

DISCUSSIONS ON PARTIAL ISOMETRIES IN BANACH SPACES AND BANACH ALGEBRAS

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ABSTRACT. The aim of this paper is twofold. Firstly, we introduce the concept of semi-partial isometry in a Banach algebra and carry out a comparison and a classification study for this concept. In particular, we show that in the context of C^* -algebras this concept coincides with the notion of partial isometry. Our results encompass several earlier ones concerning partial isometries in Hilbert spaces, Banach spaces and C^* -algebras. Finally, we study the notion of (m, p) -semi partial isometries.

1. Introduction

Let X be a complex Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators from X into itself. An operator $T \in \mathcal{L}(X)$ is said to be an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. If H is a Hilbert space, we say that $T \in \mathcal{L}(H)$ is a *partial isometry* if $\|Tx\| = \|x\|$ for all $x \in N(T)^\perp$, where $N(T)^\perp$ is the orthogonal complement of $N(T)$, or equivalently $TT^*T = T$. If T is a contraction (that is $\|T\| \leq 1$), it was shown in [13, Theorem 3.1] that T is a partial isometry if and only if T admits a contractive generalized inverse. Motivated by this fact, Mbekhta in [13] introduced the concept of partial isometry in a Banach space. A bounded linear operator T on X is called a partial isometry if it is a contraction and admits a generalized inverse which is a contraction. The drawback of this definition is that an isometry on a Banach space need not to be a partial isometry, see [13] and [16] for counter examples. In [19], the third author introduced a more general class namely the class of semi-partial isometries. A bounded linear operator T on a Banach space X is called a *semi-partial isometry* if $\|Tx\| = \text{dist}(x, N(T))$ for all $x \in X$. This class of operators, which is a natural generalization of partial isometries from Hilbert spaces to general Banach spaces, contains, among others, isometries, co-isometries, unitary operators and partial isometries (in the sense of M. Mbekhta). The concept of partial isometries in the algebra $\mathcal{L}(X)$ has been generalized and studied in the context of general Banach algebras in

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[14, 17]. In this work, semi-partial isometries in Banach algebras will be studied and several characterizations will be proved. In particular we show that the notion of partial isometry and semi-partial isometry coincide in the case of C^* -algebras. Further, an example is given to show that this property does not hold for general Banach $*$ -algebras.

The contents of this paper are as follows. In Section 2 several preliminary definitions and results are recalled. In Section 3, partial isometries on C^* -algebras are studied. The last section is devoted to the study of (m, p) -semi partial isometry.

2. Preliminaries

Let \mathcal{A} be a complex unital Banach algebra with unit e and let $a, b \in \mathcal{A}$. We say that $b \in \mathcal{A}$ is a *generalized inverse* of a , provided that $a = aba$ holds. In this case ab and ba are idempotents. In general, b is not unique. In the presence of an involution, we may require that ab and ba are Hermitian. In particular, we get the following definition in the context of C^* -algebras.

Definition. Let \mathcal{A} be a unital C^* -algebra and $a \in \mathcal{A}$. We say that b is a *Moore-Penrose inverse* of a provided that

$$a = aba, \quad b = bab, \quad (ab)^* = ab, \quad (ba)^* = ba,$$

hold.

The Moore-Penrose inverse of a is unique and will be denoted by a^\dagger . For further details, the reader is referred to [12, 13, 15, 20] and the references therein. In order to extend the concept of Moore-Penrose inverse to general Banach algebra, we should begin by defining what we mean by Hermitian element in a Banach algebra.

Definition. An element $a \in \mathcal{A}$ is said to be Hermitian if $\|\exp(ita)\| = 1$ for all $t \in \mathbb{R}$.

The set of Hermitian elements will be denoted by $\mathcal{H}(\mathcal{A})$. This set enjoys the following properties, see [6, page 47]:

- (i) $\mathcal{H}(\mathcal{A})$ is a closed real subspace of \mathcal{A} and $\mathcal{H}(\mathcal{A}) \cap i\mathcal{H}(\mathcal{A}) = \{0\}$.
- (ii) If $h, k \in \mathcal{H}(\mathcal{A})$, then $i(hk - kh) \in \mathcal{H}(\mathcal{A})$.

Now, the definition of the Moore-Penrose inverse in a general Banach algebra states as follows.

Definition ([5, Definition 1]). Let \mathcal{A} be a Banach algebra and $a \in \mathcal{A}$. We say that b is a Moore-Penrose inverse of a , if we have

$$a = aba, \quad b = bab, \quad ab \text{ and } ba \text{ are Hermitian.}$$

The Moore-Penrose inverse of a is unique and will be denoted by a^\dagger .

Set $\mathcal{J}(\mathcal{A}) = \{h+ik : h, k \in \mathcal{H}(\mathcal{A})\}$. Endowed with the norm of \mathcal{A} , $\mathcal{J}(\mathcal{A})$ is a complex Banach space. Furthermore, since $\mathcal{H}(\mathcal{A}) \cap i\mathcal{H}(\mathcal{A}) = \{0\}$, each element of $\mathcal{J}(\mathcal{A})$ has a unique representation of the form $h+ik$ with $h, k \in \mathcal{H}(\mathcal{A})$. Hence, if we define a mapping $*$ from $\mathcal{J}(\mathcal{A})$ to itself by $(h+ik)^* = h-ik$, we obtain a linear involution on $\mathcal{J}(\mathcal{A})$. That is to say that $(a+b)^* = a^*+b^*$ and $(\lambda a)^* = \bar{\lambda}a^*$ for every a and $b \in \mathcal{J}(\mathcal{A})$ and $\lambda \in \mathbb{C}$. Notice that by [6, Page 58], there exist a Banach algebra \mathcal{A} and an element $h \in \mathcal{J}(\mathcal{A})$ such that $h^2 \notin \mathcal{J}(\mathcal{A})$. Therefore, \mathcal{J} is not a Banach algebra. Accordingly, the linear involution defined above does not satisfy $(ab)^* \neq b^*a^*$ for every a and $b \in \mathcal{A}$.

Definition ([17]). We say that $a \in \mathcal{J}(\mathcal{A})$ is a partial isometry if $a = aa^*$.

For an operator $T \in \mathcal{L}(X)$ the *reduced minimum modulus* is defined by

$$\gamma(T) := \inf \{\|Tx\| : \text{dist}(x, N(T)) = 1\}, \quad (\gamma(T) = 1 \text{ if } T = 0).$$

It is well known that (see [11])

$$\gamma(T^*) = \gamma(T) \text{ and } \gamma(T) > 0 \text{ if and only if } R(T) \text{ is closed.}$$

The *reduced minimum modulus* (or the conorm) of an element a in a Banach algebra \mathcal{A} , is defined as the reduced minimum modulus of the left multiplication operator by a ([7, 12]). That is

$$\gamma(a) := \gamma(L_a) = \inf \{\|ax\| : \text{dist}(x, a^{-1}(0)) = 1\},$$

where

$$a^{-1}(0) = \{x \in \mathcal{A} : ax = 0\}$$

is the right annihilator of a .

Now, let $\mathcal{A} = \mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X . Each element T of \mathcal{A} can be viewed in two different ways. Specifically, one may consider T as an operator in the algebra $\mathcal{L}(X)$ and thus define its reduced minimum modulus as $\gamma(T) = \inf \{\|Tx\| : \text{dist}(x, N(T)) = 1\}$. Or consider it as an element of the Banach Algebra \mathcal{A} and thus define its reduced minimum modulus as $\gamma(T) = \gamma(L_T)$. Hence the following question seems to be natural: The two definitions are they identical? If not, which conditions ensure their equality? The response is negative in general (see [18]). We only have the following inequality:

$$\gamma(L_T) \leq \gamma(T) \text{ and } \gamma(R_T) \leq \gamma(T).$$

Moreover, if X is a Hilbert space we have:

$$\gamma(L_T) = \gamma(R_T) = \gamma(T).$$

It is worth observing that the above equality remains true in the setting of C^* -algebras. In fact the authors in [12] showed that if \mathcal{A} is a C^* -algebra, then we have $\gamma(a)^2 = \gamma(aa^*) = \gamma(a^*)^2$ and $\gamma(L_a) = \gamma(a)$ for all $a \in \mathcal{A}$.

We end this paragraph by the following proposition. Its proof is simple and follows immediately by applying [13, Proposition 2.2] to the operator L_a .

Proposition 2.1. *Let $a \in \mathcal{A}$ and let b be a generalized inverse of a . Then,*

$$(1) \quad \frac{1}{\|b\|} \leq \gamma(a) \leq \frac{\|p\|\|q\|}{\|b\|},$$

where $p = ab$ and $q = ba$.

3. Semi-partial isometries in C^* -algebras

In an arbitrary Banach algebra \mathcal{A} , we say that $a \in \mathcal{A}$ is a *partial isometry* (resp. *semi-partial isometry*) if L_a is a partial isometry (resp. semi-partial isometry). It follows immediately from the definition that every isometry on a Banach algebra is a semi-partial isometry and a is a semi-partial isometry if and only if $\gamma(a) = \|a\| = 1$.

As pointed out in the introduction, the notion of partial isometry in the algebra of operators has been introduced in [13]. The disadvantage of this definition is that, in general an arbitrary isometry $T \in \mathcal{L}(X)$ does not need to be a partial isometry (see [13] and [16] for more details). In [19], the author introduced the so called semi partial isometries, which is a natural generalization of partial isometries from Hilbert spaces to general Banach spaces. Our aim in the sequel is to study the relation between the two notions of partial and semi-partial isometry in the context of C^* -algebras. More precisely, our first main result reads as follows.

Theorem 3.1. *Let \mathcal{A} be a unital C^* -algebra with unit e and let $a \in \mathcal{A}$ with $0 < \|a\| \leq 1$. Then, the following assertions are equivalent:*

- (i) a is a semi-partial isometry;
- (ii) $\gamma(a) = \|a\| = 1$;
- (iii) a has a contractive Moore-Penrose inverse;
- (iv) a has a contractive generalized inverse;
- (v) a^*a is a projection.

For the proof, we need the following lemmas which are crucial for our subsequent analysis.

Lemma 3.2 ([10, Proposition 0.15]). *Let \mathcal{A} be an unital C^* -algebra with identity e and $a \in \mathcal{A}$. Then*

$$\forall \rho \geq \|a\|, \forall b \in \mathcal{A} \quad \|(e - \frac{aa^*}{\rho^2})ab\| \leq (1 - \frac{\gamma^2(a)}{\rho^2})\|ab\|.$$

Lemma 3.3 ([12, Lemma 1.2]). *Let \mathcal{A} be an unital C^* -algebra with identity e and $a \in \mathcal{A}$ having a Moore-Penrose inverse a^\dagger . Then*

$$\forall b \in \mathcal{A}, \quad \text{dist}(a^\dagger b, a^{-1}(0)) = \|a^\dagger b\|.$$

Lemma 3.4 ([10, Theorem 0.27]). *Let \mathcal{A} be an unital C^* -algebra and $a \in \mathcal{A}$. Then, a has a generalized inverse if and only if $\gamma(a) > 0$.*

Proof of Theorem 3.1. (i) \implies (ii). Firstly, notice that Corollary 1.2 of [12] implies that $\gamma(a) \leq \|a\|$. Assume that a is a semi partial isometry. Then for all $x \in \mathcal{A}$ we have $\|ax\| = \text{dist}(x, a^{-1}(0))$. Hence,

$$\gamma(a) := \inf\{\|ax\| : \text{dist}(x, a^{-1}(0)) = 1\} = 1.$$

Now, for $x = e$ we have

$$1 = \gamma(a) \leq \|a\| = \|ae\| = \text{dist}(e, a^{-1}(0)) \leq \|e\| = 1.$$

Thus, $\|a\| = \gamma(a) = 1$.

(ii) \implies (i). Let $a \in \mathcal{A}$ be such that $\gamma(a) = \|a\| = 1$. From the definition of $\gamma(a)$, we observe that $\|at\| \geq \gamma(a)\text{dist}(t, a^{-1}(0))$ for any $t \in \mathcal{A}$. Therefore, $\|at\| \geq \text{dist}(t, a^{-1}(0))$. Now, let $b \in a^{-1}(0)$, then $\|at\| = \|a(t-b)\| \leq \|t-b\|$. Hence, $\|at\| \leq \inf\{\|t-b\|, b \in a^{-1}(0)\} = \text{dist}(t, a^{-1}(0))$, which ends the proof.

(ii) \implies (iii). Assume that $\gamma(a) = \|a\| = 1$. By taking $\rho = 1$ and $b = e$ in Lemma 3.2, we infer that $(e - aa^*)a = 0$. Accordingly, $a = aa^*a$. Keeping in mind that $(aa^*)^* = aa^*$, $(a^*a)^* = a^*a$ and $\|a\| = \|a^*\|$, we infer that a^* is a contractive Moore-Penrose inverse of a .

(iii) \implies (iv). Obvious.

(iv) \implies (v). Let b be a generalized inverse of a with $\|b\| \leq 1$. Invoking Proposition 2.1, we get $\|b\|\gamma(a) \geq 1$. Hence, $\gamma(a) \geq 1$. Keeping in mind that $\gamma(a) \leq \|a\| \leq 1$, we infer that $\gamma(a) = 1$. Using again Lemma 3.2 we obtain $(e - aa^*)a = 0$. This yields $(aa^*)^2 = aa^*aa^* = aa^*$.

(v) \implies (ii). Since a^*a is a projection and $(a^*a)^* = a^*a$, then according to [10, Corollary 0.7] we have $\gamma(a^*a) = \|a^*a\| = 1$. Taking into account the fact that $\gamma(a^*a) = \gamma(a)^2$ ([10, Proposition 0.6]) and $\|a^*a\| = \|a\|^2$, we get $\gamma(a) = \|a\| = 1$. This achieves the proof. \square

Remark 3.5. It is well known that in C^* -algebras, aa^* is a projection if and only if a is a partial isometry. Thus, the above theorem shows that partial isometries and semi-partial isometries coincide in setting of C^* -algebras. Hence, Theorem 3.1 completes and extends [12, Theorem 3.1] and [12, Corollary 3.3]. Finally note that if \mathcal{A} is a C^* -algebra, then $\mathcal{J}(\mathcal{A}) = \mathcal{A}$. So, by the proof of the above theorem, we have that a is a partial isometry if and only if $a = aa^*a$.

A Banach algebra can be endowed with a given involution (we speak about Banach- $*$ algebras). Or as explained in the introduction, by defining a *linear involution* on the set $\mathcal{J}(\mathcal{A})$. So a natural question arises: Does the above Theorem holds for general Banach algebras. We will show in the sequel that this result fail to be true in general Banach algebras. We start our analysis with the following theorem.

Theorem 3.6. *Let $a \in \mathcal{J}(\mathcal{A})$ (or a in a given Banach $*$ -algebra) with $0 < \|a\| \leq 1$. Consider the following statements:*

- (i) a is partial isometry;
- (ii) There exists $b \in \mathcal{A}$ such that $\|b\| \leq 1$ and $aba = a$;
- (iii) For all $x \in \mathcal{A}$, we have $\|a\| = 1$ and $\|ax\| = \text{dist}(x, a^{-1}(0))$;

$$(iv) \quad \gamma(a) = \|a\| = 1.$$

Then, we have: (i) \implies (ii) \implies (iii) \implies (iv).

Proof. We give the proof when $a \in \mathcal{J}(\mathcal{A})$. The proof, for Banach $*$ -algebras is identical. The assertions (i) \implies (ii) and (iii) \implies (iv) are straightforward.

(ii) \implies (iii). For each $z \in a^{-1}(0)$, we have

$$\|ax\| = \|a(x - z)\| \leq \|a\|\|x - z\| \leq \|x - z\|.$$

This implies,

$$(2) \quad \|ax\| \leq \text{dist}(x, a^{-1}(0)).$$

Further, for any $x \in \mathcal{A}$, we write

$$x = bax + (1 - ba)x.$$

Taking into account that $(1 - ba)x \in a^{-1}(0)$, we obtain

$$(3) \quad \text{dist}(x, a^{-1}(0)) = \text{dist}(bax, a^{-1}(0)) \leq \|bax\| \leq \|ax\|.$$

Combining (2) and (3), we get $\|ax\| = \text{dist}(x, a^{-1}(0))$. \square

Now, we provide examples to show that in general the implication (iv) \implies (i) fails to be true.

Examples:

- (1) Let X be the Banach space of entire functions f such that $\|f\| = \sup\{|f(\sigma + it)|e^{-|t|} : \sigma, t \text{ real}\} < +\infty$.

Now, it's clear that the functions $z \mapsto \cos(z)$ and $z \mapsto \sin(z)$ belongs to X . Consider the vector subspace, $X_1 = \text{Vect}(\cos, \sin) \subset X$ and put $\mathcal{A} = \mathcal{L}(X_1)$. It is worth to notice that for $f \in X$, we have $\|f\| = \sup\{|f(\sigma)| : \sigma \text{ real}\}$. Also, by defining the operator $a : X_1 \rightarrow X_1$ by $a(f) = if'$ (where f' denotes the derivative of f). Then, h is well defined, $h \in \mathcal{A}$, a is Hermitian and $\|a\| = 1$. See [9] for proofs here.

Now, by taking into account that $a^* = a$, easy computation shows that $a^3(f) = -if^{(3)}$. From which we infer that $aa^*a \neq a$. Hence, a is not a partial isometry. On the other hand, by considering the element $b \in \mathcal{A}$ which is defined by $b(f)(z) = -i \int_{[0, z]} f(\xi) d\xi$ for every $f \in X_1$. It is easily seen that $\|b\| \leq 1$ and $aba = a$.

- (2) The next example shows that the above theorems are also not true for Banach $*$ -algebras. Consider the algebra \mathcal{A} of all 2×2 matrices over the complex field with a non-standard involution. We define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.$$

This is an involution on \mathcal{A} . Now, if we consider the elements $a = b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Easy computation shows that $aa^*a = -a$ but $aba = a$.

In the next theorem, we give another characterization of partial isometries in the setting of C^* -algebras.

Theorem 3.7. *Let \mathcal{A} be a C^* -algebra and let $a \in \mathcal{A}$ with $0 < \|a\| \leq 1$. Then the following assertions are equivalent:*

- (i) a is a partial isometry;
 - (ii) The Moore-Penrose inverse a^\dagger of a exists and satisfies
- (4)
$$a^\dagger \mathcal{A} \subset \{x \in \mathcal{A}, \|ax\| = \|x\|\}.$$
- (iii) a has a generalized inverse b satisfying
- (5)
$$b\mathcal{A} \subset \{x \in \mathcal{A}, \|ax\| = \|x\|\}.$$

Proof. (i) \implies (ii). Assume that a is a partial isometry. Then a is also a semi-partial isometry by Theorem 3.1. Proceeding as in the proof of Theorem 3.1 and using Lemma 3.2 we infer that a^* is the Moore-Penrose inverse of a . Now, we show that $a^* \mathcal{A} \subset \{x \in \mathcal{A}, \|ax\| = \|x\|\}$. To see this, let $y \in \mathcal{A}$. Since a is a semi partial isometry, then $\text{dist}(a^*y, a^{-1}(0)) = \|aa^*y\|$. Invoking Lemma 3.3, we get $\|a^*y\| = \text{dist}(a^*y, a^{-1}(0)) = \|aa^*y\|$. Thus, $\|a^*y\| = \|aa^*y\|$. This proves the inclusion.

The implication (ii) \implies (iii) is obvious. For the implication (iii) \implies (i), let b be the generalized inverse of a . For any $x \in \mathcal{A}$, by writing $x = bax + (e - ba)x$ we see that $\mathcal{A} = a^{-1}(0) \oplus b\mathcal{A}$. Now, let $x \in \mathcal{A}$. Then, there exist $y \in a^{-1}(0)$ and $z \in \mathcal{A}$ such that $x = y + bz$. Accordingly,

$$\begin{aligned} \|ax\| &= \|abz\| = \|bz\| \\ &\geq \text{dist}(bz, a^{-1}(0)) \\ &\geq \text{dist}(x, a^{-1}(0)). \end{aligned}$$

In consequence, $\gamma(a) \geq 1$. Now, taking into account the fact that $\gamma(a) \leq \|a\| \leq 1$, we infer that $\|a\| = \gamma(a) = 1$ and therefore a is a semi-partial isometry. This completes the proof of the theorem. \square

Remark 3.8. It is worth observing that if a is a partial isometry such that $a^*a = \mathbf{1}$, then the inclusion (4) becomes an equality. Moreover, by [8, Lemma 4.1] we have $a^* \mathcal{A} = a^\dagger \mathcal{A} = \{x \in \mathcal{A}, \|ax\| = \|x\|\} = \mathcal{A}$. The following example shows that in general the inclusion (4) may be strict even if the C^* -algebra is commutative.

Let $\mathcal{A} = C(K)$ be the C^* -algebra of continuous functions on the compact set $K = \{0, 1\}$. Take the element f of \mathcal{A} such that $f(0) = 0$ and $f(1) = 1$. Clearly here $\|f\| = 1$, $ff^*f = f$ and $f = f^*$. So f is a partial isometry. On the other hand, consider the element g of \mathcal{A} such that $g(0) = g(1) = 1$. Straightforward computation shows that $\|g\| = \|fg\| = 1$, but $g \notin f^* \mathcal{A}$ since $g(0) \neq 0$. Hence $f^* \mathcal{A} \subsetneq \{g : \|fg\| = \|g\|\}$.

Finally, note that if K is compact and connected then the two sets in (4) are equal since $\mathbf{1}$ is the only nonzero idempotent in the C^* -algebra $C(K)$.

4. m -semi partial isometries in Banach spaces

A bounded linear operator T on a complex Hilbert space H is called an m -isometry, if it satisfies

$$(6) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0.$$

This class of operators was introduced by Agler in the eighties and then studied by Agler and Stankus in [1, 2, 3]. Notice that (6) is equivalent to

$$(7) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \quad \forall x \in H.$$

The definition (7) does not depend on the structure of Hilbert space. This motivates the following more general definition.

Definition ([4, Definition 1.1]). Let X be a Banach space and let $T \in \mathcal{L}(X)$. Let also $m \geq 1$ and let $p \in [1, +\infty)$. T is called an (m, p) -isometry if, for any $x \in X$,

$$(8) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = 0.$$

Obviously, a $(1, 1)$ -isometry is an isometry and the Agler-Stankus definition of an m -isometry corresponds here to an $(m, 2)$ -isometry, when X is a Hilbert space.

Inspired by [4] and [19] we introduce the following concept.

Definition. Let X be a Banach space and let $T \in \mathcal{L}(X)$. Let also $m \geq 1$ and let $p \in [1, +\infty)$. T is called a semi-partial (m, p) -isometry if, for any $x \in X$,

$$(9) \quad \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = (-1)^{m+1} d^p(x, N(T)).$$

Of course, a semi-partial (m, p) -isometry is an (m, p) -isometry if and only if $N(T) = \{0\}$. Also, a semi-partial isometry [19] corresponds here to a semi-partial $(1, 2)$ -isometry. Moreover, since $d(x, N(T)) = \|x\|$ for all $x \in N(T)^\perp$, then T is a semi-partial (m, p) isometry if and only if

$$(10) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = 0, \quad \forall x \in N(T)^\perp.$$

Example 1. Let X be a Banach space and let T be the bounded linear operator defined on $X \oplus X$ by $T(x \oplus y) = 2^{-\frac{1}{p}} y \oplus 0$. It can be easily verified that $T^2 = 0$, $N(T) = X \oplus \{0\}$, $d(x \oplus y, N(T)) = \|y\|$ and $\|T(x \oplus y)\| = 2^{-\frac{1}{p}} \|y\|$. Accordingly,

$$\begin{aligned} -2\|T(x \oplus y)\|^p + \|T^2(x \oplus y)\|^p &= -\|y\|^p \\ &= -d^p(x \oplus y, N(T)). \end{aligned}$$

Thus, T is a semi-partial $(2, p)$ -isometry. However, T is neither a $(2, p)$ -isometry nor a semi-partial isometry.

Proposition 4.1. *Let T be a semi-partial (m, p) -isometry. Then,*

$$(11) \quad m^{-\frac{1}{p}} (1 + \|T\|^p)^{\frac{1-m}{p}} \leq \gamma(T) \leq \|T\|.$$

Proof. Let $x \in X$. From (9) it follows that

$$\begin{aligned} d^p(x, N(T)) &\leq \sum_{k=1}^m \binom{m}{k} \|T^k x\|^p \\ &\leq \sum_{k=0}^{m-1} \binom{m}{k+1} \|T\|^{pk} \|Tx\|^p \\ &\leq m \sum_{k=0}^{m-1} \binom{m-1}{k} \|T\|^{pk} \|Tx\|^p \\ &\leq m (1 + \|T\|^p)^{m-1} \|Tx\|^p. \end{aligned}$$

Thus,

$$(12) \quad \|Tx\| \geq m^{-\frac{1}{p}} (1 + \|T\|^p)^{\frac{1-m}{p}} d(x, N(T)).$$

Consequently,

$$(13) \quad \gamma(T) \geq m^{-\frac{1}{p}} (1 + \|T\|^p)^{\frac{1-m}{p}}.$$

This completes the proof. \square

Proposition 4.2. *Let X be a Banach space and $P \in \mathcal{L}(X)$ with $P^2 = P$ and $P \neq 0$. Then, P is a semi-partial (m, p) -isometry if and only if $\|P\| = 1$.*

Proof. Let $x \in X$. Since $P^2x = Px$ then $P^kx = Px$ for all $k \geq 1$. Hence,

$$(14) \quad \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \|P^k x\|^p = \|Px\|^p \left(\sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \right) = -(-1)^m \|Px\|^p.$$

Thus, P is a semi-partial (m, p) -isometry if and only if $\|Px\| = d(x, N(P))$ for all $x \in X$. Applying [19, Proposition 2.1] we get the desired result. \square

Proposition 4.3. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a semi-partial (m, p) -isometry. If $V \in \mathcal{L}(X)$ is an isometry with $TV = VT$, then VT is a semi-partial (m, p) -isometry.*

Proof. Since V is an isometry then for all $x \in X$ we have

$$(15) \quad \|VTx\| = \|Tx\|.$$

Thus, $N(VT) = N(T)$. Now, keeping in mind that $VT = TV$ the use of (15) gives:

$$(16) \quad \|(VT)^k x\| = \|T^k V^k x\| = \|T^k x\|$$

for all $x \in X$ and for all $k \geq 1$.

Taking into account the fact that T is a semi-partial (m, p) -isometry we get

$$\begin{aligned} \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \|(VT)^k x\|^p &= \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^p \\ &= (-1)^{m+1} d^p(x, N(T)) \\ &= (-1)^{m+1} d^p(x, N(VT)). \end{aligned}$$

Consequently, VT is a semi-partial (m, p) -isometry. \square

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