Bull. Korean Math. Soc. **54** (2017), No. 2, pp. 497–505
https://doi.org/10.4134/BKMS.b160122
pISSN: 1015-8634 / eISSN: 2234-3016

DIRECTED STRONGLY REGULAR GRAPHS AND THEIR CODES

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Abstract. The rank over a finite field of the adjacency matrix of a directed strongly regular graph is studied, with some applications to the construction of linear codes. Three techniques are used: code orthogonality, adjacency matrix determinant, and adjacency matrix spectrum.

1. Introduction

Directed Strongly Regular Graphs (DSRG) were introduced by Duval as a generalization of strongly regular graphs (SRG's) [4]. As observed in [8] a special case of these are the doubly regular tournaments or equivalently, the skew Hadamard matrices. As the latter already lead to many interesting codes [10] it is natural to consider the more general case of codes constructed from the adjacency matrix of a DSRG. Another motivation is that the codes of SRG's have been investigated in [2, 7], and the spectra of SRG and DSRG present many similarities, as evidenced by Section 3.3.

In this paper, we investigate the dimensions of these codes by using three techniques: orthogonality of codes, determinant factorization, a classical technique in design theory, and last but not least, spectrum of the adjacency matrices in the spirit of [2, 7]. Examples of codes illustrating the bounds, along with their minimum distances are given. While the codes constructed are not, in general, optimal error correcting codes in the distance vs dimension sense,

Received February 14, 2016; Revised August 27, 2016.

2010 *Mathematics Subject Classification.* Primary 94B25, 05E30.

Key words and phrases. adjacency matrix, directed graph, DSRG, error correcting code, SRG, strongly regular graph.

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (19-130-35-RG).

J.-L. Kim was supported by Basic Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03933259).

M. Shi was supported by National Natural Science Foundation of China (61202068), and the Open Research Fund of National Mobile Communications Research Laboratory, Southeast University (2015D11). Technology Foundation for Selected Overseas Chinese Scholar, Ministry of Personnel of China (05015133) and Key projects of support program for outstanding young talents in Colleges and Universities (gxyqZD2016008).

or related to classical algebraic codes, we believe they are of some interest to the combinatorics of DSRG's in the same way that the codes of [1] are relevant to the Combinatorics of designs. In particular, the problem of computing their dimension is of interest in its own right as the difficult Conjecture of Section 3.2 shows.

The material in this paper is organized as follows. Section 2 recalls the needed notions and notations. Then the three subsections of Section 3 contain the three said bounding techniques as well as code parameters. An appendix displays the adjacency matrices of DSRG's corresponding to the codes constructed.

2. Background

Let A be a v by v matrix with entries in $\{0,1\} \subset \mathbb{Z}$ and having zero diagonal. Thus A is the adjacency matrix of a directed simple graph without loops on v vertices. This graph is a *directed strongly regular* graph (DSRG) of parameters (v, k, t, λ, μ) if it satisfies the following pair of relations:

- $AJ = JA = kJ$,
- $A^2 = tI + \lambda A + \mu(J - I - A)$,

where I and J denote the identity and all-one matrices, respectively, of order v. It is easy to check that if A is the adjacency matrix of a DSRG so is $J - I - A$. These two DSRGs are said to form a *complementary pair*. When the ring of coefficients of A is a general field F instead of $\mathbb{Z}$, we still denote the matrix by A.

There is a dependency between the parameters:

Lemma 2.1. *A DSRG having parameters (v, k, t, λ, μ) satisfies the following equation*:

$$v\mu + t - \mu = k(k - \lambda + \mu).$$

Proof. Multiplying by sides of the second equation above by J and using $AJ = kJ$ we obtain

$$A^2 J = k^2 J = (t + \lambda k + \mu(v - 1 - k))J$$
$$\implies k^2 = t + \lambda k + \mu v - \mu - \mu k,$$

which implies the result. □

Lemma 2.2.

$$\det(A^2 + (\mu - \lambda)A) = \det(A)\det(A + (\mu - \lambda)I) = (t + (v - 1)\mu)(t - \mu)^{v-1}.$$

Proof. This follows since the second equation above is the same as

$$A^2 + (\mu - \lambda)A = tI + \mu(J - I) = (t - \mu)I + \mu J,$$

but by using the fact that the determinant is the product of the eigenvalues it is easy to see that

$$\det((t - \mu)I + \mu J) = (t + (v - 1)\mu)(t - \mu)^{v-1}.$$ □

3. The rank of the adjacency matrix

The following result is folklore in characteristic zero; c.f. [6].

Theorem 3.1. *A, with field of coefficients F, satisfies*

$$P(A) := (A - kI)(A^2 + (\mu - \lambda)A + (\mu - t)I) = 0.$$

Proof. Write $Q(x) = x^2 + (\mu - \lambda)x + \mu - t$. By definition of a DSRG we have that $Q(A) = \mu J$. By $AJ = JA = kJ$, we see that $(A - kI)Q(A) = 0$. □

Note that the cubic polynomial $P(x)$ (in a general commutative variable x) is an *annihilator polynomial* of A. Also, P might not be the *minimal* polynomial of A, since sometimes the minimal polynomial of A can have degree less than three; c.f. [8].

We are concerned with the rank of A over a finite field $\mathbb{F}_q$ of characteristic p. The least interesting case for coding theory is when A has full rank.

Corollary 3.2. *If $k(\mu - t) \not\equiv 0 \pmod p$, then A has rank v over $\mathbb{F}_q$. Equivalently, $\det(A) = 0$ implies that either $k \equiv 0 \pmod p$, or $t \equiv \mu \pmod p$.*

Proof. Since $P(0) = k(\mu - t)$, only in this case A can have an eigenvalue zero. □

Thus, in the rest of the paper, we will only consider the congruence cases of Corollary 3.2, or look at the linear span of a different matrix. We derive orthogonality properties of codes built from A depending on the roots of the polynomial Q.

3.1. Orthogonality properties

When X is a general matrix with v columns, we consider the (linear) code $C_X := \langle X \rangle$ (with generator matrix X) to be the row space of X. Each vector (of length v) in this row space is called a *codeword* of C_X and so v is also the *length* of C_X. The *dual code* $C_X^\perp$ is defined to be the set of all (row)-vectors orthogonal to C_X. It is well-known that for any linear code C, $\dim(C) + \dim(C^\perp) = v$.

Proposition 3.3. *If p divides μ and $Q(x) \equiv (x - a)^2 \pmod p$, then the rank of $A - aI$ over $\mathbb{F}_q$ is at most $v/2$, and the codes $\langle A - aI \rangle$ and $\langle A^T - aI \rangle$ are orthogonal to each other. In particular, if p divides μ, λ, and t, then the rank of A over $\mathbb{F}_q$ is at most $v/2$.*

Proof. With the hypotheses we made, $Q(x) = (x - a)^2$ over $\mathbb{F}_q$ and $Q(A) = 0$, so the codes spanned respectively by the rows of $A - aI$ and its transpose are orthogonal to each other and of the same rank. In particular this is true for $a = 0$ when $Q(x) = x^2$ over $\mathbb{F}_q$. □

Using the tables in [8], we can list some examples of parameters where Proposition 3.3 applies. Note that complementary pairs of graphs do not share the same $Q(x)$. These are the cases of $q = 2$ in Table 1.

TABLE 1. Parameters for Propositions 3.3 and 3.4

v,k	8,4	12,4	12,7	12,6	15,4	15,10	16,7	16,8
t,λ,μ	3,1,3	2,0,2	5,4,4	4,2,4	2,1,1	8,6,8	4,3,3	5,3,5
$Q(x)-x^2$	$2x$	$2x$	-1	$2x$	-1	$-2x$	-1	$-2x$
q	3	2	2	2	2	2	2	5

TABLE 2. Binary $[n,k,d]$ Codes from Table 1

codes	$[n,k,d]$	codes	$[n,k,d]$
$C(A_{8,4})$	$[8,3,4]_2$	$C(A^T_{8,4})$	$[8,3,4]_2$
$C(A_{12,4})$	$[12,3,4]_2$	$C(A^T_{12,4})$	$[12,3,4]_2$
$C(A_{12,7}+I)$	$[12,2,8]_2$	$C(A^T_{12,7}+I)$	$[12,2,8]^*_2$
$C(A_{12,6})$	$[12,4,4]_2$	$C(A^T_{12,6})$	$[12,4,4]_2$
$C(A_{15,4}+I)$	$[15,6,4]_2$	$C(A^T_{15,4}+I)$	$[15,6,4]_2$
$C(A_{15,10})$	$[15,5,4]_2$	$C(A^T_{15,10})$	$[15,5,4]_2$
$C(A_{16,7}+I)$	$[16,5,4]_2$	$C(A^T_{16,7}+I)$	$[16,5,4]_2$
$C(A_{16,8})$	$[16,5,4]_2$	$C(A^T_{16,8})$	$[16,5,4]_2$

We describe the codes from Table 1 in Table 2, where $A_{v,k}$ denotes the adjacency matrix of a DSRG of parameters (v,k) in Table 1 and $C(B)$ is the code spanned by the rows of B. A star indicates an *optimal* code, that is having the largest dimension for given length and distance as per the tables of [5]. In the appendix, we give the explicit $A_{v,k}$ since the table [8] does not give explicit adjacency matrices in many cases.

Now we consider the case where Q has two distinct roots.

Proposition 3.4. *Assume $a \neq b$. If p divides μ and $Q(x) \equiv (x-a)(x-b) \pmod p$, then the codes $\langle A-aI\rangle$ and $\langle A^T-bI\rangle$ are the duals of each other.*

Proof. By $(A-aI)(A-bI)=\mu J \equiv 0 \pmod p$ we get $\langle A-aI\rangle \subseteq \langle A^T-bI\rangle^\perp$. Let $u \in \langle A^T-bI\rangle^\perp$, or, equivalently, $u(A-bI)=0$, that is $uA=bu$. If $b-a$ is nonzero in $\mathbb{F}_p$ then writing $u(A-aI)=(b-a)u$, we see that $u \in \langle A-aI\rangle$. This implies $\langle A^T-bI\rangle^\perp \subseteq \langle A-aI\rangle$. The result follows. □

Parameters satisfying Proposition 3.4 are the cases of $q=3,5$ in Table 1.

3.2. Determinant bounds

Now we give rank estimates.

Proposition 3.5. *If p^r divides $\det(A) \neq 0$ exactly, then the rank of A over $\mathbb{F}_p$ is at least $v-r$. In particular, in the case $\lambda=\mu$ if p^r divides $(t+(v-1)\mu)(t-\mu)^{v-1}$ exactly, then the rank of A over $\mathbb{F}_p$ is at least $v-r$.*

Proof. The proof follows by the standard invariant factors argument using the Smith normal form of A (see [9, p. 311] for details). If $\lambda=\mu$, then $A^2=$

$(t-\mu)I+\mu J$, and consequently, by Lemma 2.2, we have $\det(A)^2 = (t+(v-1)\mu)(t-\mu)^{v-1}$. □

In practice the determinant of A is best computed by taking the product of the eigenvalues with multiplicity. There are exact formulas to compute these as a function of the 5 parameters [6, Thm. 2.2]. Some parameter values where Proposition 3.5 applies are given in Table 3.

TABLE 3. Parameters for Proposition 3.5

v,k	16,7	18,11	18,13	18,8	20,7	24,15	24,9
t,λ,μ	5,4,2	8,7,6	12,9,10	5,4,3	4,3,2	11,10,8	7,2,4
r	2	2	10	3	4	2	8
p	3	2	2	2	2	3	3

TABLE 4. Binary or Ternary $[n,k,d]$ Codes with Their Ranks from Table 3

codes	$[n,k,d]$	$\text{rank}(A_{v,k}) \geq v-r$	codes	$[n,k,d]$
$C(A_{16,7})$	$[16,14,2]_3^*$	$14 = v-r$	$C(A_{16,7}^T+I)$	$[16,3,8]_3$
$C(A_{18,11})$	$[18,16,2]_2^*$	$16 = v-r$	$C(A_{18,11}^T+I)$	$[18,2,12]_2^*$
$C(A_{18,13})$	$[18,8,5]_2$	$8 = v-r$	$C(A_{18,13}^T+I)$	$[18,10,4]_2^*$
$C(A_{18,8})$	$[18,15,2]_2$	$15 = v-r$	$C(A_{18,8}^T+I)$	$[18,4,6]_2$
$C(A_{20,7})$	$[20,16,2]_2$	$16 = v-r$	$C(A_{20,7}^T+I)$	$[20,4,8]_2$
$C(A_{24,15})$	$[24,22,2]_3^*$	$22 = v-r$	$C(A_{24,15}^T+I)$	$[24,3,8]_3$
$C(A_{24,9})$	$[24,16,4]_3$	$16 = v-r$	$C(A_{24,9}^T+I)$	$[24,9,6]_3$

We describe the codes from Table 3 in Table 4, where $A_{v,k}$ denotes the adjacency matrix of a DSRG of parameters (v,k) in Table 3, $C(B)$ is the binary or ternary code spanned by the rows of B, and the exact rank of $A_{v,k}$ is computed. A star indicates an optimal code. In the appendix, we give the explicit $A_{v,k}$ since the table [8] does not give explicit adjacency matrices in many cases.

From Table 4 we see that the rank of A over $\mathbb{F}_q$ is exactly $v-r$. Hence we conjecture the following.

Conjecture. If p^r divides $\det(A) \neq 0$ exactly, then the rank of A over $\mathbb{F}_p$ is $v-r$.

3.3. Spectrum of the adjacency matrix

We shall give very general results by adapting the case of SRG's from [3, 13.7.1]. The proof is analogous and omitted. Consider $M = A+bJ+cI$. If the spectrum of A is k,r,s, then that of M is $\theta_0 = k+bv+c, \theta_1 = r+c, \theta_2 = s+c$ with the same respective multiplicities m_0, m_1, m_2 given by $1, f, g$.

Theorem 3.6. *Put* $e = \mu + b^2 v + b(2k - \lambda + \mu)$. *With the above notation, if*

- *all* θ_i*'s are nonzero* $\pmod p$, *then* $rk_p(M) = v$.
- *exactly one of the* θ_i*'s is zero* $\pmod p$, *then* $rk_p(M) = v - m_i$.
- *exactly* θ_0, θ_1 *are zero* $\pmod p$, *then* $rk_p(M) = g$ *if* p *divides* e *and* $g + 1$ *otherwise.*
- *exactly* θ_0, θ_2 *are zero* $\pmod p$, *then* $rk_p(M) = f$ *if* p *divides* e *and* $f + 1$ *otherwise.*
- *both* θ_1, θ_2 *are zero* $\pmod p$, *then* $rk_p(M) \leq \min(f + 1, g + 1)$.

Example. Take $b = c = 0$ and $p = 2$. The DSRG of parameters $(8, 3, 2, 1, 1)$ from [8] satisfies $r = -s = 1$. In that case, $M = A$ has rank $v = 8$. On the other hand, if $p = 3$, then only θ_0 is a multiple of 3 and $rk_2(M) = 8 - 1 = 7$. Consider now the DSRG of parameters $(10, 4, 2, 1, 2)$ from [8], which satisfies $r = 0$, $s = -1$. Because $b = c = 0$, we have $e = \mu = 2$, and $rk_2(M) = g = 4$. An example when both θ_1 and θ_2 are even is obtained for the DSRG of parameters $(8, 4, 3, 1, 3)$ from [8], which satisfies $r = 0$, $s = -2$. Here we have

$$rk_2(M) \leq \min(f + 1, g + 1) = \min(3, 6) = 3.$$

4. Conclusion and open problems

In the current paper, the ranks of adjacency matrices of DSRG's have been investigated by three different techniques; first code orthogonality, then determinantal bounds like in design theory, and, eventually the real spectrum of the considered matrix. Numerical examples show that the determinantal bound should be tight. This conjecture is the main open problem of the paper. On a more speculative mode, it would be nice to have a combinatorial interpretation of the vectors of weight the minimum distance in the codes constructed from the DSRG's.

Acknowledgement. The authors acknowledge with thanks DSR technical and financial support.

Appendix

We give the explicit adjacency matrices $A_{v,k}$ for Tables 1 and 3.

For Table 1,

$$A_{8,4} = \begin{bmatrix} 00001111 \\ 00001111 \\ 01010101 \\ 10101010 \\ 01010101 \\ 10101010 \\ 11110000 \\ 11110000 \end{bmatrix} \quad A_{12,4} = \begin{bmatrix} 000011110000 \\ 000011110000 \\ 000000001111 \\ 000000001111 \\ 000000001111 \\ 000000001111 \\ 111100000000 \\ 111100000000 \\ 111100000000 \\ 111100000000 \\ 000011110000 \\ 000011110000 \end{bmatrix}$$

$$A_{12,7} = \begin{bmatrix} 011100001111 \\ 101100001111 \\ 110111110000 \\ 111011110000 \\ 111101110000 \\ 111110110000 \\ 000011011111 \\ 000011101111 \\ 000011110111 \\ 000011111011 \\ 111100001101 \\ 111100001110 \end{bmatrix} A_{12,6} = \begin{bmatrix} 001100001111 \\ 001100001111 \\ 110011110000 \\ 110011110000 \\ 111100110000 \\ 111100110000 \\ 000011001111 \\ 000011001111 \\ 000011110011 \\ 000011110011 \\ 111100001100 \\ 111100001100 \end{bmatrix} A_{15,4} = \begin{bmatrix} 010010010000010 \\ 101000001000001 \\ 010100000110000 \\ 001011000001000 \\ 100100100000100 \\ 001000001001001 \\ 000100000110100 \\ 000011000001010 \\ 100000100000101 \\ 010000010010010 \\ 000100100100100 \\ 000011010000010 \\ 100000101000001 \\ 010000010110000 \\ 001001001001000 \end{bmatrix}$$

$$A_{15,10} = \begin{bmatrix} 001101101111101 \\ 000111110111110 \\ 100011111001111 \\ 110000111110111 \\ 011001011111011 \\ 110110110110110 \\ 111011011001011 \\ 111100101110101 \\ 011111010111010 \\ 101111101001101 \\ 111011011001011 \\ 111100101110101 \\ 011111010111010 \\ 101111101001101 \\ 110110110110110 \end{bmatrix} A_{16,7} = \begin{bmatrix} 0101010100101010 \\ 1001010100101010 \\ 1001010101001010 \\ 1010010101001010 \\ 1010010101010010 \\ 1010100101010010 \\ 1010100101010100 \\ 1010101001010100 \\ 0010101001010101 \\ 0010101010010101 \\ 0100101010010101 \\ 0100101010100101 \\ 0101001010100101 \\ 0101001010101001 \\ 0101010010101001 \\ 0101010010101010 \end{bmatrix} A_{16,8} = \begin{bmatrix} 0010101011010101 \\ 0010101011010101 \\ 0100101010110101 \\ 0100101010110101 \\ 0101001010101101 \\ 0101001010101101 \\ 0101010010101011 \\ 0101010010101011 \\ 1101010100101010 \\ 1101010100101010 \\ 1011010101001010 \\ 1011010101001010 \\ 1010110101010010 \\ 1010110101010010 \\ 1010101101010100 \\ 1010101101010100 \end{bmatrix}$$

For Table 3,

$$A_{16,7} = \begin{bmatrix} 0111111100000000 \\ 1011111100000000 \\ 1101111100000000 \\ 1110111100000000 \\ 1100010011001100 \\ 1100100011001100 \\ 0011000100110011 \\ 0011001000110011 \\ 1100110001001100 \\ 1100110010001100 \\ 0011001100010011 \\ 0011001100100011 \\ 0000000011110111 \\ 0000000011111011 \\ 0000000011111101 \\ 0000000011111110 \end{bmatrix} A_{18,11} = \begin{bmatrix} 011111000000111111 \\ 101111000000111111 \\ 110111000000111111 \\ 111011111111000000 \\ 111101111111000000 \\ 111110111111000000 \\ 111111011111000000 \\ 111111101111000000 \\ 111111110111000000 \\ 000000111011111111 \\ 000000111101111111 \\ 000000111110111111 \\ 000000111111011111 \\ 000000111111101111 \\ 000000111111110111 \\ 111111000000111011 \\ 111111000000111101 \\ 111111000000111110 \end{bmatrix} A_{18,13} = \begin{bmatrix} 010111111110011011 \\ 001111111011101101 \\ 100111111101110110 \\ 111010111011110011 \\ 111001111101011101 \\ 111100111110101110 \\ 111111010011011110 \\ 111111001101101011 \\ 111111100110110101 \\ 101011011001111111 \\ 110101101100111111 \\ 011110110010111111 \\ 011101011111001111 \\ 101110101111100111 \\ 110011110111010111 \\ 011011101111111001 \\ 101101110111111100 \\ 110110011111111010 \end{bmatrix}$$

$$A_{18,8} = \begin{bmatrix} 011000111111000000 \\ 101000111111000000 \\ 110000111111000000 \\ 000011000000111111 \\ 000101000000111111 \\ 000110000000111111 \\ 000000011000111111 \\ 000000101000111111 \\ 000000110000111111 \\ 111111000011000000 \\ 111111000101000000 \\ 111111000110000000 \\ 111111000000011000 \\ 111111000000101000 \\ 111111000000110000 \\ 000000111111000011 \\ 000000111111000101 \\ 000000111111000110 \end{bmatrix} A_{20,7} = \begin{bmatrix} 00000100001100111001 \\ 00000110001010100101 \\ 00000101101000100011 \\ 00001100011001100001 \\ 10010000010101011000 \\ 10000000001100111001 \\ 11000000000110011100 \\ 10100000100100011010 \\ 01100011000010000110 \\ 01011010000011000100 \\ 01000110000010100101 \\ 11000010000010011100 \\ 01100011100000000110 \\ 00111001110000000010 \\ 00100101101000000011 \\ 10100001100100001010 \\ 10011000010101010000 \\ 01011010010011000000 \\ 00111001110001000000 \\ 00011100011001100000 \end{bmatrix}$$

$$A_{24,15} = \begin{bmatrix} 011111110000000011111111 \\ 101111110000000011111111 \\ 110111110000000011111111 \\ 111011110000000011111111 \\ 111101111111111100000000 \\ 111110111111111100000000 \\ 111111011111111100000000 \\ 111111101111111100000000 \\ 111111110111111100000000 \\ 111111111011111100000000 \\ 111111111101111100000000 \\ 111111111110111100000000 \\ 000000001111011111111111 \\ 000000001111101111111111 \\ 000000001111110111111111 \\ 000000001111111011111111 \\ 000000001111111101111111 \\ 000000001111111110111111 \\ 000000001111111111011111 \\ 000000001111111111101111 \\ 111111110000000011110111 \\ 111111110000000011111011 \\ 111111110000000011111101 \\ 111111110000000011111110 \end{bmatrix} A_{24,9} = \begin{bmatrix} 000110100011110001010000 \\ 001001010101101001010000 \\ 010011000011001100110000 \\ 100011001010101011000000 \\ 101000011100001100000101 \\ 010100101010010100000101 \\ 110001000011001100000011 \\ 110010001010101000001100 \\ 010111000001000000111010 \\ 010110100010000001010101 \\ 001100110100000000111100 \\ 110010101000000010101100 \\ 110001010000000110100011 \\ 101001010000001001010101 \\ 001100110000010011000011 \\ 101011000000100011001010 \\ 001100000101101000011100 \\ 010100000101010100101010 \\ 001100000011110001000011 \\ 101000001100110010001010 \\ 000000111010010111000001 \\ 000001010101010110100010 \\ 000000111100001100110100 \\ 000010101100110010101000 \end{bmatrix}$$

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