semi-continuous function. Consider the Hartogs domain

$$\Omega = \{(z_1,\dots,z_{n-q+1}, z') \in \mathbb{C}^{n-q+1} \times \omega \ ; \\ \frac{1}{2}\log\left(|z_1|^2 + \cdots + |z_{n-q+1}|^2\right) + u(z') < 0\}.$$

Then Ω is q-pseudoconvex, if and only if, u is q-subharmonic. Indeed, to see the necessary condition, using notation (2.5), we may observe that $u(z') = -\log\delta_\Omega\left((0,z'),(\xi_1,\dots,\xi_{n-q+1})\right)$ where $\{\xi_1,\dots,\xi_{n-q+1}\}$ is the canonical basis of $\mathbb{C}^{n-q+1}$. Conversely, assume that u is q-subharmonic and continuous. If ψ is a q-subharmonic exhaustion function of ω, then, since u is continuous and since $x \mapsto \frac{1}{|x|}$ is convex and increasing on $]-\infty, 0[$, then

$$\psi(z') + \left|\frac{1}{2}\log(|z_1|^2 + \cdots + |z_{n-q+1}|^2) + u(z')\right|^{-1}$$

is a q-subharmonic exhaustion function of Ω. If u is not assumed to be continuous, we may replace u by $u * \chi_\varepsilon$ and write $\Omega = \cup\Omega_\varepsilon$ where

$$\Omega_\varepsilon = \{(z_1,\dots,z_{n-q+1}, z'), \ \ d(z', \complement\omega) > \varepsilon, \\ \frac{1}{2}\log(|z_1|^2 + \cdots + |z_{n-q+1}|^2) + u * \chi_\varepsilon < 0\}.$$

We may conclude by application of property (3) of Proposition 2.3.

3. Levi form of the boundary of q-pseudoconvex domains

In this section we shall characterize the q-pseudoconvexity by the Levi form of the boundary $\partial\Omega$. The holomorphic tangent space is by definition the largest complex subspace which is contained in the tangent space $T_{\partial\Omega}$ to the boundary: $hT_{\partial\Omega} = T_{\partial\Omega} \cap JT_{\partial\Omega}$, where J is the almost complex structure that is the operator of multiplication by $i = \sqrt{-1}$. The holomorphic tangent space $hT_{\partial\Omega,z}$ is the complex hyperplane of vectors $\xi \in \mathbb{C}^n$ such that

$$d'\rho(z).\xi = \sum_{1\leq j\leq n} \frac{\partial\rho}{\partial z_j}\xi_j = 0. \tag{3.14}$$

The Levi form on $hT_{\partial\Omega}$ is defined at every point $z \in \partial\Omega$ by

$$L_{\partial\Omega,z}(\xi) = \frac{1}{|\nabla\rho(z)|}\sum_{1\leq j,k\leq n} \frac{\partial^2\rho}{\partial z_j \bar{z}_k}\xi_j\bar{\xi}_k, \quad \xi \in hT_{\partial\Omega,z}. \tag{3.15}$$

Let's begin this section by showing that q-pseudoconvexity of an arbitrary domain in $\mathbb{C}^n$ is a local property of the boundary. An other proof of this fact was given in [5].

Proposition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain such that every point $z_0 \in \partial\Omega$ has a neighborhood U such that $U \cap \Omega$ is q-pseudoconvex. Then Ω is q-pseudoconvex.*

Proof. Let $z_0 \in \partial\Omega$ and let $U \cap \Omega$ be a neighborhood of z_0. Since $U \cap \Omega$ is q-pseudoconvex then it is defined in a neighborhood of $\partial(\Omega \cap U)$ by a q-subharmonic function ρ_U. Let V be a neighborhood of $\partial\Omega$, then the function defined by $w = \sup_{r>0, U\subset V} \rho_{U\cap B(0,r)}$ is q-subharmonic on V. Let χ be an increasing convex function such that

$$\forall r \geq 0,\ \chi(r) > \sup_{(\Omega\smallsetminus V)\cap\bar{B}(0,r)\cap U} \rho_{U\cap B(0,r)}. \tag{3.16}$$

Since the function $z \mapsto \sum_{j=1}^n |z_j|^2 - (n-q+1)|z_n|^2$ is q-subharmonic, then by (3.16) the function

$$\psi(z) = \max\left(\chi(\sum_{j=1}^n |z|^2 - (n-q+1)|z_n|^2), w(z)\right)$$

coincides with $\chi(\sum_{j=1}^n |z|^2 - (n-q+1)|z_n|^2)$ in a neighborhood of $\Omega \setminus V$. Hence ψ is an exhaustion q-subharmonic on Ω. □

Theorem 3.2. *Let $2 \leq q \leq n$. An open subset $\Omega \subset \mathbb{C}^n$ with smooth boundary is q-pseudoconvex if and only if, for every $(n-q+1)$-dimensional complex subspace $E \subset \mathbb{C}^n$, the Levi form $L_{\partial\Omega,z|E\cap hT_{\partial\Omega,z}}$ is semipositive at every point of $\partial\Omega$.*

Proof. Consider a $(n-q+1)$-dimensional complex subspace $E \subset \mathbb{C}^n$. Without loss of generalities we may assume $E = \{\xi_1 = \cdots = \xi_q = 0\}$. Let $\delta(z) = d(z, \complement\Omega)$, $z \in \overline{\Omega}$, then the function $\rho = -\delta$ is smooth near $\partial\Omega$. Suppose that Ω is q-pseudoconvex, then the function $-\log(-\rho)$ is q-subharmonic which means that for all $z \in \Omega$ near $\partial\Omega$ and for all $\xi \in E$, we have

$$\sum_{q+1\leq j,k\leq n}\left(\frac{1}{|\rho|}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k} + \frac{1}{\rho^2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\bar{z}_k}\right)\xi_j\overline{\xi}_k \geq 0. \tag{3.17}$$

As we have

$$\sum_{q+1\le j,k\le n} \frac{1}{\rho^2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial \bar z_k}\xi_j\overline{\xi}_k = \left|\sum_{q+1\le j\le n}\frac{1}{\rho}\frac{\partial\rho}{\partial z_j}\xi_j\right|^2,$$

then inequality (3.17) gives that

$$\sum_{q+1\le j,k\le n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \ge 0 \quad \text{whenever} \quad \sum_{q+1\le j\le n}\frac{\partial\rho}{\partial z_j}\xi_j = 0$$

and this is also true at the limit on $\partial\Omega$, which means that ρ is q-subharmonic. Conversely, suppose that Ω is not q-pseudoconvex, then by Theorem 2.2, the function $-\log(\delta)$ is not q-subharmonic in any neighborhood of $\partial\Omega$. Hence there exist a $(n-q+1)$-dimensional subspace $E\subset\mathbb{C}^n$ and an orthonormal basis $\{\xi_1,\dots,\xi_{n-q+1}\}\subset E$ such that the Laplacian of the function

$$(t_1,\dots,t_{n-q+1})\mapsto \log\delta(z+t_1\xi_1+\cdots+t_{n-q+1}\xi_{n-q+1})$$

is strictly positive at point $(t_1,\dots,t_{n-q+1})=(0,\dots,0)$ for some z in the neighborhood of $\partial\Omega$. By Taylor's formula, we have

$$\begin{aligned} &\log\delta(z+t_1\xi_1+\cdots+t_{n-q+1}\xi_{n-q+1})\\ &= \log\delta(z)+\sum_{1\le j\le n-q+1}\Re(a_jt_j+b_jt_j^2)+c_j|t_j|^2+o(|t|^2), \end{aligned} \tag{3.18}$$

where $a_j,b_j\in\mathbb{C}$ and $c_j=\left(\frac{\partial^2\log\delta(z+t_1\xi_1+\cdots+t_{n-q+1}\xi_{n-q+1})}{\partial t_j\partial\bar t_j}\right)_{|t_j=0}>0$. Let $z_0\in\partial\Omega$ such that $\delta(z)=|z-z_0|$ and put

(3.19)

$$h(t_1,\dots,t_{n-q+1}) = z+\sum_{1\le j\le n-q+1}t_j\xi_j+\exp\left(\sum_{1\le j\le n-q+1}a_jt_j+b_jt_j^2\right)(z_0-z).$$

We have $h(0)=z_0$, write $\delta(z+t_1\xi_1+\cdots+t_{n-q+1}\xi_{n-q+1})=\delta(z+t\xi)$ as

$$\begin{aligned} \delta(z+t\xi) &= \delta\left[z+t\xi+\exp\left(\sum_{1\le j\le n-q+1}a_jt_j+b_jt_j^2\right)(z_0-z)\right.\\ &\qquad\left. -\exp\left(\sum_{1\le j\le n-q+1}a_jt_j+b_jt_j^2\right)(z_0-z)\right]\\ &= \left|h(t)-z_0-\left(\exp\left(\sum_{1\le j\le n-q+1}a_jt_j+b_jt_j^2\right)\right)(z_0-z)\right| \end{aligned}$$

and use the triangle inequality, by (3.18) and (3.19) we get

$$\begin{aligned}\delta(h(t)) \geq \delta(z+t\xi) - \delta(z)\left|\exp(\sum_{1\leq j\leq n-q+1} a_j t_j + b_j t_j^2)\right| \\ \geq \delta(z)\left|\exp\sum_{1\leq j\leq n-q+1}\Re(a_j t_j + b_j t^2)\right|\exp(\sum_{1\leq j\leq n-q+1}(c_j|t_j|^2)) \\ - \delta(z)\left|\exp(\sum_{1\leq j\leq n-q+1} a_j t_j + b_j t_j^2)\right| \\ \geq \delta(z)\left|\exp\left(\sum_{1\leq j\leq n-q+1} a_j t_j + b_j t_j^2\right)\right|\left[\exp\left(\sum_{1\leq j\leq n-q+1}\frac{c_j|t_j|^2}{2}\right) - 1\right] \\ \geq \delta(z)\frac{c|t|^2}{6}\end{aligned}$$

when $|t|$ is sufficiently small and $c = \min_{1\leq j\leq n-q+1} c_j$. Since $h(\delta(0)) = \delta(z_0)$ $= 0$, we get at $t = 0$ for all $1 \leq j \leq n-q+1$,

$$\frac{\partial\delta(h(t))}{\partial tj} = \sum_{1\leq k\leq n-q+1}\frac{\partial\delta}{\partial z_k}(z_0)\frac{\partial h}{\partial t_j}(0) = 0$$

and

$$\frac{\partial^2\delta(h(t))}{\partial t_j\partial\bar{t}_j} = \sum_{1\leq k,l\leq n-q+1}\frac{\partial^2\delta}{\partial z_k\partial\bar{z}_k}(z_0)\frac{\partial h}{\partial t_j}(0)\overline{\frac{\partial h}{\partial t_l}(0)} > 0.$$

Hence $\nabla h(0) \in hT_{\partial\Omega,z_0|E}$ and $L_{\partial\Omega,z_0|E}(\nabla h(0)) < 0$. □

Definition 3.1. Consider $2 \leq q \leq n$. The boundary $\partial\Omega$ is said to be weakly (resp. strongly) q-pseudoconvex, if for every $z \in \partial\Omega$ and every $(n-q+1)$-dimensional complex subspace $E \subset \mathbb{C}^n$, $L_{\partial\Omega,z}$ is semi-positive (resp. positive definite) on $E \cap hT_{\partial\Omega,z}$.

Example 3.1. Consider in $\mathbb{C}^3$, $\Omega = \{\rho < -1\}$ where $\rho(z) = 3(|z_1|^2 + |z_2|^2) - 2|z_3|^2$. Then, it is clear that Ω is 2-pseudoconvex and $0 \notin \overline{\Omega}$. Further, by (3.14) and (3.15), at every point $z \in \partial\Omega$, the holomorphic tangent space to $\partial\Omega$ is given by the equation $3\bar{z}_1\xi_1 + 3\bar{z}_2\xi_2 - 2\bar{z}_3\xi_3 = 0$ and the Levi form on $hT_{\partial\Omega,z}$ is given by

$$L_{\partial\Omega,z}(\xi) = \frac{3(|\xi_1|^2 + |\xi_2|^2) - 2|\xi_3|^2}{\sqrt{1+6|z_3|^2}}.$$

An easy computation yields that for all $j = 1, 2, 3$ we have $L_{\partial\Omega,z|E_j\cap hT_{\partial\Omega,z}} \geq 0$ where $E_j = \{\xi_j = 0\}$. Indeed, we may chose $z \in \partial\Omega$ such that $z_2 \neq 0$. For all $\xi \in E_1 \cap hT_{\partial\Omega,z}$, we have

$$\frac{|\xi_2|^2}{|\xi_3|^2} = \left(\frac{2}{3} + \frac{2+6|z_1|^2}{9|z_2|^2}\right).$$

Hence, $\frac{3|\xi_2|^2-2|\xi_3|^2}{\sqrt{1+6|z_3|^2}} \geq 0$ on $E_1 \cap hT_{\partial\Omega,z}$. Similarly, we prove that for all $\xi \in E_2 \cap hT_{\partial\Omega,z}$, we have $L_{\partial\Omega,z}(\xi) \geq 0$. Finally, it is obvious that $L_{\partial\Omega,z}$ is positive definite on $E_3 \cap hT_{\partial\Omega,z}$ but semi-positive on $E_j \cap hT_{\partial\Omega,z}$, $j = 1, 2$ so $\partial\Omega$ is weakly 2-pseudoconvex.

Example 3.2. For any $C < 0$, let consider $\Omega_C = \{z \in \mathbb{C}^n,\ K(\alpha, z) < C\}$, where K is the $(n-q+1)$-subharmonic function given by (1.2). It is clear that $z \mapsto K(\alpha, z)$ is smooth near the boundary $\partial\Omega_C$. For all $1 \leq j, k \leq n$, an easy computation yields,

$$\frac{\partial K}{\partial z_j} = -(\alpha - q)\bar{z}_j K(\alpha - 1, z)$$

and

$$\begin{cases} \dfrac{-1}{H_q(\alpha)} \dfrac{\partial^2 K}{\partial z_j \partial \bar{z}_j} = (\alpha - q)|z|^{2(\alpha-q-2)} \left(|z|^2 + (\alpha - q - 1)|z_j|^2\right) & \text{if } j = k \\ \dfrac{-1}{H_q(\alpha)} \dfrac{\partial^2 K}{\partial z_j \partial \bar{z}_k} = (\alpha - q)(\alpha - q - 1) z_j \bar{z}_k |z|^{2(\alpha-q-2)} & \text{if } j \neq k. \end{cases}$$

Let $E \subset \mathbb{C}^n$ be a q-dimensional subspace. Without loss of generalities, we may assume that E is given by the equations $\xi_{q+1} = \cdots = \xi_n = 0$. Hence, we find that, at every point $z \in \partial\Omega_C$, the intersection of the holomorphic tangent space to $\partial\Omega$ with E, is given by the equation $\sum_{1 \leq j \leq q} \bar{z}_j \xi_j = 0$ and the Levi form on $hT_{\partial\Omega,z} \cap E$ is given by

$$\begin{aligned} &L_{\partial\Omega,z_{|hT_{\partial\Omega,z}\cap E}}(\xi) \\ &= \frac{q - \alpha}{H_q(\alpha)|\nabla K(\alpha, z)|} \left(q|\xi|^2|z|^{2(\alpha-q-1)} + (\alpha - q - 1)|z|^{2(\alpha-q-2)} \left| \sum_{j=1}^{q} z_j \xi_j \right|^2 \right), \end{aligned}$$

where $|\nabla K(\alpha, z)| = (q - \alpha)|z||K(\alpha - 1, z)|$ is the modulus of the complex gradient of K. By the Cauchy-Schwartz inequality we find that

$$L_{\partial\Omega,z_{|hT_{\partial\Omega,z}\cap E}}(\xi) \geq \frac{q - \alpha}{H_q(\alpha)|\nabla K(\alpha, z)|} \left(q + (\alpha - q - 1)\right) |\xi|^2 |z|^{2(\alpha-q-1)} \geq 0.$$

The last inequality holds true on $E \cap hT_{\partial\Omega,z}$ for every q-dimensional complex subspace $E \subset \mathbb{C}^n$.

4. Kiselman's minimimum principale for q-subharmonic functions

Let v be a q-subharmonic function on $\Omega \times \Omega' \subset \mathbb{C}^n \times \mathbb{C}^p$. The partial minimum function on Ω defined by

$$u(\zeta) = \inf_{z \in \Omega} v(\zeta, z)$$

need not be q-subharmonic. Indeed, consider the following counterexample of a 2-subharmonic function in $\mathbb{C}^3 \times \mathbb{C}$ given by

(4.20)
$$v(z_1, z_2, z_3, z_4) = |z_4+\bar{z}_3+\bar{z}_2+\bar{z}_1|^2 - |z_3+z_2+z_1|^2 = |z_4|^2 + 2\Re\left(z_4(\bar{z}_3 + \bar{z}_2 + \bar{z}_1)\right).$$

We have $u(z_1, z_2, z_3) = -|z_1+z_2+z_3|^2$ and it is clear that u is not q-subharmonic for $q = 2, 3$.

However, the minimum property holds true when $v(\zeta, z)$ depends only on $\Re(z)$.

Theorem 4.1. *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^p \times \mathbb{C}^n$ be a q-pseudoconvex domain such that each slice*

$$\Omega_\zeta = \{z \in \mathbb{C}^n;\ (\zeta, z) \in \Omega\}, \quad \zeta \in \mathbb{C}^p$$

is a convex tube $\omega_\zeta + i\mathbb{R}^n$, $\omega_\zeta \subset \mathbb{C}^p$. Then, for every q-subharmonic function $v(\zeta, z)$ on Ω that does not depend on $\Im(z)$, the function $u(\zeta) = \inf_{z\in\Omega_1} v(\zeta, z)$ is q-subharmonic or locally $\equiv -\infty$ on $\Omega_2 = pr_{\mathbb{C}^n}(\Omega)$.

Proof. The idea of the proof is inspired from [1]. Consider a $(n-q+1)$-complex subspace of $\mathbb{C}^p \times \mathbb{C}^n$ such that $L = \{\zeta_{j_1} = \cdots = \zeta_{j_s} = z_{k_1} = \cdots = z_{k_t} = 0\}$ and $q = s + t$. The hypothesis implies that $v(\zeta, z)_{|L\cap\Omega}$ is convex in $x = \Re(z)$. We may, first, assume that v is smooth, q-subharmonic in (ζ, z) and $v(\zeta, z)_{|L\cap\Omega}$ is strictly convex in x and $\lim_{x\to\partial\omega_\zeta\cup\{\infty\}} v(\zeta, x) = +\infty$ for every $\zeta \in \omega'$. Then the function $x \mapsto v_{|L\cap\Omega}(\zeta, x)$ has a unique minimum point $x = g(\zeta)$ solution of the equations $\frac{\partial v}{\partial x_{k_s}} = 0$. As the matrix $\left(\frac{\partial^2 v}{\partial x_{k_t}\partial x_{k_s}}\right)$ is positive definite, the implicit function theorem shows that g is smooth. Let B a ball contained in Ω defined by the parametrization

$$L \simeq \mathbb{C}^{n-q+1} \ni (w_1, \ldots, w_{n-q+1}) \mapsto \zeta_0 + w_1a_1 + \cdots + w_{n-q+1}a_{n-q+1}$$

where $a_1, \ldots, a_{n-q+1} \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_{n-q+1}) \in B_{n-q+1}$. There exists a holomorphic function f on the unit ball $B_E(1)$ whose real part solves the Dirichlet problem

$$\Re f(t_1, \ldots, t_{n-q+1}) = g(\zeta_0 + t_1a_1 + \cdots + t_{n-q+1}a_{n-q+1}). \tag{4.21}$$

Since the function

$$(w_1, \ldots, w_{n-q+1}) \mapsto v(\zeta_0 + w_1a_1 + \cdots + w_{n-q+1}a_{n-q+1}, f(w_1, \ldots, w_{n-q+1}))$$

is subharmonic, we get the mean value inequality

$$\begin{aligned} & v(\zeta_0, f(0)) \\ \le\ & \frac{1}{area(S_E)} \int_{S_E} v(\zeta_0 + t_1a_1 + \cdots + t_{n-q+1}a_{n-q+1}, f(t_1, \ldots, t_{n-q+1}))d\sigma(t) \\ =\ & \frac{1}{area(S_E)} \int_{S_E} v(\zeta_0 + t_1a_1 + \cdots + t_{n-q+1}a_{n-q+1}, g(t_1, \ldots, t_{n-q+1}))d\sigma(t). \end{aligned}$$

The last equality holds since we have, by (4.21), $\Re f = g$ on ∂B_{n-q+1} and $v(\zeta, z) = v(\zeta, \Re(z))$ by hypothesis. We have

$$u(\zeta_0) \leq v(\zeta_0, f(0)) \quad \text{and} \quad u(\zeta) = v(\zeta, g(\zeta)) \tag{4.22}$$

hence, we see by (4.22) that u satisfies the mean value inequality, thus $u_{|L\cap\Omega'}$ is subharmonic.

Let now extend the result to an arbitrary q-subharmonic function v. We may suppose $n - q + 1 \leq p \leq n$. Let $\psi(\zeta, z)$ a positive continuous q-subharmonic function on Ω which depends only on $\Re(z)$ and is an exhaustion of $\Omega \cap (\mathbb{C}^p \times \mathbb{R}^n)$, we may choose such a function as

$$\psi(\zeta, z) = \max\left(|\sum_{j=1}^{p} \zeta_j + \sum_{j=1}^{n} \Re z_j|^2 - \sum_{j=n-q+2}^{p} |\zeta_j|^2 - \sum_{j=n-q+2}^{n} |\Re z_{n-j}|^2, -\log \delta_\Omega((\zeta, z), L) \right). \tag{4.23}$$

There is an increasing sequence $C_j \to +\infty$ such that each function obtained from (4.23) and defined by $\psi_j = (C_j - \psi * \rho_{\frac{1}{j}})^{-1}$ is an exhaustion of a q-pseudoconvex open set $\Omega_j \Subset \Omega$ whose slices are convex tubes and such that $d(\Omega_j, \complement\Omega) > \frac{2}{j}$. Let

$$v_j(\zeta, z) = v * \rho_{\frac{1}{j}}(\zeta, z) + \frac{1}{j}|\Re(z)|^2 + \psi_j(\zeta, z), \tag{4.24}$$

then (4.24) gives a decreasing sequence of q-subharmonic functions on Ω_j satisfying the previous conditions. As $v = \lim v_j$, we see that $u = \lim u_j$ is q-subharmonic. □

As we see, it is clear that the image $F(\Omega)$ of a q-pseudoconvex domain Ω by a holomorphic map F need not be q-pseudoconvex. Indeed, Consider the domain Ω defined as the following

$$\Omega = \{(z', z_5) = (z_1, \ldots, z_5) \in \mathbb{C}^5;\ \log|z_1| + v(z_2, z_3, z_4, z_5) < 0\},$$

where v is the function given by example (4.20). If $\Omega' \subset \mathbb{C}^4$ is the image of Ω by the projection map $(z', z_5) \mapsto z'$, then we have

$$\Omega' = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4;\ \log|z_1| + u(z_2, z_3, z_4) < 0\},$$

where the function u is given by $u(z_2, z_3, z_4) = \inf_{z_5 \in \mathbb{C}} v(z_2, z_3, z_4, z_5)$. It is clear that Ω' is not 2-pseudoconvex. However, we have the following result.

Proposition 4.2. *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a q-pseudoconvex open set such that all slices Ω_ζ, $\zeta \in \mathbb{C}^p$, are convex tubes in $\mathbb{C}^n$. Then the projection Ω' of Ω on $\mathbb{C}^p$ is q-pseudoconvex.*

Proof. Let v be a q-subharmonic function on Ω equal to the function ψ defined in the proof of Theorem 2.2. Then u is a q-subharmonic exhaustion function of Ω'. □

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