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AN ELABORATION OF ANNIHILATORS OF POLYNOMIALS

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Abstract. In this note we elaborate first on well-known theorems for annihilators of polynomials over IFP rings by investigating the concrete shapes of nonzero constant annihilators. We consider next a generalization of IFP which preserves Abelian property, in relation with annihilators of polynomials, observing the basic structure of rings satisfying such condition.

1. Annihilators of polynomials on IFP rings

IFP and Abelian ring property have important roles in noncommutative ring theory and module theory. We continue in this section the studies of Nielsen [22] and Shin [25], being concerned with the constant annihilators of polynomials, and introduce a generalization of IFP which preserves Abelian property.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R, let $N(R)$, $N_*(R)$, and $J(R)$ denote the set of all nilpotent elements, the prime radical, and the Jacobson radical in R, respectively. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). The right annihilator of S in R is denoted by $r_R(S)$, and by $r_R(a)$ when $S = \{a\}$. The degree of a polynomial $f(x)$ is denoted by $\deg f(x)$. The n by n full (resp. upper triangular) matrix ring over R is denoted by $\mathrm{Mat}_n(R)$ (resp. $U_n(R)$), and denote by e_{ij} the matrix with (i, j)-entry 1 and elsewhere zero. $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

A ring R (possibly without identity) is called *reduced* if $N(R) = 0$. A well-known property that unifies the commutativity and the reduced condition is the *insertion-of-factors-property*. Due to Bell [4], a ring R (possibly without identity) is called to satisfy the *insertion-of-factors-property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [21] and Shin [25] used the terms *semicommutative* and *SI* for the IFP, respectively. Commutative rings are clearly IFP, and any reduced ring is IFP by a simple computation.

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There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually called *Abelian* if each idempotent is central. A simple computation yields that IFP rings are Abelian. It is also easily checked that $N(R) = N_*(R)$ for an IFP ring R.

In the following arguments, we study annihilators of polynomials by elaborating upon Camillo and Nielsen's interesting theorems for zero-dividing polynomials on IFP rings. Recall the following two results:

[7, Theorem 5.5] (Camillo and Nielsen) Let R be an IFP ring, and let $f(x)$, $g(x) \in R[x]$ be non-zero polynomials satisfying $f(x)g(x) = 0$. If $r_{R[x]}(f(x)) \cap R = (0)$, then $\deg(f(x)) > 2$.

[22, Theorem 4] (Nielsen) Let R be an IFP ring. Given $f(x)g(x) = 0$ with $f(x), g(x) \neq 0$ then (at least) one of $r_{R[x]}(f(x)) \cap R$ or $r_{R[x]}(g(x)) \cap R$ is nonzero.

This work is able to give alternate ways to construct elements in the annihilators as we see in Theorems 1.1, 1.2 and 1.4 to follow. We demonstrate the differences between these methods via examples.

Now let R be an IFP ring, and $f(x), g(x) \in R[x]$ be nonzero polynomials satisfying $f(x)g(x) = 0$. In this situation, Camillo and Nielsen showed that if $r_{R[x]}(f(x)) \cap R = 0$ then $\deg f(x) > 2$ in [7, Theorem 5.5]. We here elaborate upon this theorem by finding nonzero elements in R contained in the right annihilator of $f(x)$ when the degree of $f(x)$ is ≤ 2. The following computation is done for the case of $\deg f(x) = 1$.

Theorem 1.1. *Let R be a ring, and $0 \neq f(x), 0 \neq g(x) \in R[x]$ be such that $f(x)g(x) = 0$ and $\deg f(x) = 1$. If R is IFP, then there exists $0 \neq r \in R$ with $f(x)r = 0$.*

Proof. Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} + b_nx^n$ with $a_1 \neq 0, b_n \neq 0$.

If $n = 1$, then it was proved by [7, Proposition 5.3]. So assume $n \geq 2$. From $f(x)g(x) = 0$, we get $a_0b_0 = a_0b_1 + a_1b_0 = \cdots = a_0b_{n-1} + a_1b_{n-2} = a_1b_{n-1} + a_0b_n = a_1b_n = 0$.

If $a_0b_n = 0$, then $f(x)b_n = 0$.

If $a_0b_n \neq 0$, then $a_1b_{n-1} \neq 0$. But $a_1b_n = 0$, and hence by IFP property we get $a_1(a_1b_{n-1}+a_0b_n) = a_1^2b_{n-1} = 0$. Here if $a_0(a_0b_n) = 0$, then $f(x)(a_0b_n) = 0$. So suppose $a_0(a_0b_n) \neq 0$. But we have $a_0^{n+1}b_n = 0$ by [7, Lemma 5.4], and so there exists $l > 1$ such that $a_0^lb_n \neq 0$ and $a_0^{l+1}b_n = 0$. Then $a_0(a_0^lb_n) = 0$. We also get $a_1(a_0^lb_n) = 0$ by IFP property. These yield $f(x)(a_0^lb_n) = 0$. □

The following computation is done for the case of $\deg f(x) = 2$.

Theorem 1.2. *Let R be a ring, and $f(x) = a_0+a_1x+a_2x^2, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ be such that $f(x)g(x) = 0$ and $a_2 \neq 0, b_n \neq 0$. If R is IFP, then there exists $0 \neq r \in R$ with $f(x)r = 0$.*

Proof. We will proceed by induction on n. We first compute the case of $n = 1$. Let R be IFP and $g(x) = b_0 + b_1x$ with $f(x)g(x) = 0$. From $f(x)g(x) = 0$, we get $a_0b_0 = 0, a_0b_1 + a_1b_0 = 0, a_1b_1 + a_2b_0 = 0, a_2b_1 = 0$. If $b_0 = 0$, then $f(x)b_1 = 0$. So assume $b_0 \neq 0$. We apply the proof of [7, Proposition 5.3 and Theorem 5.5].

If $a_1b_0 = a_2b_0 = 0$, then $f(x)b_0 = 0$.

If $a_1b_0 = 0, a_2b_0 \neq 0$, then $a_0b_1 = 0$ and $a_1b_1 \neq 0$. By IFP property, $a_0(a_1b_1) = 0$, $a_1(a_1b_1) = a_1(-a_2b_0) = 0$ and $a_2(a_1b_1) = 0$, entailing $f(x)a_1b_1 = 0$.

If $a_1b_0 \neq 0, a_2b_0 = 0$, then $a_0b_1 \neq 0$ and $a_1b_1 = 0$. By IFP property, $a_0(a_0b_1) = a_0(-a_1b_0) = 0$, $a_1(a_0b_1) = 0$ and $a_2(a_0b_1) = 0$.

If $a_1b_0 \neq 0, a_2b_0 \neq 0$, then $a_0b_1 \neq 0$ and $a_1b_1 \neq 0$. From $f(x)g(x) = 0$ we have $(a_0 + a_1 + a_2)(b_0 + b_1) = 0$. Here if $b_0 + b_1 = 0$, then $f(x)b_0(1 - x) = 0$ and so $f(x)b_0 = 0$. So we assume $b_0 + b_1 \neq 0$. Then, by IFP property, $0 = (a_0+a_1+a_2)a_0(b_0+b_1) = (a_0+a_1+a_2)a_0b_1 = a_0(-a_1b_0)+a_1a_0b_1+a_2a_0b_1 = a_1a_0b_1$, so $f(x)(a_0b_1) = 0$.

Suppose $n \geq 2$. Note that $g(x) = (x - 1)g'(x) + b$ for some $0 \neq g'(x) \in R[x]$ and $b \in R$. If $b = g(1) = 0$, then $0 = f(x)g(x) = f(x)g'(x)(x - 1)$ implies $f(x)g'(x) = 0$. Since $\deg g'(x) < \deg g(x)$, there exists $0 \neq r \in R$ such that $f(x)r = 0$ by the induction hypothesis. So assume $b = b_0 + \cdots + b_n = g(1) \neq 0$. We have $0 = f(1)g(1) = (a_0 + a_1 + a_2)b$, from $f(x)g(x) = 0$.

Let $a_1b = 0$. We already have $a_0^{n+1}g(x) = 0 = a_2^{n+1}g(x)$ (hence $a_0^{n+1}b = 0 = a_2^{n+1}b$) by [7, Lemma 5.4]. So there exists $h \geq 1$ such that $a_0^hb = 0$ and $a_0^{h-1}b \neq 0$. If $a_2(a_0^{h-1}b) = 0$, then $f(x)(a_0^{h-1}b) = 0$ by IFP property. If $a_2(a_0^{h-1}b) \neq 0$, then there exists $k \geq 1$ such that $a_2^k(a_0^{h-1}b) = 0$ and $a_2^{k-1}(a_0^{h-1}b) \neq 0$ since $a_2^{n+1}(a_0^{h-1}b) = 0$ by IFP property. So $f(x)(a_2^{k-1}a_0^{h-1}b) = 0$ by IFP property.

Let $a_1b \neq 0$. We have $(a_0 + a_1 + a_2)a_1b = 0$ by IFP property. Through a similar process to the preceding computation, we can find $s, t \geq 1$ such that $a_0^s(a_1b) = 0$, $a_0^{s-1}(a_1b) \neq 0$, and $a_2^t(a_0^{s-1}a_1b) = 0$, $a_2^{t-1}(a_0^{s-1}a_1b) \neq 0$. This yields

$$0 = (a_0 + a_1 + a_2)(a_2^{t-1}a_0^{s-1}a_1)b = a_1(a_2^{t-1}a_0^{s-1}a_1)b$$

with the help of IFP property. So we now have $f(x)(a_2^{t-1}a_0^{s-1}a_1b) = 0$.

Therefore there exists nonzero $r \in R$ such that $f(x)r = 0$ in any case. □

In [22, Section 3], Nielsen constructed an IFP ring R and found polynomials $f(x), g(x) \in R[x]$, with $\deg f(x) = 3, \deg g(x) = 1$, such as $f(x)g(x) = 0$ and there does not exists nonzero $r \in R$ with $f(x)r = 0$. It is obvious that $r_{R[x]}(f(x)) = r_{R[x]}((1 + x^k)f(x))$ for every $k \geq 1$. So we can conclude that given any $h \geq 3$ and $l \geq 1$, there exist $f(x), g(x) \in R[x]$, with $\deg f(x) = h, \deg g(x) = l$, such as $f(x)g(x) = 0$ and there does not exist nonzero $r \in R$ with $f(x)r = 0$, where R is the IFP ring in [22, Section 3].

We now construct a ring which we will use in later examples to demonstrate the differences between our constructions and those in [7].

Example 1.3. Let $A = \mathbb{Z}_2\langle\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\rangle$ be the free algebra generated by noncommuting indeterminates $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ over $\mathbb{Z}_2$. We apply the ring construction and arguments in [15, Example 2.1]. Let B be the subalgebra of A which consists of all polynomials with zero constant terms in A. Note $A = \mathbb{Z}_2 + B$. Next consider an ideal I of A generated by

$$\alpha_0\beta_0, \alpha_2\beta_2, \beta_0^2, \beta_1^2, \beta_2^2, \alpha_0\beta_1 + \alpha_1\beta_0, \alpha_1\beta_2 + \alpha_2\beta_1, \alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0,$$
$$\alpha_0 r\beta_0, \alpha_2 r\beta_2, \beta_0 r\beta_0, \beta_1 r\beta_1, \beta_2 r\beta_2,$$

and

$$(\alpha_0 + \alpha_1 + \alpha_2)(\beta_0 + \beta_1 + \beta_2), (\alpha_0 + \alpha_1 + \alpha_2) r (\beta_0 + \beta_1 + \beta_2), r_1r_2r_3r_4$$

with $r, r_1, r_2, r_3, r_4 \in B$.

Note $B^4 \subseteq I$. Let $R = A/I$. First we prove that R is IFP. Each product of indeterminates $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ is called a *monomial* and we say that an element of A is a *monomial of degree* n if it is a product of exactly n generators. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Observe that H_n is finite for any n and that the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$, then every r_i is in I).

Claim 1. If $u_1v_1 \in I$ with u_1, $v_1 \in H_1$, then $u_1rv_1 \in I$ for any $r \in B$.

Proof. By the definition of I we obtain the following cases:

$$(u_1 = c_1\alpha_0 + c_2\beta_0, v_1 = \beta_0), (u_1 = d_1\alpha_2 + d_2\beta_2, v_1 = \beta_2), (u_1 = \beta_1, v_1 = \beta_1),$$
$$\text{or } (u_1 = \alpha_0 + \alpha_1 + \alpha_2, v_1 = \beta_0 + \beta_1 + \beta_2),$$

where $c_1, c_2, d_1, d_2 \in \mathbb{Z}_2$. So we complete the proof, using the definition of I again. □

Claim 2. If $uv \in I$ with $u, v \in B$, then $urv \in I$ for any $r \in B$.

Proof. Observe that $u = u_1 + u_2 + u_3 + u_4$, $v = v_1 + v_2 + v_3 + v_4$ and $r = r_1 + r_2 + r_3 + r_4$ for some $u_1, v_1, r_1 \in H_1$, $u_2, v_2, r_2 \in H_2$, $u_3, v_3, r_3 \in H_3$, and some $u_4, v_4, r_4 \in I$. Note that $H_i \subseteq I$ for $i \geq 4$. So $urv = u_1r_1v_1 + h$ for some $h \in I$. $uv \in I$ implies $u_1v_1 \in I$ since I is homogeneous; hence $u_1r_1v_1 \in I$ by Claim 1. Consequently $urv \in I$. □

Let $y, z \in A$ with $yz \in I$ and $r \in A$. Note that $y = c + y'$, $z = d + z'$ for some $c, d \in \mathbb{Z}_2$ and some $y', z' \in B$. So $yz = cd + cz' + y'd + y'z' \in I$; hence $c = 0$ or $d = 0$. Assume $c = 0$. Then $y'd + y'z' \in I$. If $d \neq 0$, then $y' \in I$ because I is homogeneous and $d \in \mathbb{Z}_2$, entailing $y'z' \in I$. Moreover $y'rz' \in I$ by Claim 2. Thus $yrz = y'rd + y'rz' \in I$. If $d = 0$, then $y'z' \in I$, entailing that $y'rz' \in I$ by Claim 2. Thus, $yrz = y'rz' \in I$. The computation for the case of $d = 0$ and $c \neq 0$ is similar. Therefore R is IFP.

Next identify $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ with their images in R for simplicity, and consider nonzero polynomials $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, $g(x) = \beta_0 + \beta_1 x + \beta_2 x^2$. Then

$$\begin{aligned} f(x)g(x) = \alpha_0\beta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0)\, x + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0)\, x^2 \\ + (\alpha_1\beta_2 + \alpha_2\beta_1)\, x^3 + \alpha_2\beta_2 x^4 = 0. \end{aligned}$$

We can find a nonzero constant right annihilator of $f(x)$ by help of the proof of Theorem 1.2. Say $g(x) = (x-1)\, g_1(x) + \beta$, where $0 \neq g_1(x) \in R[x]$ and $\beta \in R$. Then $\beta(= g(1)) = \beta_0 + \beta_1 + \beta_2 \neq 0$, $\alpha_1\beta \neq 0$, and $(\alpha_0 + \alpha_1 + \alpha_2)\, \alpha_1\beta = 0$ (from $f(1)g(1) = 0$) by IFP property. Thus, we consider two sequences $\{\alpha_0^s \alpha_1 \beta \mid s \geq 1\}$ and $\{\alpha_2^t \alpha_0^s \alpha_1 \beta \mid t \geq 1, s \geq 1\}$. Then

$$\alpha_0^2 \alpha_1 \beta = 0 \neq \alpha_0^1 \alpha_1 \beta \text{ and } \alpha_2^1 \alpha_0^1 \alpha_1 \beta = 0 \neq \alpha_2^0 \alpha_0^1 \alpha_1 \beta.$$

Therefore we can find a nonzero element $r = \alpha_2^0 \alpha_0^1 \alpha_1 \beta$ such that $f(x)r = 0$ by help of the proof of Theorem 1.2.

McCoy proved in [20] that if two polynomials annihilate each other over a commutative ring, then each polynomial has a nonzero annihilator in the base ring. However Nielsen showed in [22, Section 3] that McCoy's result need not hold over IFP rings (of course noncommutative), and next proved the following through [22, Lemmas 1, 3 and Theorem 4]. We here find another direct proof of this result independently.

Theorem 1.4. (1) *Let R be an IFP ring. Given $f(x)g(x) = 0$ with $f(x), g(x) \in R[x]$, we have that $r_{R[x]}(f(x)) \cap R \neq 0$ or $r_{R[x]}(g(x)) \cap R \neq 0$.* (*Similarly, for the left annihilators.*)

(2) *Let R be an IFP ring. Given $f(x)g(x) = 0$ with $f(x), g(x) \in R[x]$, we have that* (*at least*) *one of $r_{R[x]}(f(x))$ or $r_{R[x]}(g(x))$ contains a nonzero ideal of R.* (*Similarly, for the left annihilators.*)

Proof. (1) Let R be an IFP ring and $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i$, $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$. For our purpose of the proof, we can suppose $a_0, a_m, b_0, b_n \in R \backslash \{0\}$ without loss of generality.

Now assume on the contrary that $r_{R[x]}(f(x)) \cap R = 0$ and $r_{R[x]}(g(x)) \cap R = 0$. Since $r_{R[x]}(f(x)) \cap R = 0$ and $b_n \neq 0$, $f(x)b_n \neq 0$ and so we can find a_{k_1} such that $a_{k_1} b_n \neq 0$. Next since $r_{R[x]}(g(x)) \cap R = 0$ and $a_{k_1} b_n \neq 0$, $g(x) a_{k_1} b_n \neq 0$ and so we can find b_{k_2} such that $b_{k_2} a_{k_1} b_n \neq 0$. Note $g(x) b_{k_2} a_{k_1} b_n \neq 0$. We proceed alternatively in this manner. Then for any $s \geq 0$ we can find

$$c_{s+1} = a_{k_{s+1}} b_{k_s} \cdots a_{k_3} b_{k_2} a_{k_1} b_{k_0} \neq 0$$

such that $g(x) c_{s+1} \neq 0$, where $b_{k_0} = b_n$. From

$$f(x)g(x) = \sum_{k=0}^{m+n} \sum_{i+j=k} a_i b_j x^{i+j} = 0,$$

we have the following equalities:

$$a_0b_0 = 0, \tag{1}$$

$$a_0b_1 + a_1b_0 = 0, \tag{2}$$

$$a_sb_{k-s} + a_{s+1}b_{k-s-1} + \cdots + a_{k-t-1}b_{t+1} + a_{k-t}b_t = 0, \tag{3}$$

$$a_{m-1}b_n + a_mb_{n-1} = 0, \tag{4}$$

$$a_mb_n = 0, \tag{5}$$

where $0 \leq s, \ldots, k-t \leq m$ and $0 \leq k-s, \ldots, t \leq n$. Since R is IFP, we have $a_0Rb_0 = 0$ from the equality (1). Multiplying the equality (2) by b_0 on the right side, we get $a_1b_0b_0 = 0$ since $a_0Rb_0 = 0$, entailing $a_1b_0a_1b_0 = 0$ since R is IFP. This yields $a_1b_0, a_0b_1 \in N(R)$. Summarizing, we have that

$$a_ib_j \in N(R) \text{ (equivalently, } Ra_iRb_jR \subseteq N(R) \text{ since } R \text{ is IFP) for } i+j = 0, 1.$$

Inductively we assume that $a_ib_j \in N(R)$ for $i+j = 0, 1, \ldots, k-1$ with $k \leq m+n$. Since R is IFP, we also get $Ra_iRb_jR \subseteq N(R)$ for $i+j = 0, 1, \ldots, k-1$. We will use freely the elementary fact that $N(R) = N_*(R)$. Multiplying the equality (3) on the right side by b_t, we get

$$a_{k-t}b_tb_t = -(a_sb_{k-s}b_t + a_{s+1}b_{k-s-1}b_t + \cdots + a_{k-t-1}b_{t+1}b_t) \in N(R)$$

since $Ra_iRb_jR \subseteq N(R)$ for $i+j = 0, 1, \ldots, k-1$. Say $(a_{k-t}b_tb_t)^l = 0$. Since R is IFP, we also get $(a_{k-t}b_ta_{k-t}b_t)^l = 0$ and this yields $a_{k-t}b_t \in N(R)$. Next multiplying the equality (3) on the right side by $b_{t+1}, \ldots,$ and b_{k-s-1} in turn, we can similarly obtain

$$a_{k-t-1}b_{t+1}b_{t+1}, \ \ldots, \ a_{s+1}b_{k-s-1}b_{k-s-1} \in N(R) \tag{6}$$

since $Ra_iRb_jR \subseteq N(R)$ for $i+j = 0, 1, \ldots, k-1$. Since R is IFP, it can be obtained from (6) that

$$Ra_{k-t}Rb_tR, \ Ra_{k-t-1}Rb_{t+1}R, \ \ldots, \ Ra_{s+1}Rb_{k-s-1}R$$

are all contained in $N(R)$, entailing $Ra_sRb_{k-s}R \subseteq N(R)$. This implies that $Ra_iRb_jR \subseteq N(R)$ for all i and j with $i+j = k$, and so the induction process gives us the following:

$$Ra_iRb_jR \subseteq N(R) \text{ for all } i \text{ and } j \text{ with } 0 \leq i+j \leq m+n.$$

Then there exists $v \geq 1$ such that $(a_ib_j)^v = 0$ for all i, j. Further, we get $(Ra_iRb_jR)^v = 0$ for all i, j since R is IFP.

Now let $t = (2mn+2)(v+1)$. Then we can find c_{t+1} such that $g(x)c_{t+1} \neq 0$ as above. Here some $a_{i_0}b_{j_0}$ occurs at least v-times in the nonzero product c_{t+1} since there are mn numbers of a_ib_j's. But since $(Ra_iRb_jR)^v = 0$ for all i, j, we also have $c_{t+1} = 0$ and $g(x)c_{t+1} = 0$, a contradiction. This completes the proof.

(1) is equivalent to (2) when R is an IFP ring. □

Note that we can construct c_{s+1} starting by $a_m \neq 0$ in the proof of Theorem 1.4. Analyzing the proof of Theorem 1.4, we can also obtain a kind of direct method to find constant annihilators of zero-dividing polynomials over IFP rings, in relation with coefficients of the polynomials. Let R be an IFP ring and $f(x)g(x) = 0$ for $0 \neq f(x) \sum_{i=0}^{m} a_i x^i, 0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$.

(**Method 1.** (start by $f(x)b_n$)) We first start from $b_{k_0} = b_n$, and take largest k_1 such that $a_{k_1} b_{k_0} \neq 0$ (if any from $f(x)b_n$). Note $k_1 < m$ since $a_m b_n = 0$. Next take largest k_2 such that $b_{k_2} a_{k_1} b_{k_0} \neq 0$ (if any from $g(x)a_{k_1} b_{k_0}$). Proceed in this manner. Then we can find $0 \neq c \in R$ such that $f(x)c = 0$ or $0 \neq d \in R$ such that $g(x)d = 0$, where

$$c = b_{k_t} a_{k_{t-1}} \cdots a_{k_3} b_{k_2} a_{k_1} b_{k_0} \text{ and } g(x)(a_{k_{t-1}} \cdots a_{k_3} b_{k_2} a_{k_1} b_{k_0}) \neq 0$$

and

$$d = a_{k_{t+1}} b_{k_t} \cdots a_{k_3} b_{k_2} a_{k_1} b_{k_0} \text{ and } f(x)(b_{k_t} \cdots a_{k_3} b_{k_2} a_{k_1} b_{k_0}) \neq 0$$

with $t \geq 0$, respectively.

(**Method 2.** (start by $g(x)a_m$)) We first start from $a_{k_0} = a_m$, and take largest k_1 such that $b_{k_1} a_{k_0} \neq 0$ (if any from $g(x)a_m$). Next take largest k_2 such that $a_{k_2} b_{k_1} a_{k_0} \neq 0$ (if any from $f(x)b_{k_1} a_{k_0}$). Proceed in this manner. Then we can find $0 \neq c \in R$ such that $f(x)c = 0$ or $0 \neq d \in R$ such that $g(x)d = 0$, where

$$c = b_{k_{t+1}} a_{k_t} \cdots b_{k_3} a_{k_2} b_{k_1} a_{k_0} \text{ and } g(x)(a_{k_t} \cdots b_{k_3} a_{k_2} b_{k_1} a_{k_0}) \neq 0$$

and

$$d = a_{k_t} b_{k_{t-1}} \cdots b_{k_3} a_{k_2} b_{k_1} a_{k_0} \text{ and } f(x)(b_{k_{t-1}} \cdots b_{k_3} a_{k_2} b_{k_1} a_{k_0}) \neq 0$$

with $t \geq 0$, respectively.

(**Nielsen's method**) By [22, Lemma 3] and the proof of [22, Theorem 4], there exist nonnegative integers $l_0, \ldots, l_n$ such that $b_s b_n^{l_n} \cdots b_0^{l_0} \neq 0$ and $f(x) b_s b_n^{l_n} \cdots b_0^{l_0} = 0$ for some $s \in \{0, \ldots, n\}$, or $b_n^{l_n} \cdots b_0^{l_0} \neq 0$ and $g(x) b_n^{l_n} \cdots b_0^{l_0} = 0$.

In the following we actually apply the preceding three methods.

Example 1.5. Let R and $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, $g(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ be as in Example 1.3. Then R is IFP and

$$\begin{aligned} f(x)g(x) = \alpha_0\beta_0 &+ (\alpha_0\beta_1 + \alpha_1\beta_0)\, x + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0)\, x^2 \\ &+ (\alpha_1\beta_2 + \alpha_2\beta_1)\, x^3 + \alpha_2\beta_2 x^4 = 0. \end{aligned}$$

Now we construct a nonzero annihilator of $f(x)$ or one of $g(x)$ over R using three given methods.

Method 1: Take $b_{k_0} = \beta_2$. Then $f(x)b_{k_0} = \alpha_0\beta_2 + \alpha_1\beta_2 x \neq 0$. Now we turn to next stage.

$$g(x)\alpha_1\beta_2 = \beta_0\alpha_1\beta_2 + \beta_1\alpha_1\beta_2 x$$

is a nonzero polynomial. So we multiply $f(x)$ by new nonzero element $\beta_1\alpha_1\beta_2$ on the right side. Then it is zero since $A^4 \subseteq I$. Thus $\beta_1\alpha_1\beta_2 \in r_{R[x]}\,(f(x)) \cap R \neq 0$.

Method 2: Take $a_{k_0} = \alpha_2$. Then $g(x)a_{k_0} \neq 0$. In particular, $\beta_2\alpha_2 \neq 0$. Now we turn to next stage. $f(x)\beta_2\alpha_2 = \alpha_0\beta_2\alpha_2 + \alpha_1\beta_2\alpha_2 x \neq 0$. So we multiply $g(x)$ by new nonzero element $\alpha_1\beta_2\alpha_2$ on the right side. Then it is zero since $A^4 \subseteq I$. Thus $\alpha_1\beta_2\alpha_2 \in r_{R[x]}(g(x)) \cap R \neq 0$.

Nielsen's method: We first find nonnegative integers l_0, l_1, and l_2, according to [22, Lemma 3]. Since $\beta_0^2 = 0$, we have $f(x)\beta_0^2 = 0 \neq f(x)\beta_0$. Thus, $l_0 = 1$. To find nonnegative integer l_1 we multiply $f(x)$ on the right side by $\beta_1\beta_0$, $\beta_1^2\beta_0$, ... in turn. Then $f(x)\beta_1^1\beta_0 \neq 0 = f(x)\beta_1^2\beta_0$, since $\beta_1^2 = 0$. This means that $l_1 = 1$. We also obtain $l_2 = 0$ from $f(x)\beta_2^0\beta_1^1\beta_0 \neq 0 = f(x)\beta_2^1\beta_1^1\beta_0$. Next we consider second stage. To determine the annihilator element, we multiply $g(x)$ by $\beta_2^{l_2}\beta_1^{l_1}\beta_0^{l_0}$ then

$$g(x)\beta_2^{l_2}\beta_1^{l_1}\beta_0^{l_0} = g(x)\beta_2^0\beta_1^1\beta_0^1 = \beta_2\beta_1\beta_0 x^2 \neq 0.$$

Thus, there exists nonzero element $\beta_2\beta_1\beta_0$ in R such that $f(x)\beta_2\beta_1\beta_0 = 0$, that is $\beta_2\beta_1\beta_0 \subseteq r_{R[x]}(f(x)) \cap R \neq 0$.

Computing other kinds of examples, one may notice that each of the preceding methods has cases for which it is convenient to find nonzero constant annihilators of zero-dividing polynomials over an IFP ring.

We consider next a generalization of IFP which preserves Abelian property, based on the following.

Proposition 1.6. *For a ring R the following conditions are equivalent*:

(1) *R is IFP*;

(2) *If $x^{m_1}y^{m_2} = 0$ for $x, y \in R$ and some positive integers m_1, m_2, then $x^{n_1}Ry^{n_2} = 0$ for some positive integers n_1, n_2 with $n_1 \leq m_1, n_2 \leq m_2$*;

(3) *If $x^{m_1}y^{m_2} = 0$ for $x, y \in R$ and some positive integers m_1, m_2, then $x^{n_1}Ry^{n_2} = 0$ for some integers n_1, n_2 with $0 \leq n_1 \leq m_1, 0 \leq n_2 \leq m_2$.*

Proof. (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(1): Suppose that the condition (3) holds. Let $xy = 0$ for $x, y \in R$. Then $x^{n_1}Ry^{n_2} = 0$ for some integers n_1, n_2 with $0 \leq n_1 \leq 1, 0 \leq n_2 \leq 1$. If $n_1 = 0$, then $n_2 = 1$ and this yields $y = 0$. If $n_2 = 0$, then $n_1 = 1$ and this yields $x = 0$. If $n_1 = n_2 = 1$, then $xRy = 0$. Thus we get $xRy = 0$ in any case. □

We here introduce a generalization of the IFP condition, deleting the condition "$n_1 \leq m_1, n_2 \leq m_2$" in Proposition 1.6.

Definition 1.7. *A ring R (possibly without identity) is called π-IFP if $x^mRy^n = 0$ for some positive integers m, n whenever $xy = 0$ for $x, y \in R$.*

IFP rings are π-IFP evidently, but the converse need not hold by the following. Let A be a ring and $n \geq 2$. Following the literature, we consider a subring of $\mathrm{Mat}_n(A)$ by

$$D_n(A) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in \mathrm{Mat}_n(A) \mid a, a_{ij} \in A \right\}.$$

Let A be a division ring and $R = D_n(A)$ for $n \geq 4$. Suppose $xy = 0$ for $x, y \in R$. Then the diagonal entries of x, y are both zero and so $x^n = y^n = 0$, entailing $x^n R y^n = 0$. Thus R is π-IFP. However R is not IFP by [15, Example 1.7].

Lemma 1.8. (1) *Any π-IFP ring is Abelian.*

(2) *Every nil ring is π-IFP as a ring without identity.*

(3) *R is π-IFP if and only if $x^n R y^n = 0$ for some positive integer n whenever $xy = 0$ for $x, y \in R$.*

(4) *If R is a reduced ring and $x_1 x_2 \cdots x_n = 0$ for $x_1, \ldots, x_n \in R$, then $x_{\sigma(1)} R x_{\sigma(2)} R \cdots R x_{\sigma(n)} = 0$ for any permutation σ of $\{1, 2, \ldots, n\}$.*

(5) *Subrings of π-IFP rings (possibly without identity) are π-IFP.*

(6) *Let R be a π-IFP ring. If $a^2 = 0$ for $a \in R$, then aR and Ra are both nil.*

Proof. (1) Let R be a π-IFP ring and $e^2 = e \in R$. Then from $e(1-e) = 0$ we obtain $e^m R(1-e)^n = 0$ for some positive integers m, n, entailing $eR(1-e) = 0$. Similarly we get $(1-e)Re = 0$ from $(1-e)e = 0$.

(2) Let R be a nil ring and suppose that $xy = 0$ for $x, y \in R$. Then there exist $m = m(x), n = n(y)$ such that $x^m = 0 = y^n$. This yields $x^m R y^n = 0$.

(3) It suffices to show the sufficiency. Let $xy = 0$ for $x, y \in R$. Then since R is π-IFP, $x^m R y^n = 0$ for some positive integers m, n. Say $m \leq n$. Then $x^n R y^n = x^{n-m} x^m R y^n = 0$.

(4) is obtained with the help of [17, 19] and (5) is obvious.

(6) Let $a \in R$ with $a^2 = 0$. Then $raar = 0$ for all $r \in R$. Since R is π-IFP, $(ra)^m R(ar)^n = 0$ for some $m, n \geq 1$. This yields $(ra)^{m+n+1} = (ra)^m r(ar)^n a = 0$, entailing that Ra is nil. This also yields that aR is nil. $\square$

However there exist π-IFP rings which are not Abelian when they do not have the identity. Let A be any domain and consider the subring $B = \left(\begin{smallmatrix} A & A \\ 0 & 0 \end{smallmatrix}\right)$ of $U_2(A)$. Then B is π-IFP but non-Abelian as can be seen by the computation that $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & 0 \end{smallmatrix}\right) = 0$ if and only if $a = 0$, where $0 \neq \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_1 & b_1 \\ 0 & 0 \end{smallmatrix}\right) \in B$.

We will use Lemma 1.8(3, 5) freely. For any ring A, $\mathrm{Mat}_n(A)$ and $U_n(A)$ are both non-Abelian when $n \geq 2$, and so they cannot be π-IFP by Lemma 1.8(1).

Homomorphic images of a π-IFP ring need not be π-IFP, in contrast to Lemma 1.8(5). In fact, considering the domain $R = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, the ring of quaternions with integer coefficients, R/pR is isomorphic to the $\mathrm{Mat}_2(\mathbb{Z}_p)$

by the argument in [8, Exercise 2A], where p is any odd prime integer. But $\mathrm{Mat}_2(\mathbb{Z}_p)$ is non-Abelian and so is not π-IFP by Lemma 1.8(1).

In the following we see that the converse of Lemma 1.8(1) is not true in general. Armendariz [3, Lemma 1] proved that a reduced ring R satisfies the property that $a_ib_j = 0$ for all i, j whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$. Rege et al. [23] called a ring (not necessarily reduced) *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz. Armendariz rings are Abelian by the proof of [1, Theorem 6] or [12, Corollary 8]. Following [14], a ring R is called *power-serieswise Armendariz* if it satisfies the property that $a_ib_j = 0$ for all i, j whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[[x]]$. Power-serieswise Armendariz rings are clearly Armendariz but the converse need not hold by [14, Example 2.1]. Power-serieswise Armendariz rings are IFP by [14, Lemma 2.3(2)]. Armendariz and IFP are independent of each other by [12, Examples 2, 14] and [23, Proposition 4.6]. So one may conjecture that Armendariz are π-IFP. However the following example provides counterexamples.

Let K be a field and R_1, R_2 be K-algebras. Use $R_1 *_K R_2$ to denote the ring coproduct of R_1 and R_2 (see Antoine [2] and Bergman [5, 6] for details). Given a ring R, $U(R)$ means the group of units in R.

Example 1.9. There exist Abelian rings which are not π-IFP. Let K be a field and $A = K\langle a, b, c\rangle$ be the free algebra with noncommuting indeterminates a, b, c over K. We give subalgebras of A the following relations.

(1) Let $R_1 = K[b] = K\langle b\rangle$ and $R_2 = K\langle a \mid a^2 = 0\rangle$. Then $R_1 *_K R_2 \cong K\langle a, b \mid a^2 = 0\rangle$ is Armendariz (hence Abelian) by [2, Theorem 4.7] since $U(K[b]) = K\backslash\{0\}$. But $K\langle a, b \mid a^2 = 0\rangle$ is not π-IFP as can be seen by the computation that

$$baab = 0 \text{ but } 0 \neq (ba)^m b(ab)^n \in (ba)^m R(ab)^n \text{ for all } m, n \geq 1.$$

(2) Consider $A_1 = K\langle a, b, c \mid a^2 = 0\rangle$ and $A_2 = K\langle a, b, c \mid ab = 0\rangle$. Note that $K\langle a, b \mid ab = 0\rangle$ can be viewed as a subring of $K\langle a, b \mid a^2 = 0\rangle$ through the monomorphism with $a \mapsto ba$ and $b \mapsto ab$, in the proof of [2, Example 4.10]. Then A_1 is isomorphic to $K\langle b, c\rangle *_K K\langle a \mid a^2 = 0\rangle$ which is Armendariz by [2, Theorem 4.7] since $U(K\langle b, c\rangle) = K\backslash\{0\}$; and A_2 is isomorphic to a subring of A_1. Note that A_1 is not π-IFP by the same computations as in (1). A_2 is also not π-IFP since

$$ab = 0 \text{ but } 0 \neq a^m cb^n \in a^m Rb^n \text{ for all } m, n \geq 1.$$

In the following we see a concrete computation that shows A_2 being Armendariz. This is applicable to find idempotents in A_1.

Let $R = A_2$, and $0 \neq f(x) = \sum_{i=0}^{m} \alpha_ix^i$, $0 \neq g(x) = \sum_{j=0}^{m} \beta_jx^j \in R[x]$ satisfying $f(x)g(x) = 0$. Then $f(x)$ and $g(x)$ can be rewritten by

$$\begin{aligned} f(x) = {} & f_0 + f_1a + f_2b + f_3c + af_4a + af_5b + af_6c + bf_7a + bf_8b + bf_9c \\ & + cf_{10}a + cf_{11}b + cf_{12}c \end{aligned}$$

and

$$g(x) = g_0 + g_1a + g_2b + g_3c + ag_4a + ag_5b + ag_6c + bg_7a + bg_8b + bg_9c + cg_{10}a + cg_{11}b + cg_{12}c,$$

where $f_0, g_0 \in K[x]$, $f_1, g_1 \in K[a][x]$, $f_2, g_2 \in K[b][x]$, $f_3, g_3 \in K[c][x]$, $f_h, g_h \in K\langle a,b,c\rangle[x]$ for $h = 4, 5, \ldots, 12$, and every nonzero sum-factor (if any) of coefficients of f_k, g_k (resp., f_ℓ, g_ℓ) must contain c (resp., a or b) for $k = 4, 5, 8$ (resp., $\ell = 12$).

We first get $f_0g_0 = f_1ag_1a = f_2bg_2b = f_3cg_3c = 0$ since each of them is unique in the expansion of $f(x)g(x)$. So we have

$f_0 = 0$ or $g_0 = 0$; $f_1a = 0$ or $g_1a = 0$; $f_2b = 0$ or $g_2b = 0$; and $f_3c = 0$ or $g_3c = 0$.

Note that $f_0g_1a + f_1ag_0 = 0$, $f_0g_2b + f_2bg_0 = 0$, and $f_0g_3c + f_3cg_0 = 0$; hence

$$f_0g_1a = f_1ag_0 = f_0g_2b = f_2bg_0 = f_0g_3c = f_3cg_0 = 0 \tag{7}$$

since $f_0 = 0$ or $g_0 = 0$.

Assume $f_0 \neq 0$. Then $g_0 = g_1a = g_2b = g_3c = 0$ by the relation (1), and so $g(x) = ag_4a + ag_5b + ag_6c + bg_7a + bg_8b + bg_9c + cg_{10}a + cg_{11}b + cg_{12}c$.

Then, from $f(x)g(x) = 0$, we have

$$0 = f_0ag_4a + f_1a(ag_4a + bg_7a + cg_{10}a) + af_4a(ag_4a + bg_7a + cg_{10}a) + af_5b(ag_4a + bg_7a + cg_{10}a) + af_6c(ag_4a + bg_7a + cg_{10}a).$$

But $af_4a(ag_4a + bg_7a + cg_{10}a)$ is unique in the right hand side of this equality, entailing $0 = af_4a(ag_4a + bg_7a + cg_{10}a) = af_4aag_4a + af_4acg_{10}a$. Here we similarly get $af_4aag_4a = af_4acg_{10}a = 0$. If $ag_4a \neq 0$, then $af_4a = 0$ and

$$0 = f_0ag_4a + f_1a(ag_4a + bg_7a + cg_{10}a) + af_5b(ag_4a + bg_7a + cg_{10}a) + af_6c(ag_4a + bg_7a + cg_{10}a) \tag{8}$$

$$= f_0ag_4a + (f_1a + af_5b + af_6c)ag_4a + (f_1a + af_5b + af_6c)bg_7a + (f_1a + af_5b + af_6c)cg_{10}a. \tag{9}$$

But $(f_1a + af_5b + af_6c)ag_4a$ is unique in the right hand side of the equality (3), entailing $0 = (f_1a + af_5b + af_6c)ag_4a = f_1aag_4a + af_5bag_4a + af_6cag_4a$. This also yields $f_1aag_4a = af_5bag_4a = af_6cag_4a = 0$. But since $ag_4a \neq 0$ we obtain $f_1a = af_5b = af_6c = 0$ and this entails $f_0ag_4a = 0$. But since $ag_4a \neq 0$ we have $f_0 = 0$, a contradiction. Thus we must have $ag_4a = 0$ and $g(x) = ag_5b + ag_6c + bg_7a + bg_8b + bg_9c + cg_{10}a + cg_{11}b + cg_{12}c$. Proceeding in this method, we finally obtain $g(x) = 0$. So $f_0 = 0$.

Next we consider the assumption of $f_2b \neq 0$. Then we also obtain $g(x) = 0$ similarly. Proceeding with this process, we finally obtain

$$f_0 = f_2b = f_3c = af_5b = af_6c = bf_8b = bf_9c = cf_{11}b = cf_{12}c = 0.$$

This entails

$$f(x) = f_1a + af_4a + bf_7a + cf_{10}a.$$

Suppose $f_1a \neq 0$. Then, from $f(x)g(x) = 0$, we obtain

$$g_0 = g_1a = g_3c = ag_4a = ag_5b = ag_6c = cg_{10}a = cg_{11}b = cg_{12}c = 0$$

by using the method above, entailing

$$g(x) = g_2b + bg_7a + bg_8b + bg_9c.$$

Next suppose that $f_1a = 0$ and $af_4a \neq 0$. Then we also obtain $g(x) = g_2b + bg_7a + bg_8b + bg_9c$. Proceeding in this method, we can obtain $g(x) = g_2b + bg_7a + bg_8b + bg_9c$ in any case. Therefore every α_i (resp., β_j) is of the form sa with $s \in R$ (resp., bt with $t \in R$), and so $\alpha_i\beta_j = 0$ for all i and j.

By Example 1.9, we can say that Armendariz and π-IFP are independent of each other.

Note. We find the structure of idempotents in the rings in Example 1.9. Let R be any ring which is constructed in Example 1.9 and $f \in R$. We can write $f = \alpha + f_0$ such that $\alpha \in K$ and $f_0 \in R$ with zero constant term. Put $f^2 = f$. Then $\alpha + f_0 = \alpha^2 + 2\alpha f_0 + f_0^2$, so $\alpha = \alpha^2$. This yields that $\alpha = 0$ or $\alpha = 1$.

Case 1. $\alpha = 0$.

From $\alpha = 0$, we have $f_0 = f_0^2$. We can express f_0 by

$$f_0 = g_1 + \cdots + g_k \text{ with } g_\ell \in R \text{ for } \ell = 1, \ldots, k$$

such that the degree of g_i is less than one of g_{i+1} for $i = 1, \ldots, k-1$. Then

$$g_1 + \cdots + g_k = f_0 = f_0^2 = g_1^2 + g_1g_2 + g_2g_1 + \cdots + g_k^2$$

and so we must get $g_1 = 0$, entailing $f_0 = g_2 + \cdots + g_k$. Thus we can also obtain $g_2 = \cdots = g_k = 0$ inductively, entailing $f = f_0 = 0$.

Case 2. $\alpha = 1$.

From $\alpha = 1$, we have $f_0 + f_0^2 = 0$ and $(-f_0)^2 = -f_0$. We also obtain $f_0 = 0$ by a similar method to Case 1. Thus $f = 1 + f_0 = 1$.

Thus $0, 1$ are all idempotents in R by Cases 1 and 2.

In the following we can see a method by which one can always construct π-IFP rings but not IFP, over given any IFP ring.

Theorem 1.10. (1) *A ring R is π-IFP if and only if $D_n(R)$ is π-IFP for all $n \geq 1$.*

(2) *Let N be a nil algebra over a field F and $R = F + N$. Then $D_n(R)$ is π-IFP for all $n \geq 1$.*

Proof. (1) Suppose that R is a π-IFP ring. Let $D = D_n(R)$ and $x = (a_{ij}), y = (b_{st}) \in D$ such that $xy = 0$ and $a_{ii} = a, b_{ss} = b$. From $xy = 0$, we have $ab = 0$. Since R is π-IFP, we get $a^hRb^h = 0$ for some $h \geq 1$. Now we can write $v = x^h$ and $w = y^h$. For given $z \in D$, vzw and v^2zw^2 are elements of D such that their diagonal entries are all zero. Moreover, the $(i, i+1)$-entries of v^2zw^2 are contained in $a^hRb^h = 0$.

We claim that every (i,j)-entries of $v^k z w^k$ are all zero for $j-i<k$ and $k=1,2,\dots,n$. Assume that this holds for $2\le k<l$, that is, every (i,j)-entries of $v^{l-1}zw^{l-1}$ are all zero for $j-i<l-1$. Consider the case of $k=l$. Set $u=(c_{ij})=v^{l-1}zw^{l-1}$. Then every (i,j)-entries of vuw must be zero for $j-i<l-1$ by the hypothesis. For any $i=1,2,\dots$, $(i,i+l-1)$-entry of vuw is equal to $a^h c_{i(i+l-1)} b^h=0$. Thus, every (i,j)-entries of $v^l z w^l$ are all zero for $j-i<l$. By induction, we have the claim. This implies $v^n D w^n = x^{hn} D y^{hn}=0$ and so $D_n(R)$ is π-IFP. The converse is obvious.

(2) Let $D=D_n(R)$ and $x=(a_{ij}), y=(b_{st})\in D$ such that $xy=0$ and $a_{ii}=a, b_{ss}=b$. We can write $x=x_1+x_2, y=y_1+y_2$ for $x_1,x_2,y_1,y_2\in D$ such that the diagonals of x_1,y_1 are a,b respectively, and elsewhere zero. Then $x_1y_1=0$ and note that $z^n=0$ for all $z\in N_n(R)$. We can write $a=a_1+a_2, b=b_1+b_2$ for $a_1,b_1\in F$ and $a_2,b_2\in N$. Say $a_2^m=0$, $b_2^k=0$. $x_1y_1=0$ (hence $ab=0$) gives the relations $a_1b_1=0, a_1b_2+a_2b_1+a_2b_2=0$. So $a_1=0$ or $b_1=0$. Let $a_1=0$. Then $a^m=a_2^m=0$ yields $x_1^m=0$; hence we have $x^{mn}=(x^m)^n=0$ since $x^m=(x_1+x_2)^m=x_1^m+x_3=x_3$ for some $x_3\in N_n(R)$. Thus $x^{mn}Ry^h=0$ for all $h\ge 1$. The computation of the case $b_1=0$ is similar. □

Let R be any ring and $D=D_n(R)$ with $n\ge 4$. Since $e_{12}e_{34}=0$ but $e_{12}e_{23}e_{34}=e_{14}$, $e_{12}De_{34}\ne 0$ implies that D is not IFP. However D is π-IFP by Theorem 1.10(1) when R is IFP.

To see another example, let A be any algebra over a field F and $N=N_n(A)$ for $n\ge 4$. Then N is a nil algebra over F. Note that $1\le i<j$ for any $e_{ij}\in N$. Take e_{12},e_{34} in N. Then $e_{12}e_{34}=0$ but $e_{12}e_{23}e_{34}=e_{14}$, entailing $e_{12}Ne_{34}\ne 0$. Next letting $R=F+N$, then R is π-IFP by Theorem 1.10(2). However R is not IFP by the computation above.

Note. If a ring R is IFP, then $x^nD_n(R)y^n=0$ by applying the proof of Theorem 1.5 (indeed, $h=1$ if R is IFP), whenever $xy=0$ for $x,y\in D_n(R)$. But if R is a reduced ring, then we obtain $x^{n-2}D_n(R)y^{n-2}=0$ for $n\ge 3$ by the following computation. Let $D=D_n(R)$. Then D is IFP by [15, Proposition 1.2] when $n=1,2,3$. We first compute the case of $n=4$. Let

$$x=\begin{pmatrix} a & a_{12} & a_{13} & a_{14}\\ 0 & a & a_{23} & a_{24}\\ 0 & 0 & a & a_{34}\\ 0 & 0 & 0 & a\end{pmatrix},\quad y=\begin{pmatrix} b & b_{12} & b_{13} & b_{14}\\ 0 & b & b_{23} & b_{24}\\ 0 & 0 & b & b_{34}\\ 0 & 0 & 0 & b\end{pmatrix}\in R$$

such that $xy=0$. Then we have $ab=0$, $ab_{12}+a_{12}b=0$, $ab_{13}+a_{12}b_{23}+a_{13}b=0$, $ab_{23}+a_{23}b=0$, $ab_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b=0$, $ab_{24}+a_{23}b_{34}+a_{24}b=0$, and $ab_{34}+a_{34}b=0$. We use the reduced condition of R and Lemma 1.3(4) freely. By the computation in the proof of [15, Proposition 1.2], we get $aRb=0$, $aRb_{12}=0$, $a_{12}Rb=0$, $aRb_{13}=0$, $a_{12}Rb_{23}=0$, $a_{13}Rb=0$, $aRb_{23}=0$, and $a_{23}Rb=0$. Multiplying $ab_{34}+a_{34}b=0$ by b, we get $a_{34}Rb=0$ and $aRb_{34}=0$. Multiplying $ab_{24}+a_{23}b_{34}+a_{24}b=0$ by a, we get $aRb_{24}=0$ and $a_{23}b_{34}+a_{24}b=0$. Multiplying $a_{23}b_{34}+a_{24}b=0$ by a_{23}, we get $a_{23}Rb_{34}=0$

and $a_{24}Rb = 0$. Next multiplying $ab_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b = 0$ by b, we get $a_{14}Rb = 0$; and multiplying $ab_{14} + a_{12}b_{24} + a_{13}b_{34} = 0$ by b, we get $aRb_{14} = 0$. Thus, using these results, we have

$$xry = \begin{pmatrix} 0 & 0 & 0 & a_{12}\alpha b_{24} + a_{12}\epsilon b_{34} + a_{13}\alpha b_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $r = \left(\begin{smallmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & \epsilon & \sigma \\ 0 & 0 & \alpha & \pi \\ 0 & 0 & 0 & \alpha \end{smallmatrix}\right) \in R$. So every element of x^2Ry^2 is of the form

$$\begin{pmatrix} 0 & 0 & 0 & a(a_{12}\alpha b_{24} + a_{12}\epsilon b_{34} + a_{13}\alpha b_{34})b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

We next compute the case of $n = 5$ to find a formula. Let

$$x = \begin{pmatrix} a & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a & a_{34} & a_{35} \\ 0 & 0 & 0 & a & a_{45} \\ 0 & 0 & 0 & 0 & a \end{pmatrix}, \quad y = \begin{pmatrix} b & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b & b_{23} & b_{24} & b_{25} \\ 0 & 0 & b & b_{34} & b_{35} \\ 0 & 0 & 0 & b & b_{45} \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \in R$$

such that $xy = 0$. Through a similar computation to the case of $n = 4$, we obtain $aRb = 0$, $aRb_{12} = 0$, $aRb_{13} = 0$, $aRb_{23} = 0$, $a_{12}Rb_{23} = 0$, $a_{12}Rb = 0$, $a_{13}Rb = 0$, $aRb_{14} = 0$, $aRb_{15} = 0$, $aRb_{23} = 0$, $aRb_{24} = 0$, $aRb_{25} = 0$, $aRb_{34} = 0$, $aRb_{35} = 0$, $aRb_{45} = 0$, and $a_{12}Rb = 0$, $a_{13}Rb = 0$, $a_{14}Rb = 0$, $a_{15}Rb = 0$, $a_{23}Rb = 0$, $a_{24}Rb = 0$, $a_{25}Rb = 0$, $a_{34}Rb = 0$, $a_{35}Rb = 0$, $a_{45}Rb = 0$, $a_{34}Rb_{45} = 0$. Let $r \in D$. Then, using these results, we have

$$xry = \begin{pmatrix} 0 & 0 & 0 & u & v \\ 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $u, v, w \in R$. So

$$x^2ry^2 = \begin{pmatrix} 0 & 0 & 0 & aub & g \\ 0 & 0 & 0 & 0 & awb \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $g \in R$. This yields $x^3Dy^3 = 0$.

Let $x = (a_{ij}), y = (b_{st}) \in D_n(R)$ with $a_{ii} = a, b_{ss} = b$ such that $xy = 0$. Then we inductively have

$$aRb = 0,\ aRb_{st} = 0,\ a_{ij}Rb = 0,\ \text{and } a_{(n-2)(n-1)}Rb_{(n-1)(n)} = 0$$

for $1 \leq i, s \leq n-1$ and $2 \leq j, t \leq n$. These results imply $xry = (c_{hk})$ for $r \in D$ such that $c_{hk} = 0$ for $h = n - \ell$ and $k \leq 3 + (n - 1 - \ell)$ ($\ell = 0, 1, \ldots, n-1$). Then $x(xry)y = x(c_{hk})y = (d_{hk})$ such that

$$d_{14}, d_{25}, \ldots, d_{(n-3)n} \in aRb = 0.$$

Continuing this computation, we finally obtain $x^{n-2}Dy^{n-2} = 0$.

It is well-known that semiprime IFP rings are reduced. But this is not valid for π-IFP rings by the following.

Example 1.11. We refer the ring in [13, Theorem 2.2(2)]. Let S be a reduced ring, n be a positive integer and $R_n = D_{2^n}(S)$. Each R_n is a π-IFP ring by Theorem 1.10(1). Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of D. Note $R = \cup_{n=1}^{\infty} R_n$. We claim that R is π-IFP. Suppose $AB = 0$ for $A, B \in R$. Then $A, B \in R_m$ for some $m \geq 1$. Since R_m is π-IFP, $A^h R_m B^k = 0$ for some $h, k \geq 1$. Considering $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ and $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, we can get $A^h R_\ell B^k = 0$ for all $\ell \geq m$, entailing $A^h R B^k = 0$ since $R = \cup_{n=1}^{\infty} R_n$. So R is a non-reduced π-IFP ring. But R is semiprime by [13, Theorem 2.2(2)].

Let R be a ring and I be an ideal of R. Suppose that I is π-IFP as a ring without identity and R/I is π-IFP. Then it is natural to conjecture that R is also π-IFP. However the answer is negative. Consider $R = U_n(A)$ ($n \geq 2$) for an IFP ring A and let

$$I = \{m \in U_n(A) \mid \text{ every diagonal entry of } m \text{ is zero}\}.$$

Then R is not π-IFP by Lemma 1.8(1). But R/I is IFP by Proposition 2.3(4) to follow, and I is π-IFP by Lemma 1.8(2). Note that I is IFP when A is a reduced ring and $n \leq 3$. In the following we consider a stronger condition for I than "(π-)IFP" for I.

Proposition 1.12. *Let R be a ring and I be a proper ideal of R. If R/I is π-IFP and I is a reduced ring without identity, then R is π-IFP.*

Proof. Let $ab = 0$ for $a, b \in R$. Then we have $a^m R b^m \subseteq I$ since R/I is π-IFP. We apply the proof of [12, Theorem 6]. Note that $ab = 0$ yields $(bIa)^2 = 0$. Since I is reduced, $bIa = 0$. This gives $aRbI = 0$ since $((aRb)I)^2 = aRbIaRbI = aR(bIa)RbI = 0$. Recall $a^m R b^m \subseteq I$. Then $(a^m R b^m)^2 \subseteq aRbI$ implies $(a^m R b^m)^2 = 0$. Since I is reduced, we get $a^m R b^m = 0$. Thus R is π-IFP. □

Considering the condition "I is reduced" in Proposition 1.12, it is natural to conjecture that R is π-IFP when I is IFP. However $U_2(F)$ (with F a field) provides a counterexample as can be seen by the computation in [12, Example 5].

We see an application of Proposition 1.12 in the following. Let D be a domain and

$$R = \left\{ \begin{pmatrix} (a,0) & (a,0) & (0,0) \\ (0,0) & (0,b) & (0,c) \\ (0,0) & (0,0) & (d,b) \end{pmatrix} \mid a,b,c,d \in D \right\},$$

a subring of $U_3(D \oplus D)$. Consider $I = \left\{ \begin{pmatrix} (a,0) & (a,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (d,0) \end{pmatrix} \in R \right\}$. Then I is a proper ideal of R that is reduced as a ring. Moreover $R/I \cong D_2(D)$ that is π-IFP by Theorem 1.10(1), and hence R is π-IFP by Proposition 1.12. Note that the identity of R is $\begin{pmatrix} (1,0) & (1,0) & (0,0) \\ (0,0) & (0,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix}$.

2. Examples of π-IFP rings

In this section we concern several kinds of rings, either concluding that they are π-IFP or finding necessary conditions under which they can be π-IFP. Following to Huh et al. [11], a ring is called *locally finite* if every finite subset generates a finite subring. It is obvious that every locally finite ring is of finite characteristic. Finite rings are clearly locally finite. Note that an algebraic closure of a finite field is locally finite but not finite. A ring R is usually called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is usually called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$.

Let R be a ring. Due to Marks [18], R is called *NI* if $N(R)$ forms an ideal in R, i.e., $N(R) = N^*(R)$, where $N^*(R)$ means the upper nilradical of R. IFP rings are shown to be NI through a simple computation. NI rings need not be π-IFP as can be seen by $U_2(A)$ over any reduced ring A. A prime ideal P of R is usually called *completely prime* if R/P is a domain. Due to Rowen [24, Definition 2.6.5], an ideal P of R is called *strongly prime* if P is prime and R/P has no nonzero nil ideals. Maximal ideals and completely prime ideals are clearly strongly prime. $N^*(R)$ is the unique maximal nil ideal of R by [24, Proposition 2.6.2], and $N^*(R) = \{a \in R \mid RaR \text{ is a nil ideal of } R\} = \bigcap\{P \mid P \text{ is a strongly prime ideal of } R\} = \bigcap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}$ by help of [24, Proposition 2.6.7]. Hong and Kwak showed that R is NI if and only if every minimal strongly prime ideal of R is completely prime in [9, Corollary 13]. While, Shin proved that $N_*(R) = N(R)$ if and only if every minimal prime ideal of R is completely prime in [25, Proposition 1.11].

Proposition 2.1. (1) *Let R be a locally finite ring. Then R is Abelian if and only if R is π-IFP. Especially finite Abelian rings are π-IFP.*

(2) *Suppose that every finitely generated subring of a ring R is semiperfect. If R is π-IFP, then R is NI.*

(3) *Locally finite π-IFP rings are NI.*

Proof. (1) It suffices to prove the necessity by Lemma 1.8(1). Let $ab = 0$ for $a, b \in R$. Since R is locally finite, a^m is an idempotent for some $m \geq 1$ by the proof of [12, Proposition 16]. If R is Abelian, then we can get $a^m Rb = 0$ from $ab = 0$. It is an immediate consequence that finite Abelian rings are π-IFP.

(2) Let R be a π-IFP ring such that every finitely generated subring of R is semiperfect. Let $a, b \in N(R)$ and $r \in R$. Set S be the subring of R generated by $1, a, b, r$. Then S is semiperfect by hypothesis. So S is semilocal, and moreover $S/J(S)$ is Abelian by Lemma 1.8(1) since idempotents can be lifted modulo $J(R)$. This yields that $S/J(S)$ is a finite direct sum of division rings, entailing that $S/J(S)$ is reduced and $J(S) = N(S)$. Thus we have $a - b, ra, ar \in J(S) \subseteq N(R)$ since $a, b \in N(S)$. This result concludes that R is NI.

(3) is shown by (2) and [16, Proposition 3.6.1]. □

Let A be any locally finite Abelian ring. Then $D_n(A)$ is π-IFP for any $n \geq 1$ by Proposition 2.1 and [10, Lemma 2]. There exist many locally finite NI rings but not π-IFP as can be seen by $U_2(B)$ over an algebraic closure B of a finite field.

Proposition 2.2. *Let N be a nil ring.*

(1) *Adjoining an identity, the ring $R = \mathbb{Z} + N$ is π-IFP.*

(2) *If N is a K-algebra over a commutative domain K, then $K+N$ is π-IFP.*

Proof. (1) Let $0 \neq a = a_1 + a_2, b = b_1 + b_2 \in R$ with $a_1, b_1 \in \mathbb{Z}$ and $a_2, b_2 \in N$. If $ab = 0$, then $a_1 = 0$ or $b_1 = 0$, so we get that $a^m = 0$ or $b^m = 0$ for some $m \geq 1$. This yields $a^m Rb^m = 0$, so R is π-IFP. The proof of (2) is similar. □

Proposition 2.3. *Let I be an indexing set and R_i be rings for $i \in I$.*

(1) *Let I be finite. Then the direct product of R_i's is π-IFP if and only if so is every R_i.*

(2) *Let I be infinite. Then the direct sum of R_i's is π-IFP* (*as a ring without identity*) *if and only if so is every R_i.*

(3) *Let I be infinite. Then if the direct product of R_i's is π-IFP, then so is every R_i.*

(4) *The direct product of R_i's is IFP if and only if so is every R_i.*

Proof. (1) Let $I = \{1, \ldots, n\}$ and R be the direct product of R_i's. Suppose that every R_i is π-IFP. Let $x = (x_i), y = (y_i) \in R$ such that $xy = 0$. Then $x_i y_i = 0$ for all i, and since R_i is π-IFP we get $x_i^{m_i} R_i y_i^{m_i} = 0$. Put $m = \max\{m_1, \ldots, m_n\}$. Then $x_i^m R_i y_i^m = 0$ for all i and this yields $x^m Ry^m = 0$. Conversely let $ab = 0$ in R_j for $j \in I$. Let $x = (x_i), y = (y_i) \in R$ such that $x_i = a, y_i = b$ for $i = j$ and $x_i = 0, y_i = 0$ for $i \neq j$. Then $xy = 0$. Since R is π-IFP, $x^m Ry^m = 0$ for some $m \geq 1$. This gives $a^m R_j b^m = 0$. The proofs of (2), (3) and (4) are similar to (1). □

The converse of Proposition 2.3(3) need not hold by the following.

Example 2.4. Let A be an IFP ring and $R_n = D_n(A)$ for $n \geq 6$. Then every R_n is π-IFP by Theorem 1.5(1). Set R be the direct product of R_i's for $i = 6, 8, \ldots, 2k, \ldots$ $(k = 3, 4, \ldots)$. Take $x = (x_i), y = (y_i) \in R$ such that

$$x_i = e_{12} + \cdots + e_{(\frac{i}{2}-1)\frac{i}{2}} \text{ and } y_i = e_{(\frac{i}{2}+1)\frac{i}{2}+2} + \cdots + e_{(i-1)i}.$$

Then $xy = 0$, but $x_i^{\frac{i}{2}} = y_i^{\frac{i}{2}} = 0$ and $x_i^t \neq 0$, $y_i^t \neq 0$ for any $t < \frac{i}{2}$. Thus x, y are both non-nilpotent such that $x^m = (a_i)$, $y^m = (b_i)$ with

$$a_{2(m+1)} = e_{1(1+m)}, a_{2(m+2)} = e_{1(1+m)} + e_{2(2+m)}, \ldots$$

and

$$b_{2(m+1)} = e_{(m+2)(2(m+1))}, b_{2(m+2)} = e_{(m+3)(2(m+2)-1)} + e_{(m+4)(2(m+2))}, \ldots.$$

Thus the computation

$$a_{2(m+1)}e_{(1+m)(m+2)}b_{2(m+1)} = e_{1(1+m)}e_{(1+m)(m+2)}e_{(m+2)(2(m+1))} = e_{1(2(m+1))}$$

yields $x^m z y^m = (c_i)$ for $z \in R$ with $z_{2(m+1)} = e_{(1+m)(m+2)}$ and $z_i = 0$ for $i \neq 2(m+1)$ such that $c_{2(m+1)} = e_{1(2(m+1))}$ and $c_i = 0$ for $i \neq 2(m+1)$. This yields $x^m R y^m \neq 0$ for all $m \geq 1$, concluding that R is not π-IFP.

In the following we show that the π-IFP condition does not go up to polynomial rings.

Example 2.5. The construction follows Smoktunowicz [26]. Let $\bar{A}$ be the algebra of polynomials with zero constant terms in noncommuting indeterminates a, b, c over a countable field K. Then $\bar{A}$ can be enumerated, say $\bar{A} = \{f_1, f_2, \ldots\}$. By the argument in the proof of [26, Theorem 12], there are natural numbers $m_1, m_2, \ldots$ such that (i) $m_1 > 10^8$, $m_{i+1} > m_i 2^{i+101}$ for $i \geq 1$, (ii) each m_i divides m_{i+1} and (iii) $m_i > 3^{2\deg(f_i)}(\deg(f_i))^2 40^2$ for $i \geq 1$. Let I be the ideal of $\bar{A}$ generated by $\{f_i^{10m_{i+1}} \mid i = 1, 2, \ldots\}$ and $N = \bar{A}/I$. Then clearly N is a nil ring, so $R = K + N$ is also NI. Moreover, R is π-IFP by Proposition 2.2(2). Somktunowicz showed that $\bar{a} + \bar{b}x + \bar{c}y$ is not nilpotent in [26, Theorem 12], where x, y are commuting indeterminates over R. This implies that $R[x, y]$ is not NI since $\bar{a}, \bar{b}, \bar{c}$ are all nilpotent in R. This result also yields the following two situations:

(1) If $R[x]$ is π-IFP, then we have a π-IFP ring but not NI;

(2) If $R[x]$ is not π-IFP, then we have a π-IFP ring over which the polynomial ring is not π-IFP.

Here the statement (2) is shown to be true, essentially by help of Smoktunowicz. We can say that $\bar{a}^{2t} = 0$ and $\bar{a}^{2t-1} \neq 0$ for some $t \geq 2$, based on the construction of m_i's. Consider the polynomial $f(x, y) = \bar{a}^t(\bar{a} + \bar{b}x + \bar{c}y) = \bar{a}^{t+1} + \bar{a}^t\bar{b}x + \bar{a}^t\bar{c}y \in \bar{a}^t R[x, y]$. Note that each of $\bar{a}^{t+1}, \bar{a}^t\bar{b}, \bar{a}^t\bar{c}$ is nonzero. Then $f(x, y)$ is not nilpotent by applying the proof of [26, Theorem 12]. Thus $R[x, y]$ cannot be π-IFP by Lemma 1.8(6), entailing that $R[x]$ is not π-IFP.

Proposition 2.6. *Let R be an IFP ring and $f(x) = a_0 + a_1x, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. If $f(x)g(x) = 0$, then $f(x)^{2(n+1)}R[x]g(x) = 0$.*

Proof. Let $f(x)g(x) = 0$ for $f(x) = a_0+a_1x$, $g(x) = \sum_{j=0}^{n} b_jx^j$. Then $a_i^{n+1}b_j = 0$ for all i, j by [7, Lemma 5.4]. Since R is IFP, we obtain $f(x)^{2(n+1)}R[x]g(x) = 0$ because some a_i occurs at least $n+1$ times in every monomial of the expansion of $f(x)^{2(n+1)}$. □

The π-IFP condition also does not go up to formal power series rings by Example 2.5 and Lemma 1.8(5). But we can see such an actual example in the following.

Example 2.7. The π-IFP condition does not go up to formal power series rings. Let F be a field and

$$N_n = \{a \in U_{2n}(F) \mid \text{ the diagonal entries of } a \text{ are all zero}\}$$

for $n \geq 1$. Next set $N = \oplus_{n=1}^{\infty} N_{n+1}$. Then N is a nil algebra over F and so $R = F + N$ is π-IFP by Theorem 1.10(2).

Take $a_n = (\alpha_i), b_n = (\beta_i) \in N$ $(n \geq 1)$ such that

$$\alpha_n = e_{12} + e_{23} + \cdots + e_{(n-1)n}, \beta_n = e_{(n+1)(n+2)} + e_{(n+2)(n+3)} + \cdots + e_{(2n-1)2n},$$

and $\alpha_i = 0, \beta_i = 0$ for $i \neq n$. Then $\alpha_i\beta_i = 0$ for all i and $a_nb_n = 0$; moreover $a_sb_t = 0$ for $s \neq t$. Now let

$$f(x) = \sum_{j=1}^{\infty} a_jx^j \text{ and } g(x) = \sum_{j=1}^{\infty} b_jx^j \in R[[x]].$$

Then $f(x)g(x) = 0$. Note that

$$f(x)^m = (\gamma_i)x^{m^2} + \cdots \neq 0 \text{ and } g(x)^m = (\delta_i)x^{m^2} + \cdots \neq 0$$
$$\text{with } \gamma_{m^2} = e_{1(1+m)}, \delta_{m^2} = e_{(2+m)(2+2m)}$$

for all $m \geq 1$. So, letting $(\sigma_i) \in R$ such that $\sigma_{m^2} = e_{(1+m)(2+m)}$ and $\sigma_i = 0$ for $i \neq m^2$, we have

$$(\zeta_i)x^{2m^2} + \cdots \in f(x)^m(\sigma_i)g(x)^m$$

with $\zeta_{m^2} = e_{1(2+2m)}$. Thus $f(x)^mR[[x]]g(x)^m \supseteq f(x)^mRg(x)^m \neq 0$ for all $m \geq 1$, concluding that $R[[x]]$ is not π-IFP.

We end this note by raising the following.

Questions. (1) Is a π-IFP ring NI when it is not locally finite?

(2) If R is an IFP ring, then is $R[x]$ π-IFP?

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References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[2] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140.

[3] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[4] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[5] G. M. Bergman, *Coproducts and some universal ring constructions*, Trans. Amer. Math. Soc. **200** (1974), 33–88.

[6] ———, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. **200** (1974), 1–32.

[7] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.

[8] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney, 1989.

[9] C.Y. Hong and T. K. Kwak, *On minimal strongly prime ideals*, Comm. Algebra **28** (2000), no. 10, 4867–4878.

[10] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52.

[11] C. Huh, N. K. Kim, and Y. Lee, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 195–210.

[12] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[13] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), no. 1, 135–146.

[14] N. K. Kim, K. H. Lee, and Y. Lee, *Power series rings satisfying a zero divisor property*, Comm. no. 1-3, Algebra **34** (2006), no. 6, 2205–2218.

[15] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3 207–223.

[16] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham, 1966.

[17] ———, *On the representations of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

[18] G. Marks, *On 2-primal Ore extensions*, Comm. Algebra **29** (2001), no. 5, 2113–2123.

[19] G. Mason, *Reflexive ideals*, Comm. Algebra **9** (1981), no. 17, 1709–1724.

[20] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), 286–295.

[21] L. Motais de Narbonne, *Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents*, Proceedings of the 106th National Congress of Learned Societies, pp. 71–73, (Perpignan, 1981), Bib. Nat., Paris, 1982.

[22] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141.

[23] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[24] L. H. Rowen, *Ring Theory*, Academic Press, Inc., San Diego, 1991.

[25] G. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.

[26] A. Smoktunowicz, *Polynomial rings over nil rings need not be nil*, J. Algebra **233** (2000), no. 2, 427–436.

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