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ON THE STRUCTURE OF FACTOR LIE ALGEBRAS

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Abstract. The Lie algebra analogue of Schur's result which is proved by Moneyhun in 1994, states that if L is a Lie algebra such that $\dim L/Z(L) = n$, then $\dim L_{(2)} = \frac{1}{2}n(n-1) - s$ for some non-negative integer s. In the present paper, we determine the structure of central factor (for $s = 0$) and the factor Lie algebra $L/Z_2(L)$ (for all $s \geq 0$) of a finite dimensional nilpotent Lie algebra L, with n-dimensional central factor. Furthermore, by using the concept of n-isoclinism, we discuss an upper bound for the dimension of $L/Z_n(L)$ in terms of $\dim L_{(n+1)}$, when the factor Lie algebra $L/Z_n(L)$ is finitely generated and $n \geq 1$.

1. Introduction

In 1904, Schur [14] proved that if the center of a group G has finite index, then the derived subgroup G' is also finite. Also, Wiegold [16] showed that if the order of central factor group of G is p^n, then G' is a p-group of order at most $p^{\frac{1}{2}n(n-1)}$. The structure of a group and its central factor, with regards to the order of derived subgroup, has been already studied by many authors (see [9, 12, 16]). Berkovich [4] studied the structure of $G/Z(G)$, where G is a p-group such that $|G/Z(G)| = p^n$ and $|G'| = p^{\frac{1}{2}n(n-1)}$. Later in 2004, Kim [10] characterized the structure of a p-group G such that $|G/Z(G)| = p^n$ and $|G'| = p^{\frac{1}{2}n(n-1)-1}$. Also, Hekster [8] showed that if G is a finitely generated group and $n \geq 1$, then $G/Z_n(G)$ is finite if and only if $\gamma_{n+1}(G)$ is finite, where $Z_n(G)$ and $\gamma_{n+1}(G)$ are n-th and $(n+1)$-st terms of the upper and lower central series of G, respectively.

Throughout of this paper, all Lie algebras are over a fixed field $\mathbb{F}$ and $Z_n(L)$ denotes the n-th term of the upper central series of a Lie algebra L defined inductively by $Z_1(L) = Z(L)$ and $Z_n(L)/Z_{n-1}(L) = Z(L/Z_{n-1}(L))$ for $n \geq 2$. Also, $L_{(n)}$ denotes the n-th term of the lower central series of L defined by $L_{(1)} = L$ and $L_{(n)} = [L_{(n-1)}, L]$ for $n \geq 2$, where $[\ ,\ ]$ denotes the Lie bracket. Moreover, $A(n)$ denotes the n-dimensional abelian Lie algebra and the *Heisenberg Lie algebra* $H(m)$ is the $2m + 1$ dimensional real Lie algebra

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with the basis $\{a_1, \ldots, a_m, b_1, \ldots, b_m, c\}$ and the Lie brackets defined by

$$[a_i, a_j] = [b_i, b_j] = [a_i, c] = [b_i, c] = [c, c] = 0 \text{ and } [a_i, b_j] = c\delta_{ij},$$

where δ_{ij} is the Kronecker delta (see [7] for more details).

Nilpotent Lie algebras have played an important role in mathematics over the last 30 years in the classification theory of Lie algebras. Furthermore, the characterization of $L/Z_n(L)$ ($n \in \mathbb{N}$) has always been one of the most popular problems among Lie algebra experts. The Lie algebra analog of Schur's was proved by Moneyhun [11]: if L is a Lie algebra such that $\dim L/Z(L) = n$, then $\dim L_{(2)} = \frac{1}{2}n(n-1) - s$ for some non-negative integer s. Moreover, Batten et al. [3] proved that if L is a finite dimensional nilpotent Lie algebra and $\dim L/Z(L) = n$, then $\dim L_{(2)} = \frac{1}{2}n(n-1) - s$ and $\dim(L/Z(L))_{(2)} \leq 1 + s$ for some non-negative integer s. In the present paper, we determine the structure of $L/Z_2(L)$, where L is a finite dimensional nilpotent Lie algebra such that $\dim L/Z(L) = n$ and $\dim L_{(2)} = \frac{1}{2}n(n-1) - s$ for all non-negative integers s. In particular, we characterize the structure of central factor of L, when $s = 0$. We show that $L/Z_2(L)$ must be a nilpotent Lie algebra of dimension not equal to 1. Nilpotent Lie algebras of dimension less than 6, over a field of any characteristic, are classified in [5, 6]. Also, the nilpotent Lie algebras of dimension greater than 7, under the special conditions are characterized in [1, 15].

Moreover, Salemkar and Mirzaei [13] proved a Lie algebra version of the Hekster's result and showed that if L is a finitely generated Lie algebra, then $L/Z_n(L)$ is finite dimensional if and only if $L_{(n+1)}$ is finite dimensional. In the last section, we give an upper bound for the dimension of $L/Z_n(L)$, when $L_{(n+1)}$ is finite dimensional and $L/Z_n(L)$ is finitely generated. We show that

$$\dim L/Z_n(L) \leq d^n \cdot \dim L_{(n+1)},$$

where d is the minimal number of generators of $L/Z_n(L)$ and $n \geq 1$. Note that the first author and Saeedi proved the above result for $n = 1$ in [2]. Here, we use the idea of n-isoclinism discussed in [13], which gives us a different method from the technique applied in [2].

2. Preliminary results

In this section, we discuss some preliminary results, which will be used in the proof of the main theorems.

Lemma 2.1 ([3, 11]). *Let L be a Lie algebra such that $\dim L/Z(L) = n$. Then $\dim L_{(2)} \leq \frac{1}{2}n(n-1) - s$ for some non-negative integer s. Moreover, if L is a finite dimensional nilpotent Lie algebra, then $\dim(L/Z(L))_{(2)} \leq 1 + s$.*

Definition 2.2. A Lie algebra L is called *capable*, if there exists a Lie algebra H such that $L \cong H/Z(H)$.

The following result for capable Lie algebras is proved by the first author and Saeedi in [2], which has an important role in the proof of Theorems 3.1 and 3.2.

Theorem 2.3 ([2]). *Let L be a capable Lie algebra such that* $\dim L_{(2)} = m$. *Then* $\dim L/Z(L) \leq 2m^2$.

All nilpotent Lie algebras of dimension at most 5 are classified over an arbitrary field $\mathbb{F}$.

Theorem 2.4 ([6]). *Let L be a finite dimensional nilpotent Lie algebra. Then*

(a) *L is 1-dimensional if and only if $L \cong A(1)$.*
(b) *L is 2-dimensional if and only if $L \cong A(2)$.*
(c) *L is 3-dimensional if and only if $L \cong A(3)$ or $H(1)$.*
(d) *L is 4-dimensional if and only if $L \cong A(4)$, $H(1) \oplus A(1)$ or*

$$\langle x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle.$$

(e) *L is 5-dimensional if and only if L is isomorphic to one of the following Lie algebras:*
  (1) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle$.
  (2) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle$.
  (3) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle$.
  (4) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_5, [x_3, x_4] = x_5 \rangle = H(2)$.
  (5) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle$.
  (6) $L \cong \langle x_1, x_2, \ldots, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$.
  (7) $L \cong A(5)$ *or* $H(1) \oplus A(2)$.

Also in [5], all of the nilpotent Lie algebras of dimension 6 are characterized.

3. On the central factor and $L/Z_2(L)$ of a nilpotent Lie algebra

Now, we are ready to prove our main theorems.

Theorem 3.1. *Let L be a finite dimensional nilpotent Lie algebra such that* $\dim L/Z(L) = n$ *and* $\dim L_{(2)} = \frac{1}{2}n(n-1)$. *Then one of the following holds:*
(i) *$L/Z(L)$ is abelian.*
(ii) *$L/Z(L) \cong H(1) \oplus A(n-3)$.*

Proof. Define $K := (L/Z(L))_{(2)}$. Then Lemma 2.1 implies that $\dim K = 0, 1$. If $\dim K = 0$, then $L/Z(L)$ is abelian. If $\dim K = 1$, then $K \subseteq Z(L/Z(L))$ since $L/Z(L)$ is nilpotent (as quotients of nilpotent Lie algebras are nilpotent). Therefore, it is straightforward to see that

$$L/Z(L) \cong H(m) \oplus A(n-2m-1) \tag{1}$$

for some m. Since $L/Z(L)$ is a capable Lie algebra, by Theorem 2.3, we have $D := \dim(L/Z(L))/Z(L/Z(L)) \leq 2$. Thus

$$\dim Z(L/Z(L)) + D = \dim L/Z(L) = n,$$

and note that $\dim Z(L/Z(L)) = n - 2m$, from Equation (1). If $D < 2$, then this implies that $m < 1$, which contradicts $\dim K = 1$. Hence we must have $D = 2$ and $m = 1$. □

Theorem 3.2. *Let L be a finite dimensional nilpotent Lie algebra such that $\dim L/Z(L) = n$ and $\dim L_{(2)} = \frac{1}{2}n(n-1) - s$ for some integer $s \geq 1$. Then one of the following holds*:

(a) *$L/Z(L)$ is an n-dimensional abelian Lie algebra.*
(b) *$L/Z(L) \cong H(1) \oplus A(n-3)$.*
(c) *$L/Z_2(L)$ is an i-dimensional nilpotent Lie algebra, where $2 \leq i \leq 2$ $(1+s)^2$, and $\dim Z_2(L)/Z(L) = n - i$.*

Proof. By Lemma 2.1, we have $\dim(L/Z(L))_{(2)} \leq 1+s$. If $\dim(L/Z(L))_{(2)} \leq 1$, then $L/Z(L)$ is abelian or by Theorem 3.1, $L/Z(L)$ is isomorphic to $H(1) \oplus A(n-3)$. Now fix $m := \dim(L/Z(L))_{(2)}$. Then by Theorem 2.3

$$\dim \frac{L}{Z_2(L)} = \dim \frac{L/Z(L)}{Z_2(L)/Z(L)} = \dim \frac{L/Z(L)}{Z(L/Z(L))} \leq 2m^2. \tag{2}$$

Now from Equation (2), we get

$$\dim L/Z(L) - \dim Z_2(L)/Z(L) = \dim L/Z_2(L) = i,$$

where $2 \leq i \leq 2m^2$. Thus, we have $\dim Z_2(L)/Z(L) = n - i$. □

4. An upper bound for the dimension of $L/Z_n(L)$

First, we present the notion of n-isoclinism, which is the key part of our method in the proof of last main theorem. The n-isoclinism is the isoclinism with respect to the variety of all nilpotent Lie algebras L for which $L = Z_n(L)$.

Definition 4.1. Let L and H be Lie algebras. Then an *n-isoclinism* ($n \geq 1$) between L and H is a pair of isomorphisms (α, β) with $\alpha : L/Z_n(L) \longrightarrow H/Z_n(H)$ and $\beta : L_{(n+1)} \longrightarrow H_{(n+1)}$ such that the following diagram commutes:

$$\begin{array}{ccc} L/Z_n(L) \oplus \cdots \oplus L/Z_n(L) & \longrightarrow & L_{(n+1)} \\ {\scriptstyle \alpha^{n+1}} \downarrow & & \downarrow {\scriptstyle \beta} \\ H/Z_n(H) \oplus \cdots \oplus H/Z_n(H) & \longrightarrow & H_{(n+1)} \end{array}$$

where horizontal maps are defined by

$$(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}) \longmapsto [\cdots [x_1, x_2], \ldots, x_{n+1}]$$

such that $\bar{x}_i = x_i + Z_n(L)$ and $\bar{x}_i = x_i + Z_n(H)$ in the top and bottom horizontal maps, respectively (see [8, 13] for more details). If there exists such an n-isoclinism, we say that L is *n-isoclinic* to H and write $L \sim_n H$.

Salemkar and Mirzaei investigated the n-isoclinism of Lie algebras in [13].

Lemma 4.2 ([13]). *If L is a Lie algebra with a subalgebra H such that $L = H + Z_n(L)$, then $L \sim_n H$. Conversely, if the factor Lie algebra $L/Z_n(L)$ is finite dimensional and $L \sim_n H$, then $L = H + Z_n(L)$.*

Proposition 4.3 ([13]). *Let L be a finitely generated Lie algebra. Then the following statements are equivalent.*
(i) *$L/Z_n(L)$ is finite dimensional.*
(ii) *$L_{(n+1)}$ is finite dimensional.*
(iii) *$(L/Z(L))_{(n)}$ is finite dimensional.*

Now, we prove the last theorem. Recall that in [2], the following result is proved for $n = 1$.

Theorem 4.4. *Let L be a Lie algebra such that $L_{(n+1)}$ is finite dimensional and $L/Z_n(L)$ is finitely generated. Then*

$$\dim L/Z_n(L) \leq d^n \cdot \dim L_{(n+1)},$$

where d is the minimal number of generators of $L/Z_n(L)$.

Proof. We proceed inductively. Suppose that $n = 1$. Fix $x_1, \ldots, x_d \in L$ such that $\{x_1+Z(L), \ldots, x_d+Z(L)\}$ generates $L/Z(L)$. Let $H = \langle x_1, \ldots, x_d\rangle$, and so $L = H+Z(L)$. By Lemma 4.2, we have $L \sim_1 H$, and hence $L/Z(L) \cong H/Z(H)$ and $L_{(2)} \cong H_{(2)}$. Therefore, we may replace L by H. Let $y \in H$ and define

$$f : \frac{H}{\cap_{i=1}^{d} C_H(x_i)} \longrightarrow \underbrace{H_{(2)} \oplus H_{(2)} \oplus \cdots \oplus H_{(2)}}_{d\ times}$$
$$y + \cap_{i=1}^{d} C_H(x_i) \longmapsto ([y, x_1], [y, x_2], \ldots, [y, x_d]),$$

where $C_H(x_i)$ is the centralizer of x_i in H. The definition of $\cap_{i=1}^{d} C_H(x_i)$ implies that f is well-defined and one may easily check that it is an injective linear transformation. Thus

$$\dim L/Z(L) = \dim H/Z(H) = \dim H/\cap_{i=1}^{d} C_H(x_i) \leq d \cdot \dim L_{(2)}.$$

Now, assume that the claim holds for $(n-1)$, i.e., $\dim L/Z_{n-1}(L) \leq d^{n-1} \cdot \dim L_{(n)}$. Fix $x_1, \ldots, x_d \in L$ such that $\{x_1 + Z_n(L), \ldots, x_d + Z_n(L)\}$ generates $L/Z_n(L)$. Let $H = \langle x_1, \ldots, x_d\rangle$, and so $L = H + Z_n(L)$. Then Lemma 4.2 implies that H is n-isoclinic to L. Trivially, $H/Z(H)$ is finitely generated. It follows from Proposition 4.3 and finiteness of $\dim H_{(n+1)}$ that $\dim(H/Z(H))_{(n)}$ is finite. Therefore, $H/Z(H)$ satisfies the induction hypothesis. Hence

$$\dim H/Z_n(H) = \dim \frac{H/Z(H)}{Z_{n-1}(H/Z(H))} \leq d^{n-1} \cdot \dim(H/Z(H))_{(n)}.$$

On the other hand, by the second isomorphism theorem, we have

$$(H/Z(H))_{(n)} \cong \frac{H_{(n)}}{H_{(n)} \cap Z(H)} \cong \frac{H_{(n)}}{\cap_{i=1}^{d} C_{H_{(n)}}(x_i)}.$$

The above isomorphisms and defining an analogous map to f, imply that

$$\begin{aligned}\dim(H/Z(H))_{(n)} &= \dim\frac{H_{(n)}}{\cap_{i=1}^{d} C_{H_{(n)}}(x_i)} \\ &\leq d \cdot \dim H_{(n+1)},\end{aligned}$$

and since $L \sim_n H$, we have

$$\begin{aligned}\dim L/Z_n(L) = \dim H/Z_n(H) &\leq (d \cdot \dim H_{(n+1)}) \cdot d^{n-1} \\ &= d^n \cdot \dim H_{(n+1)} \\ &= d^n \cdot \dim L_{(n+1)},\end{aligned}$$

which completes the proof. □

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