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AN UPPER BOUND ON THE CHEEGER CONSTANT OF A DISTANCE-REGULAR GRAPH

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Abstract. We present an upper bound on the Cheeger constant of a distance-regular graph. Recently, the authors found an upper bound on the Cheeger constant of distance-regular graph under a certain restriction in their previous work. Our new bound in the current paper is much better than the previous bound, and it is a general bound with no restriction. We point out that our bound is explicitly computable by using the valencies and the intersection matrix of a distance-regular graph. As a major tool, we use the discrete Green's function, which is defined as the inverse of β-Laplacian for some positive real number β. We present some examples of distance-regular graphs, where we compute our upper bound on their Cheeger constants.

1. Introduction

A notion of the Cheeger constant of a graph has an important geometric meaning in graph theory. The Cheeger constant of a graph is closely related to the problem of separating a graph into two large components by making a small edge-cut. In fact, the Cheeger constant of a connected graph is strictly positive. If the Cheeger constant of a connected graph is "small", then it means that there are two large sets of vertices with "few" edges between them. On the other hand, if a graph has "large" Cheeger constant, then it indicates that there are two sets of vertices with "many" edges between these two subsets. In general, computation of the Cheeger constant of a graph is a hard task. Only limited research has been done for finding the Cheeger constant of a graph. We are interested in finding bounds of Cheeger constants of graphs.

We begin with introducing some definitions in graph theory. Let $\Gamma = (V, E)$ be a simple and connected graph, where V is the vertex set of Γ and E is the

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edge set of Γ. Let S be a nonempty subset of V. The *edge boundary* of S, denoted by ∂S, is defined as follows:

$$\partial S = \{\{x, y\} \in E \mid x \in S \text{ and } y \in V - S\}.$$

The *volume* of S, denoted by $\mathrm{vol}(S)$, is defined as follows:

$$\mathrm{vol}(S) = \sum_{u \in S} k_u,$$

where k_u is the valency of u in Γ. The *Cheeger ratio* of S, denoted by h_S, is defined as

$$h_S = \frac{|\partial S|}{\min\{\mathrm{vol}(S),\ \mathrm{vol}(\Gamma) - \mathrm{vol}(S)\}}.$$

The *Cheeger constant* of Γ, denoted by h_Γ, is defined as

$$h_\Gamma = \min\{h_S \mid S \subseteq V\}.$$

Recent developments in [3, 4, 14] regarding distance-regular graphs show that there is a close connection between the Cheeger constant and vertex (or edge) connectivity. From Propositions A, B, and C we see that, for a distance-regular graph, there are close connections between the Cheeger constant and vertex and edge connectivity.

Proposition A ([3]). *Let Γ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency k, and the only disconnecting sets of k edges are the sets of edges incident with a single vertex.*

Proposition B ([4]). *Let Γ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals k, and the only disconnecting sets of vertices of size not more than k are the point neighbourhoods.*

Proposition C ([14]). *Let $\Gamma = (V, E)$ be a simple graph with the vertex-connectivity $\kappa(\Gamma)$ and the edge-connectivity $\lambda(\Gamma)$. Then*

$$\frac{2\kappa(\Gamma)}{|V|} \leq \frac{2\lambda(\Gamma)}{|V|} \leq \inf \frac{|\partial S|}{|S|} \leq \kappa(\Gamma) \leq \lambda(\Gamma),$$

where S is a subset of V with $|S| \leq \frac{|V|}{2}$.

The Cheeger constants [9, 15] are related to the eigenvalues of the Laplacians of distance-regular graphs, and their eigenvalues are also involved with the *intersection numbers* of distance-regular graphs [13, 16]. However, in general, it is a hard task to compute the Cheeger constant of a distance-regular graph. Distance-regular graphs introduced by Biggs [2] are connected with coding theory and design theory; well-known examples of distance-regular graphs are the Hamming graphs and the Johnson graphs. In [11, 12], by using the relationship between a discrete Green's function and the Cheeger constant, we obtain an upper bound on the Cheeger constant of a distance-regular graph under a certain condition.

We find a general upper bound on the Cheeger constant of a distance-regular graph with no additional condition. Furthermore, our bound is a much more improved one comparing with the bound in [12] under the same additional condition; in Example 7 and Example 8, we show that our bound is much more improved one comparing with the bound in [12] under the same additional condition: $\beta v r_d^{(\beta)} > \frac{\lambda_1}{1+\lambda_1}$.

We point out that our bound is explicitly computable by using the valencies and the *intersection matrix* of a distance-regular graph; first, our bound is expressed in terms of q-numbers, and in general, it is not easy to compute the q-numbers. For resolving this problem, we obtain an alternative expression of our bound using the valencies and the intersection matrix of a distance-regular graph. In Example 10 and Example 11, we compute the upper bound on the Cheeger constant using the alternative expression in Theorem 2 and Remark 6. As a major tool, we use the discrete Green's function, which is defined as the inverse of β-Laplacian for some positive real number β. We present some examples which show our upper bound on the Cheeger constant for some distance-regular graphs.

We discuss our main result in more detail for the rest of this section. In this paper, we study distance-regular graphs. Let $\Gamma = (V, E)$ be a distance-regular graph of order v, diameter d and valency k. Let A_1 be the adjacency matrix of Γ and P be the transition probability matrix of Γ. Two adjacent vertices x, y are denoted by $x \sim y$. For a function $f : V \to \mathbb{R}$, we define a Laplace operator Δ by $\Delta f(x) = \frac{1}{k}\sum_{y\sim x}(f(x) - f(y))$. Then $\Delta = I - \frac{1}{k}A_1$. Let $\mathcal{L}_\beta$ be the β-*normalized Laplacian* $\beta I + \Delta$. For $\beta > 0$, let $\mathcal{G}_\beta$ be a discrete Green's function denoted by the symmetric matrix which satisfies $\mathcal{L}_\beta \mathcal{G}_\beta = I$; that is, $\mathcal{G}_\beta$ is defined as the inverse of the β-Laplacian $\mathcal{L}_\beta$ [5, 6, 7]. As in [11], for any positive real number β, let $r_i^{(\beta)}$ $(i = 0, 1, \ldots, d)$ denote the components of a Green's function $\mathcal{G}_\beta$. We define α_i to be the limit of a sequence $\{\alpha_i^{(\beta)}\}$ as β goes to 0^+, where

$$\alpha_i^{(\beta)} = \frac{\beta^2 v r_i^{(\beta)}}{1 - \beta v r_i^{(\beta)}} \ (i = 0, 1, \ldots, d). \tag{1}$$

In fact, we can express α_i's by the eigenvalues λ_j of the Laplacian $\mathcal{L}_\beta$ and the q-numbers $q_j(i)$ of the P-polynomial scheme [1, 8, 10, 11]:

$$\alpha_i = \frac{1}{-q_1(i)\frac{1}{\lambda_1} - \cdots - q_d(i)\frac{1}{\lambda_d}}. \tag{2}$$

We also see that $0 < \alpha_d < \alpha_{d-1} < \cdots < \alpha_e < \lambda_1$ for some e.

The authors obtain the following result [12] on an upper bound on the Cheeger constant of a distance-regular graph with a certain restricted condition as follows.

Theorem A ([12]). *Let Γ be a distance-regular graph with diameter d and $\beta v r_d^{(\beta)} > \frac{\lambda_1}{1+\lambda_1}$ for $\beta \le \alpha_d$. Let λ_1 be the smallest eigenvalue of the Laplacian.*

Then we have

$$\lambda_1 h_\Gamma < \alpha_d < \alpha_{d-1} < \cdots < \alpha_e < \lambda_1$$

for $e \in \mathbf{C}'_\beta$.

Main results of this paper are the following Theorem 1, Theorem 2 and Corollary 3. Theorem 1 presents an upper bound on the Cheeger constant of a distance-regular graph. In Theorem 2, we find an explicit expression for the bound given in Theorem 1 by using the valencies k_j and the basis of nullspace $\mathcal{N}(L_{sub}^{(\alpha_d)})$. This shows that our new bound is a computable bound using the valencies and the intersection matrix of a distance-regular graph. Corollary 3 shows that our generalized bound in Theorem 1 and Theorem 2 improves the bound given in Theorem A [12] under the same additional condition.

Theorem 1. *Let* Γ *be a distance-regular graph of diameter* d. *Then we have the following upper bound*:

$$h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}},$$

where $\alpha_d = \lim_{\beta\to 0^+} \frac{\beta^2 v r_d^{(\beta)}}{1-\beta v r_d^{(\beta)}}$ *and* $\alpha_d^{(\alpha_d)} = \frac{\alpha_d^2 v r_d^{(\alpha_d)}}{1-\alpha_d v r_d^{(\alpha_d)}}$.

Theorem 2. *Let* Γ *be a distance-regular graph of order* v *and diameter* d. *Let* $(u_0^{(\alpha_d)}, u_1^{(\alpha_d)}, \ldots, u_d^{(\alpha_d)})$ *be a basis of* $\mathcal{N}(L_{sub}^{(\alpha_d)})$ *with* $u_d^{(\alpha_d)} = 1$ *as in Lemma 4. Then we have*

$$h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} = \alpha_d\big(\frac{1}{v}\sum_{j=0}^{d} k_j u_j^{(\alpha_d)} - 1\big),$$

where k_j *are valencies as in Lemma 4 and* α_d *is the same as in Lemma 5.*

Corollary 3. *Let* Γ *be a distance-regular graph, and* h_Γ *be a Cheeger constant of* Γ. *If* $\beta v r_d^{(\beta)} > \frac{\lambda_1}{1+\lambda_1}$ *for* $\beta \le \alpha_d$, *then we have*

$$h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} < \frac{\alpha_d}{\lambda_1},$$

where λ_1 *is the smallest positive eigenvalue of the Laplacian and* α_d *is the same as in* (1).

In Section 2, we introduce some notations and facts about distance-regular graph and some properties of the Green's function $\mathcal{G}_\beta$. In Section 3, we find a new upper bound on the Cheeger constant of a distance-regular graph. We also obtain an alternative expression of our upper bound by using the valencies k_j and the basis of nullspace $\mathcal{N}(L_{sub}^{(\alpha_d)})$. Finally, in Section 4, we present some examples about our upper bound on the Cheeger constant of some distance-regular graphs.

2. Preliminaries and Green's function

We introduce definitions of the distance-regular graphs and the P-polynomial schemes. A connected graph Γ with diameter d is called a *distance-regular graph* if there are constants c_i, a_i, b_i such that for all $i = 0, 1, \ldots, d$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are c_i at distance $i-1$ from x, a_i at distance i, and b_i at distance $i+1$. It follows that Γ is a regular graph with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for all $i = 0, 1, \ldots, d$. By these equations, the intersection numbers a_i can be expressed in terms of the others, and it is a standard to put these others in the so-called *intersection array* $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$. We describe the relations by its adjacency matrices A_i $(i = 0, 1, \ldots, d)$ which are $v \times v$ matrices defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

[1, 8]. Let X be a nonempty finite set and $R = \{R_0, R_1, \ldots, R_d\}$ be a family of relations defined on X. We say that the pair (X, R) is a symmetric association scheme with d classes if it satisfies the following conditions.

(1) $A_0 = I$ (indentity matrix).
(2) $A_0 + A_1 + \cdots + A_d = J$ (all 1 matrix).
(3) $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$, where p_{ij}^k is the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$.
(4) $A_j^t = A_j$.
(5) $A_iA_j = A_jA_i$.

A symmetric association scheme $\mathfrak{X} = (X, R)$ is called a *P-polynomial scheme* with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exist some complex coefficient polynomials $v_i(x)$ of degree i $(i = 0, 1, \ldots, d)$ such that $A_i = v_i(A_1)$, where A_i is the adjacency matrix with respect to R_i.

It is known [1, 8] that a distance-regular graph is equivalent to a P-polynomial scheme $\mathfrak{X}$ with respect to some relations $R_0,\ R_1, \ldots, R_d$ on a vertex set V with $|V| = v$. Thus, we can define the Green's function over a P-polynomial scheme, and then by using the Green's function we will obtain an upper bound on the Cheeger constant of a distance-regular graph.

The first intersection matrix B_1 of a distance-regular graph is a tridiagonal matrix with non-zero off diagonal entries:

$$B_1 = \begin{pmatrix} 0 & k & 0 & 0 & \cdots & 0 \\ 1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & \cdots & \cdots & 0 & c_d & a_d \end{pmatrix} \quad (b_i \neq 0, c_i \neq 0).$$

Let $\mathcal{A}$ be the algebra spanned by the adjacency matrices $A_0, A_1, \ldots, A_d$. Then $\mathcal{A}$ is called the *Bose-Mesner algebra* of $\mathfrak{X}$, and $\mathcal{A}$ has two distinguished bases $\{A_i\}$ and $\{E_i\}$, where the latter consists of primitive idempotent matrices. For A_i and E_i, we can express one in terms of the other as the following:

$$A_j = \sum_{i=0}^{d} p_j(i) E_i,\ E_j = \frac{1}{|X|}\sum_{i=0}^{d} q_j(i) A_i$$

for $j = 0, 1, \ldots, d$. The $(d+1)\times(d+1)$ matrix $\mathbf{P} = (p_j(i))$ (respectively, $\mathbf{Q} = (q_j(i))$) is called *the first eigenmatrix* (respectively, the second eigenmatrix) of the P-polynomial scheme $\mathfrak{X}$, where $p_j(i)$ (respectively, $q_j(i)$) is a p-number (respectively, a q-number).

As defined in Section 1, the Green's function $\mathcal{G}_\beta$ is the inverse of the β-Laplacian $\mathcal{L}_\beta$. For $\beta > 0$, we thus have $\mathcal{G}_\beta(\beta I + I - P) = I$. Therefore, we get

$$\mathcal{L}_\beta = \sum_{j=0}^{d}\left(\beta + 1 - \frac{1}{k_1}p_1(j)\right)E_j,\ \mathcal{G}_\beta = \sum_{j=0}^{d}\left(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\right)E_j.$$

Since $E_j = (1/v)\sum q_j(i)A_i$ and $\lambda_j = 1 - p_1(j)/k_1$, we get

$$\begin{aligned}
\mathcal{G}_\beta &= \sum_{j=0}^{d}\Big(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\Big)\sum_{i=0}^{d} q_j(i)\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}(1/v)\Big(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\Big)q_j(i)A_i\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}(1/v)\Big(\frac{1}{\beta + 1 - \frac{p_1(j)}{k_1}}\Big)q_j(i)A_i\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}(1/v)\Big(\frac{1}{\beta+\lambda_j}\Big)q_j(i)A_i.
\end{aligned}$$

That is, $\mathcal{G}_\beta$ is a linear combination of adjacency matrices A_i as follows:

$$\mathcal{G}_\beta = r_0^{(\beta)}A_0 + r_1^{(\beta)}A_1 + \cdots + r_d^{(\beta)}A_d,$$

where $r_i^{(\beta)} = \frac{1}{v}(\frac{1}{\beta} + q_1(i)\frac{1}{\beta+\lambda_1} + \cdots + q_d(i)\frac{1}{\beta+\lambda_d})$ $(i = 0, 1, \ldots, d)$.

In [10, 11], a $d\times(d+1)$ matrix $L_{sub}^{(\beta)}$ is introduced as a matrix obtained by the removal of the first row of $B_1 - k_1(\beta+1)I$.

Lemma 4 ([11])**.** *For $\beta > 0$, let $\mathcal{G}_\beta = r_0^{(\beta)}A_0 + r_1^{(\beta)}A_1 + \cdots + r_d^{(\beta)}A_d$ be the Green's function of a distance-regular graph Γ of order v. Then we have*

(a) *$\mathcal{G}_\beta$ can be expressed as $\mathcal{G}_\beta = tu_0^{(\beta)}A_0 + tu_1^{(\beta)}A_1 + \cdots + tu_d^{(\beta)}A_d$ for some nonzero $t \in \mathbb{R}$, where $(u_0^{(\beta)}, u_1^{(\beta)}, \ldots, u_d^{(\beta)})$ is the unique basis of the nullspace $\mathcal{N}(L_{sub}^{(\beta)})$ of $L_{sub}^{(\beta)}$ with $u_d^{(\beta)} = 1$.*

(b) $k_0r_0^{(\beta)} + k_1r_1^{(\beta)} + \cdots + k_dr_d^{(\beta)} = \frac{1}{\beta}$, *where* k_j *is the valency of* A_j *for* $j = 0, 1, \ldots, d$.

(c) $r_0^{(\beta)} > r_1^{(\beta)} > \cdots > r_d^{(\beta)} > 0$.

(d) $\lim_{\beta\to 0^+} |r_i^{(\beta)} - r_j^{(\beta)}| = 0$ *for* $0 \le i, j \le d$.

3. A new improved bound on the Cheeger constant

In this section we prove Theorem 1 and Theorem 2. We need the following lemma for the proof of Theorem 2 and Theorem 3. We consider a set $\mathbf{C}_\beta = \{i \mid \frac{1}{\beta} - vr_i^{(\beta)} > 0\}$ as a subset of $\{0, 1, 2, \ldots, d\}$; then $\mathbf{C}_\beta$ is a non-empty set by Lemma 4. When β is sufficiently close to 0^+, we consider a set $\mathbf{C}'_\beta = \{i \mid \beta vr_i^{(\beta)}(\beta + \lambda_1) < \lambda_1\}$; then $\mathbf{C}'_\beta$ is a subset of $\mathbf{C}_\beta$.

Lemma 5 ([11])**.** *For* $\beta > 0$, *let* Γ *be a distance-regular graph of order* v, *and let* $\mathcal{G}_\beta = r_0^{(\beta)}A_0 + r_1^{(\beta)}A_1 + \cdots + r_d^{(\beta)}A_d$ *be a Green's function of* Γ. *We recall that* $\alpha_i^{(\beta)} := \frac{\beta^2 vr_i^{(\beta)}}{1-\beta vr_i^{(\beta)}}$ $(i = 0, 1, \ldots, d)$ *as given in Eq.* (1). *Then for* $i \in \mathbf{C}'_\beta$, *we have the following*:

(a) $\lim_{\beta\to 0^+} \beta vr_i^{(\beta)} = 1^-$, $\lim_{\beta\to 0^+} \beta^2 vr_i^{(\beta)} = 0^+$.

(b) $\alpha_i^{(\beta)}$ *is decreasing in* $i \in \mathbf{C}'_\beta$.

(c) *There exists* $i \in \mathbf{C}'_\beta$ *such that* $\lim_{\beta\to 0^+} \alpha_i^{(\beta)} = \alpha_i < \lambda_1$.

(d) $\alpha_i^{(\beta)}$ *is decreasing in* $\beta > 0$.

Proof of Theorem 1. Let S be a subset of the vertex set V of Γ with $\mathrm{vol}(S) \le \mathrm{vol}(V)/2$ and $\frac{|\partial S|}{\mathrm{vol}(S)} \ne h_\Gamma$. Let S' be a subset of V with $\frac{|\partial S'|}{\mathrm{vol}(S')} = h_\Gamma$. Then there exists some β with $0 < \beta < 1$ such that

$$\frac{|\partial S|\beta}{\mathrm{vol}(S)} = \frac{|\partial S'|}{\mathrm{vol}(S')}. \tag{3}$$

We first note that for a positive integer n,

$$\beta < \frac{\alpha_d^{(\alpha_d\beta^n)}}{\alpha_d^{(\alpha_d)}}; \tag{4}$$

this follows immediately from $\frac{\alpha_d^{(\alpha_d)}}{\alpha_d} < \frac{\alpha_d^{(\alpha_d\beta^n)}}{\alpha_d\beta}$, which is clear since $\alpha_d^{(x)}$ is decreasing in x by Lemma 5.

We claim that for any $\epsilon > 0$, there exists a positive integer N_ε such that

$$\alpha_d^{(\alpha_d\beta^n)} < \alpha_d^2 + \varepsilon \tag{5}$$

for any $n \ge N_\varepsilon$; we use Lemma 5 for the proof as follows. Let $f_n = \alpha_d\beta^n vr_d^{(\alpha_d\beta^n)}$. Then $\lim_{n\to\infty} f_n = 1^-$ by Lemma 5(a). Thus, for sufficiently large positive integer n, we obtain the following approximations:

$$f_n(\beta^n + \alpha_d) \approx \alpha_d$$

$$\Rightarrow f_n\beta^n \approx \alpha_d(1-f_n)$$
$$\Rightarrow \alpha_d^{(\alpha_d\beta^n)} = \frac{f_n\beta^n\alpha_d}{1-f_n} \approx \alpha_d^2;$$

so our claim in Eq. (5) follows.

From Eq. (4) and Eq. (5), we thus have that

$$\beta < \frac{\alpha_d^2+\varepsilon}{\alpha_d^{(\alpha_d)}}. \tag{6}$$

Taking $\varepsilon > 0$ to be such that $\varepsilon < \big(\frac{\mathrm{vol}(S)}{|\partial S|}-1\big)\alpha_d^2$ (noting that the right hand side of this inequality is positive), we obtain

$$h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}};$$

this is because from Eq. (3) and Eq. (6), we get the following:

$$h_\Gamma = \frac{|\partial S|\beta}{\mathrm{vol}(S)} < \frac{|\partial S|}{\mathrm{vol}(S)}\Big(\frac{\alpha_d^2+\varepsilon}{\alpha_d^{(\alpha_d)}}\Big) < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}.$$

Consequently, the result follows as desired. □

Proof of Theorem 2. From Theorem 1, we have

$$h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}.$$

By Lemma 5, we have

$$\alpha_d = \lim_{\beta\to 0^+}\frac{\beta^2 vr_d^{(\beta)}}{1-\beta vr_d^{(\beta)}},\ \alpha_d^{(\alpha_d)} = \frac{\alpha_d^2 vr_d^{(\alpha_d)}}{1-\alpha_d vr_d^{(\alpha_d)}}.$$

Thus,

$$\frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} = \frac{1-\alpha_d vr_d^{(\alpha_d)}}{vr_d^{(\alpha_d)}} = \frac{1}{vr_d^{(\alpha_d)}} - \alpha_d, \tag{7}$$

where $vr_d^{(\alpha_d)} = \frac{1}{\alpha_d} + q_1(d)\frac{1}{\alpha_d+\lambda_1} + \cdots + q_d(d)\frac{1}{\alpha_d+\lambda_d}$. From Lemma 4, we have

$$\sum_{j=0}^{d} k_j r_j^{(\alpha_d)} = \frac{1}{\alpha_d},$$

and this implies that

$$r_d^{(\alpha_d)}\sum_{j=0}^{d} k_j u_j^{(\alpha_d)} = \frac{1}{\alpha_d};$$

so we get

$$\alpha_d\sum_{j=0}^{d} k_j u_j^{(\alpha_d)} = \frac{1}{r_d^{(\alpha_d)}}.$$

It thus follows that

$$\frac{1}{vr_d^{(\alpha_d)}} - \alpha_d = \alpha_d\big(\frac{1}{v}\sum_{j=0}^{d} k_j u_j^{(\alpha_d)} - 1\big). \tag{8}$$

The result follows immediately by combining Eq. (7) with Eq. (8). □

The following remark shows that $u_j^{(\beta)}$ can be expressed by a determinant of a submatrix $L_j^{(\beta)}$ of $\mathcal{N}(L_{sub}^{(\beta)})$, and α_d can be expressed in terms of a basis $(u_0^{(\beta)}, u_1^{(\beta)}, \dots, u_d^{(\beta)})$ of $\mathcal{N}(L_{sub}^{(\beta)})$ and the valencies k_j's as in Lemma 4.

Remark 6. (1) [10, 11] For $\beta > 0$, let $L_0^{(\beta)}$ be the $d \times d$ matrix obtained by the removal of the first column of $L_{sub}^{(\beta)}$ as in Lemma 4. Let $L_j^{(\beta)}$ be the $(d-j)\times(d-j)$ matrix obtained by the removal from the first row(respectively, column) to the j-th row(respectively, column) of $L_0^{(\beta)}$, and let $(u_0^{(\beta)}, u_1^{(\beta)}, \dots, u_d^{(\beta)})$ be a basis of $\mathcal{N}(L_{sub}^{(\beta)})$ with $u_d^{(\beta)} = 1$. Then we have

$$u_j^{(\beta)} = (-1)^{d-j}\frac{\det(L_j^{(\beta)})}{c_{j+1}c_{j+2}\cdots c_d}, \ j = 0, 1, \dots, d-1,$$

where $\det(L_d^{(\beta)}) = 1$.

(2) [11] Let Γ be a distance-regular graph of order v, and let $\mathcal{G}_\beta = r_0^{(\beta)}A_0 + r_1^{(\beta)}A_1 + \cdots + r_d^{(\beta)}A_d$ be a Green's function of Γ for $\beta > 0$. Then we have

$$\alpha_d = \lim_{\beta\to 0^+}\frac{\beta v}{\sum_{j=0}^{d} k_j u_j^{(\beta)} - v} \text{ and } \alpha_d < \alpha_d^{(\beta)} + \beta.$$

Proof of Corollary 3. Since $\beta v r_d^{(\beta)} > \frac{\lambda_1}{1+\lambda_1}$ for $\beta \le \alpha_d$, we have

$$\lambda_1 < \frac{\beta v r_d^{(\beta)}}{1-\beta v r_d^{(\beta)}} = \frac{\alpha_d^{(\beta)}}{\beta}.$$

Letting $\beta = \alpha_d$, we get $\frac{\alpha_d}{\alpha_d^{(\alpha_d)}} < \frac{1}{\lambda_1}$. It thus follows

$$\frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} < \frac{\alpha_d}{\lambda_1}.$$

□

4. Examples

In this section we present some examples regarding our upper bound on the Cheeger constants for some distance-regular graphs. In particular, Example 7 and Example 8 show that our bound is much more improved one comparing with the bound in [11, 12] under the same additional condition.

Theorem 1 shows an upper bound on the Cheeger constant h_Γ in terms of α_d and $\alpha_d^{(\beta)}$. From Equations (2) and (3), if we know the q-numbers of the given P-polynomial scheme, then we can find α_d and $\alpha_d^{(\beta)}$ immediately.

In the Hamming scheme $H(d,q)$ (respectively, Johnson scheme $J(m,d)$), the p-number $p_j(i)$ is defined by the Krawtchouk polynomial (respectively, the Eberlein polynomial) [1]. Since $\mathbf{PQ} = v\mathbf{I}$, we can obtain the q-numbers $q_j(i)$ of the Hamming scheme $H(d,q)$ and the Johnson scheme $J(m,d)$. We present the following two examples for showing this case.

Example 7. Let Γ be the graph of the Hamming scheme $H(d,q)$ with respect to A_1. Then Γ is a distance-regular graph with q^d vertices, valency $d(q-1)$ and d diameter. We consider two cases: (a) $d = 5,\ q = 4$ and (b) $d = 7,\ q = 3$, and in each case, our upper bound on the Cheeger constant is as follows:

(a) $H(5,4) : v = 1024,\ k_1 = 15,\ \lambda_1 = 4/15,\ \alpha_d = 16/137$.
And, for $\beta \le \alpha_d,\ \beta v r_d^{(\beta)} \ge \alpha_d v r_d^{(\alpha_d)} \approx 0.41256 > \frac{\lambda_1}{1+\lambda_1} \approx 0.21053$.
Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.166291 < \frac{\alpha_d}{\lambda_1} \approx 0.43795$.

(b) $H(7,3) : v = 2187,\ k_1 = 14,\ \lambda_1 = \frac{3}{14},\ \alpha_d = \frac{10}{121}$.
And, for $\beta \le \alpha_d,\ \beta v r_d^{(\beta)} \ge \alpha_d v r_d^{(\alpha_d)} \approx 0.404240 > \frac{\lambda_1}{1+\lambda_1} \approx 0.176471$.
Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.12180 < \frac{\alpha_d}{\lambda_1} \approx 0.385675$.

Example 8. Let Γ be a graph of the Johnson scheme $J(m,d)$ with respect to A_1. Then Γ is a distance-regular graph with $\binom{m}{d}$ vertices, valency $d(m-d)$ and d diameter. We consider two cases: (a) $m = 6,\ d = 3$ and (b) $m = 11,\ d = 5$. In each case, our upper bound on the Cheeger constant is as follows:

(a) $J(8,4) : v = 126,\ k_1 = 20,\ \lambda_1 = 9/20,\ \alpha_d = 252/1325$.
And, for $\beta \le \alpha_d,\ \beta v r_d^{(\beta)} \ge \alpha_d v r_d^{(\alpha_d)} \approx 0.415036 > \frac{\lambda_1}{1+\lambda_1} \approx 0.310345$.
Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.268058 < \frac{\alpha_d}{\lambda_1} \approx 0.422642$.

(b) $J(11,5) : v = 462,\ k_1 = 30,\ \lambda_1 = 11/30,\ \alpha_d = 11088/79091$.
And, for $\beta \le \alpha_d,\ \beta v r_d^{(\beta)} \ge \alpha_d v r_d^{(\alpha_d)} \approx 0.40805 > \frac{\lambda_1}{1+\lambda_1} \approx 0.268293$.
Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.203375 < \frac{\alpha_d}{\lambda_1} \approx 0.382344$.

Example 9. Let Γ be a Taylor graph with intersection array $(275, 112, 1; 1, 112, 275)$. Then Γ is a distance-regular graph with vertices 552, valency 275 and 3 diameter. Also, $\lambda_1 = 4/5,\ \alpha_3 = 3864/12475,\ \alpha_3^{(\alpha_3)} = 0.2265$. Thus we have an upper bound on the Cheeger constant of Γ as follows:

$$h_\Gamma < \frac{\alpha_3^2}{\alpha_3^{(\alpha_3)}} \approx 0.42357.$$

In Theorem 2, we find an alternative upper bound, which is explicitly computable, by using α_d, the valencies k_j, and the basis of nullspace $\mathcal{N}(L_{sub}^{(\beta)})$. In Example 10 and Example 11, we compute the upper bound on the Cheeger constant using the alternative expression in Theorem 2 and Remark 6.

Example 10. Let Γ be a graph with respect to A_1 of a Johnson scheme $J(8,4)$. Then Γ is a distance-regular graph with 70 vertices and valency 16. Also, the valencies of $J(8,4)$ are $1, 16, 36, 16, 1$ and

$$L_{sub}^{(\beta)} = \begin{pmatrix} 1 & 6-16(\beta+1) & 9 & 0 & 0 \\ 0 & 4 & 8-16(\beta+1) & 4 & 0 \\ 0 & 0 & 9 & 6-16(\beta+1) & 1 \\ 0 & 0 & 0 & 16 & -16(\beta+1) \end{pmatrix}.$$

Since

$$\alpha_d = \frac{1}{-q_1(d)\frac{1}{\lambda_1} - \cdots - q_d(d)\frac{1}{\lambda_d}},$$

$(q_1(4), \ldots, q_4(4)) = (-7, 20, -28, 14)$ and $(\lambda_1, \ldots, \lambda_4) = (\frac{8}{16}, \frac{14}{16}, \frac{18}{16}, \frac{20}{16})$, we get $\alpha_d = 315/1522$. Let $\beta = 315/1522$. By Remark 6, a basis $(u_0^{(\alpha_d)}, \ldots, u_4^{(\alpha_d)})$ for $\mathcal{N}(L_{sub}^{(\alpha_d)})$ is

$$\Big(\frac{10692972602391}{335381132641}, \frac{3108779427}{881422162}, \frac{969476}{579121}, \frac{1837}{1522}, 1\Big).$$

Thus, by Theorem 2, we have

$$h_\Gamma < (315/1522)\Big(\frac{1}{70}\sum_{j=0}^{4} k_j u_j^{(\alpha_d)} - 1\Big) \approx 0.292388.$$

Example 11. Let X be a set of $d \times n$ matrices over $GF(p^t)$ $(d \le n)$. We define the i-th relation R_i on X by $(x, y) \in R_i$ if and only if $\text{rank}(x - y) = i$. Then $\mathfrak{X} = (X, \{R_i\})$ $(0 \le i \le d)$ is a P-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$. Let $p = 2, t = 1, d = 4, n = 5$. Then $L_{sub}^{(\beta)}$ is obtained as follows:

$$\begin{pmatrix} 1 & 44-465(\beta+1) & 420 & 0 & 0 \\ 0 & 6 & 123-465(\beta+1) & 336 & 0 \\ 0 & 0 & 28 & 245-465(\beta+1) & 192 \\ 0 & 0 & 0 & 120 & 345-465(\beta+1) \end{pmatrix}.$$

We have $|X| = v = 1048576$, $k_0 = 1$, $k_1 = 465$, $k_2 = 32550$, $k_3 = 390600$ and $k_4 = 624960$ by using $k_i = \frac{k_1 b_1 b_2 \cdots b_{i-1}}{c_2 c_3 \cdots c_i}$ $(i = 2, 3, \ldots, d)$.

Let $\beta = \frac{1}{100}$. Then by Lemma 4 and Remark 6 (1), we obtain the unique basis of $\mathcal{N}(L_{sub}^{(\beta)})$ as follows:

$$(u_0^{(\beta)}, u_1^{(\beta)}, u_2^{(\beta)}, u_3^{(\beta)}, u_4^{(\beta)}) = \Big(\frac{3921317781669}{358400000}, \frac{486743013}{17920000}, \frac{661683}{448000}, \frac{831}{800}, 1\Big).$$

Thus, by Remark 6, we find

$$\frac{(1048576)\frac{1}{100}}{(1)u_0^{(\beta)} + (465)u_1^{(\beta)} + (32550)u_2^{(\beta)} + (390600)u_3^{(\beta)} + (624960)u_4^{(\beta)} - 1048576}$$
$$\approx 0.195023.$$

Thus, we have $\alpha_d < \widetilde{\alpha_d} \approx 0.195023 + 0.01 = 0.205023$.

Let $\beta = 0.205023$. Then, we obtain the unique basis of $\mathcal{N}(L_{sub}^{(\widetilde{\alpha_d})})$ as follows:

$$u_0^{(\widetilde{\alpha_d})} = \frac{227848229494208860060049660207}{512000000000000000000000000}, \ u_1^{(\widetilde{\alpha_d})} = \frac{2234200208459744136613}{2560000000000000000},$$

$$u_2^{(\widetilde{\alpha_d})} = \frac{85453392459301}{6400000000000}, \ u_4^{(\widetilde{\alpha_d})} = \frac{14355713}{8000000}, \ u_4^{(\widetilde{\alpha_d})} = 1.$$

Thus, from Theorem 2, we obtain

$$h_\Gamma < (\widetilde{\alpha_d})\Big(\frac{1}{1048576}\sum_{j=0}^{4} k_j u_j^{(\widetilde{\alpha_d})} - 1\Big) \approx 0.305557.$$

Remark 12. In general, it is a hard task to compute the Cheeger constant of the graph, and there is not much known about the actual value of the Cheeger constant of a graph. As far as we know, the only known case is the Cheeger constant of the Hamming graph $H(d, q)$ with q even, which is $\frac{q}{2n(q-1)}$. For instance, we consider two cases $H(5, 2)$ and $H(5, 4)$ for comparing our bound with the actual Cheeger constant:

(a) $H(5, 2) : v = 512$, $k_1 = 5$, $\lambda_1 = 2/5$, $\alpha_d = 24/137$, $\alpha_d^{(\alpha_d)} \approx 0.123033$. Thus, $h_\Gamma = \frac{1}{5} = 0.2 < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.249437$.

(b) $H(5, 4) : v = 1024$, $k_1 = 15$, $\lambda_1 = 4/15$, $\alpha_d = 16/137$, $\alpha_d^{(\alpha_d)} \approx 0.083724$. Thus, $h_\Gamma = \frac{2}{15} = 0.133333\cdots < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} \approx 0.166291$.

As we can see from these examples, our bound is close to the Cheeger constant, but it is not sharp yet.

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