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q-FREQUENT HYPERCYCLICITY IN AN ALGEBRA OF OPERATORS

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Abstract. We study a notion of q-frequent hypercyclicity of linear maps between the Banach algebras consisting of operators on a separable infinite dimensional Banach space. We derive a sufficient condition for a linear map to be q-frequently hypercyclic in the strong operator topology. Some properties are investigated regarding q-frequently hypercyclic subspaces as shown in [5], [6] and [7]. Finally, we study q-frequent hypercyclicity of tensor products and direct sums of operators.

1. Introduction

Let X be an F-space and let $\mathcal{L}(X)$ be the space of all continuous linear operators on X. An operator $T \in \mathcal{L}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that the T-orbit $O(T, x) = \{T^n x \mid n = 0, 1, 2, \ldots\}$ is dense in X. Such a vector $x \in X$ is called a *hypercyclic vector* for T. In [6], Chan has defined a hypercyclicity of a linear mapping on $\mathcal{B}(\mathcal{H})$ in the strong operator topology, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space $\mathcal{H}$. He provided a sufficient condition for a bounded linear operator on a Hilbert space to have a closed infinite dimensional subspace of hypercyclic vectors, which is called a *hypercyclic subspace*. In [5], the notion of a hypercyclic subspace is extended to the case of frequently hypercyclic subspace. Subsequently, we use similar methods appeared in [6] to investigate the existence of frequently hypercyclic subspaces.

The q-frequent hypercyclicity in a locally convex space was first introduced in [11] and also can be seen as a particular type of (m_k)-hypercyclicity appeared in [3]. In terms of frequency, one can see that the q-frequent hypercyclicity lies between hypercyclicity and frequent hypercyclicity. A considerable amount of properties on the q-frequent hypercyclicity resembles the ones occurred in

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frequent hypercyclicity. So, it is of interest to consider some questions raised in [5], [6], and [7] for the case of q-frequent hypercyclicity. In this paper, we study the notion of q-frequent hypercyclicity in the Banach algebra of all bounded operators on a separable infinite dimensional Banach space under the strong operator topology, which has been studied in [12] regarding left multiplication and conjugate operators. We find a sufficient condition for q-frequent hypercyclicity. Using this result, we study a q-frequently hypercyclic subspace problem as done in [5] and [6].

2. q-frequently hypercyclic operators

When $x \in X$ is a hypercyclic vector for an operator $T \in \mathcal{L}(X)$, it is known that the return set $\mathbf{N}(x, U) = \{n \in \mathbb{N} \mid T^n x \in U\}$ is non-empty for any non-empty open set U in X. A sufficient condition in order for an operator to be hypercyclic was initially found by [14] and was strengthened in [8]. The frequent hypercyclicity corresponds to the largeness of return sets $\mathbf{N}(x, U)$, in the sense of how frequently the T-orbit intersects with an open set U. Let us first recall that the *lower density* of a subset A in $\mathbb{N}$ given by

$$\underline{\mathrm{dens}}(A) = \liminf_{N \to \infty} \frac{|A \cap [1, N]|}{N},$$

where $|A \cap [1, N]|$ denotes the cardinality of the set $A \cap [1, N]$.

Definition 2.1. Let X be a topological vector space. An operator $T \in \mathcal{L}(X)$ is said to be *frequently hypercyclic* if there is a vector $x \in X$ such that for every non-empty open set U, $\mathbf{N}(x, U)$ has positive lower density. Such a vector x is called *frequently hypercyclic* for T.

Let A be an infinite subset of $\mathbb{N}$ and let $(n_k)_{k \in \mathbb{N}}$ be an enumeration of A. We may assume that the sequence $(n_k)_{k \in \mathbb{N}}$ is increasing. Then it is easy to see that A has positive lower density if and only if there is a constant $C > 0$ such that

$$n_k \leq Ck \quad \text{for all } k \geq 1.$$

Thus, a vector $x \in X$ is frequently hypercyclic for T if and only if for each non-empty open subset U of X, there are a strictly increasing sequence (n_k) and some constant $C > 0$ such that

$$T^{n_k} x \in U \quad \text{and} \quad n_k \leq Ck$$

for all $k \in \mathbb{N}$. This implies that $n_k = O(k)$ as $k \to \infty$. For a separable F-space X whose topology is defined by an F-norm $\| \cdot \|$, a vector $x \in X$ is *frequently hypercyclic* for T if there exists a sequence $(R_l)_{l \geq 1}$ of subsets of $\mathbb{N}$ with a positive lower density such that, for any $n \in R_l$ and $\epsilon > 0$

$$\|T^n x - x_l\| < \epsilon,$$

where $(x_l)_{l\geq 1}$ is a countable dense subset of X. In fact, such a sequence (R_l) of sets with a separation property exists, and is provided in the following lemma. The proof can be found in [2, 10].

Lemma 2.2. *Let (N_l) be a strictly increasing sequence of positive integers. Then there exists a sequence $(R_l)_{l\geq 1}$ of pairwise disjoint subsets of $\mathbb{N}$ such that*

(a) *each subset R_l has positive lower density;*
(b) *for any $l \geq 1$ and $n \in R_l$, $n \geq N_l$;*
(c) *$|n - m| \geq N_k + N_l$ for $(n, m) \in R_k \times R_l$ with $n \neq m$.*

Analogous to the hypercyclicity criterion, one has the useful criterion for frequently hypercyclic operators which was proved in [1] and [4].

Theorem 2.3 (Frequent Hypercyclicity Criterion)**.** *Let X be a separable F-space and let $T \in \mathcal{L}(X)$. If there exist a dense subset X_0 of X and a map $A : X_0 \to X_0$ such that, for any $x \in X_0$*

(i) *$\sum_{n=0}^{\infty} T^n x$ and $\sum_{n=0}^{\infty} A^n x$ are unconditionally convergent,*
(ii) *$TA = I$, the identity on X_0,*

then T is frequently hypercyclic.

The q-frequent hypercyclicity introduced in [11] is to control the frequency of the T-orbit intersecting with each open set.

Definition 2.4. Let $q \in \mathbb{N}$ and let X be a separable F-space. A continuous linear operator $T \in \mathcal{L}(X)$ is said to be *q-frequently hypercyclic* if there is a vector $x \in X$ such that for each non-empty open set U in X, the return set $\mathbf{N}(x, U)$ has positive q-lower density:

$$q\text{-}\underline{\text{dens}}(\mathbf{N}(x,U)) := \liminf_{k\to\infty} \frac{|\mathbf{N}(x,U) \cap [0, k^q]|}{k} > 0. \tag{1}$$

Such a vector x is said to be *q-frequently hypercyclic* for T.

In other words, $x \in X$ is a q-frequently hypercyclic vector for T if and only if for each non-empty open subset U of X, there are a strictly increasing sequence (n_k) and some constant $C > 0$ such that

$$T^{n_k} x \in U \quad \text{and} \quad n_k \leq Ck^q$$

for all $k \in \mathbb{N}$. We see that q-frequent hypercyclicity may be a generalization of frequent hypercyclicity. For example, the frequently hypercyclic operators are the 1-frequently hypercyclic operators. Also, note that the q-frequent hypercyclicity may be seen as a particular type of (m_k)-hypercyclicity defined in [3]. Let $\|\cdot\|$ be an F-norm defining the topology of X and let $(x_l)_{l\in\mathbb{N}}$ be a countable dense subset of X. Then a vector $x \in X$ is q-frequently hypercyclic for T if there exists a sequence $(R_l)_{l\geq 1}$ of subsets in $\mathbb{N}$ of positive lower density such that, for any $k \in R_l$ and $\epsilon > 0$

$$\|T^{k^q} x - x_l\| < \epsilon.$$

Let us recall that a series $\sum x_n$ in a normed space is said to be *unconditionally convergent* if for every permutation σ of $\mathbb{N}$, $\sum x_{\sigma(n)}$ is convergent, see [13] for details. As given in [4], a collection of series $\sum_{n=1}^{\infty} x_{n,k}$, $k \in J$, is called *unconditionally convergent, uniformly in* $k \in J$ if for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for any finite set $F \subset [N, \infty) \cap \mathbb{N}$ and every $k \in J$ we have

$$\left\| \sum_{n \in F} x_{n,k} \right\| < \varepsilon.$$

Analogous to the frequent hypercyclicity criterion, we have a criterion for q-frequently hypercyclic operators and its proof is given in [11] and [12].

Theorem 2.5 (q-Frequent Hypercyclicity Criterion)**.** *Let X be a separable F-space, $T \in \mathcal{L}(X)$, and let $q \in \mathbb{N}$. Suppose that there is a dense subset X_0 of X and a map $A : X_0 \to X_0$ such that for each $x \in X_0$,*

(i) $\sum\limits_{n=0}^{k} T^{k^q-(k-n)^q} x$ *converges unconditionally, uniformly in* $k \geq 0$,

(ii) $\sum\limits_{n=0}^{\infty} A^{(n+k)^q-k^q} x$ *converges unconditionally, uniformly in* $k \geq 0$,

(iii) $T^{n^q} A^{n^q} x = x$.

Then T is q-frequently hypercyclic.

3. SOT-q-frequent hypercyclicity criterion

Throughout this section, X and $\mathcal{B}(X)$ denote a separable infinite dimensional Banach space and the algebra of all bounded linear operators on X, respectively, unless specified otherwise. Among many other topologies on $\mathcal{B}(X)$, we use only two of them. First, we consider the operator norm topology on $\mathcal{B}(X)$. But, in general $\mathcal{B}(X)$ is not separable under the operator norm topology, so that we will also consider the strong operator topology. We add the prefix "SOT" in front of the corresponding term when we use the strong operator topology. For example, a dense subset in the strong operator topology is denoted by a SOT-dense subset. Inspired by the definitions given in [7], we introduce a notion of SOT-q-frequent hypercyclicity.

Definition 3.1. A bounded linear mapping $\Phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is said to be *SOT-q-frequently hypercyclic* if there is an operator $T \in \mathcal{B}(X)$ such that for each non-empty SOT-open subset $\mathfrak{U}$ of $\mathcal{B}(X)$, the return set

$$\mathcal{N}(T, \mathfrak{U}) = \{n \in \mathbb{N} \mid \Phi^n(T) \in \mathfrak{U}\}$$

has the property (1) in Definition 2.4.

Analogous to the SOT-frequent hypercyclicity criterion given in [5], we define the corresponding criterion.

Definition 3.2. A linear map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies the *SOT-q-frequent hypercyclicity criterion* if there exist a countable SOT-dense subset $\mathcal{B}_0$ of $\mathcal{B}(X)$ and a map $\Psi : \mathcal{B}_0 \to \mathcal{B}_0$ such that for all $A \in \mathcal{B}_0$

(i) $\sum_{n=0}^{k} \Phi^{k^q-(k-n)^q}(A)$ converges unconditionally, uniformly in $k \geq 0$, in the operator norm,

(ii) $\sum_{n=0}^{\infty} \Psi^{(n+k)^q-k^q}(A)$ converges unconditionally, uniformly in $k \geq 0$, in the operator norm,

(iii) $\Phi^{n^q} \circ \Psi^{n^q}(A) = A$.

We follow the same steps as in [4], [10] and [12] to prove the SOT-q-frequent hypercyclicity criterion.

Theorem 3.3. *Let $q \in \mathbb{N}$. If a linear map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies the SOT-q-frequent hypercyclicity criterion, then Φ is SOT-q-frequently hypercyclic.*

Proof. We enumerate the SOT-dense set $\mathcal{B}_0$ in Definition 3.2 as $\{A_1, A_2, \ldots\}$. It follows from (ii) in Definition 3.2 that for each $n \geq 1$, there is $N_n \in \mathbb{N}$ such that for all $1 \leq j \leq n$ and any finite set $F \subseteq [N_n, \infty) \cap \mathbb{N}$

$$\left\| \sum_{r \in F} \Psi^{(r+k)^q-k^q}(A_j) \right\| \leq \frac{1}{n2^n} \tag{2}$$

uniformly in $k \geq 0$. By (i) in Definition 3.2, for any $F \subset [N_n, \infty) \cap \{1, \ldots, k\}$, we have

$$\left\| \sum_{r \in F} \Phi^{k^q-(k-r)^q}(A_j) \right\| \leq \frac{1}{n2^n}. \tag{3}$$

We may assume that $(N_n)_{n \geq 1}$ is strictly increasing. By Lemma 2.2, there exists a sequence (S_n) of disjoint subsets in $\mathbb{N}$ such that

(a) each set S_n has positive lower density,
(b) for each $n \geq 1$, $\ell \in S_n$ implies $\ell \geq N_n$,
(c) for any $(\ell, \ell') \in S_n \times S_{n'}$, $|\ell - \ell'| \geq N_n + N_{n'}$.

We define an operator A in $\mathcal{B}(X)$ by

$$A = \sum_{j=1}^{\infty} \sum_{k \in S_j} \Psi^{k^q}(A_j). \tag{4}$$

By using the inequality (2) for the case of $k = 0$, one can show that the series in (4) is unconditionally convergent and so A is well-defined. Indeed, fix $n \in \mathbb{N}$. For any finite set $F \subset \mathbb{N}$,

$$\sum_{j=1}^{\infty} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) = \sum_{j=1}^{n} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) + \sum_{j=n+1}^{\infty} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j). \tag{5}$$

It follows from (2) that for a finite subset $F \subset [N_n, \infty) \cap \mathbb{N}$ and $1 \leq j \leq n$, we have that

$$\left\| \sum_{j=1}^{n} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) \right\| \leq \sum_{j=1}^{n} \left\| \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) \right\| \leq \sum_{j=1}^{n} \frac{1}{n2^n} = \frac{1}{2^n}.$$

If $k \in S_j$, then $k \geq N_j$ and $S_j \cap F \subset \{N_j, N_j+1, N_j+2, \ldots\}$. Thus, for $j \geq n+1$, we have

$$\left\| \sum_{j=n+1}^{\infty} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) \right\| \leq \sum_{j=n+1}^{\infty} \left\| \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) \right\| \leq \sum_{j=n+1}^{\infty} \frac{1}{j2^j} \leq \frac{1}{2^n}.$$

Since $\| \sum_{j=1}^{\infty} \sum_{k \in S_j \cap F} \Psi^{k^q}(A_j) \| \leq 2/2^n$ and n was arbitrary, the series is unconditionally convergent and A is well-defined.

To show that A is SOT-q-frequently hypercyclic for Φ, we will prove that for any $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\|\Phi^{k^q}(A) - A_n\| < \epsilon$. Let $n \geq 1$ be fixed. For any $k \in S_n$, we have that

$$\begin{aligned}
\Phi^{k^q}(A) &= \Phi^{k^q}\left(\sum_{j=1}^{\infty} \sum_{i \in S_j} \Psi^{i^q}(A_j) \right) = \sum_{j=1}^{\infty} \sum_{i \in S_j} \Phi^{k^q} \circ \Psi^{i^q}(A_j) \\
&= \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k > i}} \Phi^{k^q} \circ \Psi^{i^q}(A_j) + \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k < i}} \Phi^{k^q} \circ \Psi^{i^q}(A_j) + \Phi^{k^q} \circ \Psi^{k^q}(A_n),
\end{aligned}$$

so that

$$\begin{aligned}
\Phi^{k^q}(A) - A_n &= \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k > i}} \Phi^{k^q} \circ \Psi^{i^q}(A_j) + \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k < i}} \Phi^{k^q} \circ \Psi^{i^q}(A_j) \\
&= \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k < i}} \Psi^{i^q - k^q}(A_j) + \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k > i}} \Phi^{k^q - i^q}(A_j),
\end{aligned}$$

where the last equality follows from the condition (iii). For the first sum in the above, we divide the sum into two parts as in (5) and by using the inequality (2), we get

$$\left\| \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k < i}} \Psi^{i^q - k^q}(A_j) \right\| \leq \frac{2}{2^n}.$$

Similarly, for $(i, k) \in S_j \times S_n$ with $i < k$, we also have that

$$\left\| \sum_{j=1}^{\infty} \sum_{\substack{i \in S_j \\ k > i}} \Phi^{k^q - i^q}(A_j) \right\| \leq \frac{2}{2^n}.$$

Since $\mathcal{B}_0$ is SOT-dense in $\mathcal{B}(X)$ and the strong operator topology is weaker than the norm topology, A is q-frequently hypercyclic for Φ in SOT. Therefore, Φ is SOT-q-frequently hypercyclic. □

For $T \in \mathcal{B}(X)$, we consider the left multiplication operator $\Phi_T : \mathcal{B}(X) \to \mathcal{B}(X)$ defined by $\Phi_T(A) = TA$ for all $A \in \mathcal{B}(X)$. In general, $\mathcal{B}(X)$ is not separable under the operator norm topology, so that we consider orbits of Φ_T under the strong operator topology. In fact, the q-frequent hypercyclicity of left multiplication and conjugate operators in $\mathcal{B}(X)$ as well as in the p-th Schatten von-Neumann class $S_p(\mathcal{H})$ of operators have been studied in great details in [12].

Proposition 3.4. *If $T \in \mathcal{B}(X)$ satisfies the q-frequent hypercyclicity criterion, then the left multiplication operator Φ_T by T also satisfies the SOT-q-frequent hypercyclicity criterion.*

Proof. Let X_0 be a dense subset of X and $A : X_0 \to X_0$ be a map given in the q-Frequent Hypercyclicity Criterion. By the same analogy given in [9], one can choose a linearly independent and countable subset of X_0 which spans the linear space generated by X_0.

Extending A linearly to span(X_0), we may assume that X_0 is a subspace and A is a linear map. We denote $\mathcal{F}(X, X_0)$ the space of finite rank operators on X whose ranges lie in X_0. For any $B \in \mathcal{F}(X, X_0)$, the map $AB : X \to X_0$ is also of a finite rank.

Let $\Phi_A : \mathcal{F}(X, X_0) \to \mathcal{F}(X, X_0)$ be a map given by $\Phi_A(B) = AB$. By Theorem 2 of [7], $\mathcal{F}(X, X_0)$ contains a countable SOT-dense subset (B_k) in $\mathcal{B}(X)$. Note that the set $\mathcal{B}_0 = \{\Phi_A^n(B_k) \mid n, k \geq 1\}$ is a countable SOT-dense subset of operators in $\mathcal{F}(X, X_0)$. Since $T^{n^q}A^{n^q} = I$ on X_0, we have that $\Phi_T^{n^q} \circ \Phi_A^{n^q}(B) = B$ for all $B \in \mathcal{B}_0$, which shows that the condition (iii) in Definition 3.2 holds.

To prove whether the conditions (i) and (ii) in Definition 3.2 hold, it suffices to consider operators of form $B = f(\cdot)x$ with $f \in X^*$ and $x \in X_0$. For such an operator B, we have that

$$\sum_{n=0}^{k} (\Phi_T)^{k^q-(k-n)^q}(B) = \sum_{n=0}^{k} f(\cdot)T^{k^q-(k-n)^q}(x) = f(\cdot)\sum_{n=0}^{k} T^{k^q-(k-n)^q}(x),$$

which converges unconditionally, uniformly in $k \geq 0$ in $\mathcal{B}(X)$ under the operator norm topology. Furthermore, we have the equalities

$$\sum_{n=0}^{\infty} (\Phi_A)^{n^q}(B) = \sum_{n=0}^{\infty} f(\cdot)A^{n^q}(x) = f(\cdot)\sum_{n=0}^{\infty} A^{n^q}(x),$$

and

$$\sum_{n=0}^{\infty} (\Phi_A)^{(n+k)^q-k^q}(B) = \sum_{n=0}^{\infty} f(\cdot)A^{(n+k)^q-k^q}(x) = f(\cdot)\sum_{n=0}^{\infty} A^{(n+k)^q-k^q}(x),$$

which also converge unconditionally, uniformly in $k \geq 0$ in $\mathcal{B}(X)$ under the operator norm topology. Therefore, Φ_T satisfies the SOT-q-frequent hypercyclicity criterion. $\square$

Corollary 3.5. *Let $T \in \mathcal{B}(X)$. If $A \in \mathcal{B}(X)$ is a SOT-q-frequently hypercyclic vector for the left multiplication operator Φ_T and if x is a non-zero vector in X, then $A(x)$ is a q-frequently hypercyclic vector for T.*

Proof. Let $y \in X$ and $\epsilon > 0$ be given. There is an operator $B : X \to X$ such that $B(x) = y$. Since the set $\mathfrak{U} = \{U \in \mathcal{B}(X) : \|U(x) - y\| < \epsilon\}$ is a SOT-neighborhood of B and A is a SOT-q-frequently hypercyclic vector for Φ_T, there exist a strictly increasing sequence (n_k) and a constant $C > 0$ such that

$$\Phi_T^{n_k}(A) \in \mathfrak{U} \quad \text{and} \quad n_k \leq C \cdot k^q.$$

Thus, we have that $\|T^{n_k}(A(x)) - y\| < \epsilon$ and $n_k \leq C \cdot k^q$, which implies that $A(x)$ is a q-frequently hypercyclic vector for T. $\square$

In [5], Bonilla and Grosse-Erdmann studied frequently hypercyclic subspaces. A *frequently hypercyclic subspace* (respectively, *q-frequently hypercyclic subspace*) for $T \in \mathcal{B}(X)$ is a closed and infinite-dimensional linear subspace of X consisting entirely - except for a zero vector - of frequently hypercyclic vectors (respectively, q-frequently hypercyclic vectors) for T. Menet [16] studied weighted shifts on ℓ^p for the existence and the non-existence of frequently hypercyclic subspaces. In the following proposition, we find a sufficient condition for the existence of q-frequently hypercyclic subspaces, which is similar to Theorem 4 of [5].

Proposition 3.6. *Let $q \in \mathbb{N}$. If $T \in \mathcal{B}(X)$ satisfies the q-frequent hypercyclicity criterion and if there exists an infinite dimensional closed subspace X_0 of X such that $(T^{k^q}(x))$ converges to 0 for all $x \in X_0$, then T has a q-frequently hypercyclic subspace.*

Proof. Since T satisfies the q-frequent hypercyclicity criterion, by Proposition 3.4 and Corollary 3.5 the left multiplication operator Φ_T admits a SOT-q-frequently hypercyclic vector $A \in \mathcal{B}(X)$. Since any non-zero scalar multiple of A is also SOT-q-frequently hypercyclic for Φ_T, we may assume that $\|A\| = 1/2$. For any $x \in X$, we have that

$$\|x + A(x)\| \geq \|x\| - \|A(x)\| \geq \frac{1}{2}\|x\|,$$

which indicates that the operator $B = I + A$ is bounded below. Therefore, $X_B = B(X_0)$ is an infinite dimensional closed subspace of X. For any $y \in X_B$, we have that $y = x_0 + Ax_0$ for some $x_0 \in X_0$. Let x be any vector in X. Then we get the inequality

$$\begin{aligned} \|T^{k^q} y - x\| &= \|T^{k^q} A x_0 + T^{k^q} x_0 - x\| \\ &\leq \|T^{k^q} A x_0 - x\| + \|T^{k^q} x_0\|. \end{aligned}$$

Since Ax_0 is a q-frequently hypercyclic vector for (T^{k^q}), for any $\varepsilon > 0$, $\|T^{k^q}Ax_0 - x\| < \frac{\varepsilon}{2}$ for some k and by (2), for such k, $T^{k^q}x_0 \to 0$ for any $x_0 \in B_0$. For sufficiently large k, $\|T^{k^q}x\| < \frac{\varepsilon}{2}$ and hence $\|T^{k^q}y - x\| < \varepsilon$ for such k, which implies that y is a q-frequently hypercyclic vector for T. □

Corollary 3.7. *Suppose that $T \in \mathcal{B}(X)$ satisfies the q-frequent hypercyclicity criterion and* $\dim\ker(T-\lambda) = \infty$ *for some λ with $|\lambda| < 1$. Then T has a q-frequently hypercyclic subspace.*

Proof. We see that $X_0 = \ker(T-\lambda)$ is a closed infinite dimensional subspace of X. For each $x \in X_0$, we have that $T^{k^q}(x) = \lambda^{k^q}(x)$ converges to 0 as k goes to ∞. By Proposition 3.6, T has a q-frequently hypercyclic subspace. □

4. q-frequent hypercyclicity on projective tensor product

In [15], sufficient conditions for hypercyclicity on tensor products of operators were studied and a characterization of hypercyclicity of backwards shifts was obtained. In this section, we consider the q-frequent hypercyclicity for tensor products of operators.

Given Banach spaces X, Y, the *projective norm* is a norm defined by

$$\|z\|_\gamma = \inf\left\{\sum_{j=1}^{n}\|x_j\|\|y_j\| : z = \sum_{j=1}^{n} x_j \otimes y_j \in X \otimes Y\right\},$$

where $X \otimes Y$ is the algebraic tensor product. The completion of $X \otimes Y$ with respect to $\|\cdot\|_\gamma$ is called the *Banach space projective tensor product*, which is denoted by $X \otimes_\gamma Y$. For $T_1 \in \mathcal{B}(X)$ and $T_2 \in \mathcal{B}(Y)$, we define $T_1 \otimes T_2$ on $X \otimes Y$ by $T_1 \otimes T_2(x \otimes y) = T_1(x) \otimes T_2(y)$. The extension to $X \otimes_\gamma Y$ is also denoted by $T_1 \otimes T_2$.

Proposition 4.1. *Let X and Y be separable infinite dimensional Banach spaces. If $T \in \mathcal{B}(X)$ satisfies the q-frequent hypercyclicity criterion, then $T \otimes I_Y$ on $X \otimes_\gamma Y$ also satisfies the q-frequent hypercyclicity criterion.*

Proof. Let X_0 be a dense subset of X and let A be a mapping from X_0 into X_0 as in Theorem 2.5. We may also assume that X_0 is a subspace and A is a linear mapping on X_0. Then $X_0 \otimes Y$ is a dense subspace of $X \otimes_\gamma Y$ and $A \otimes I_Y$ is a linear mapping from $X_0 \otimes Y$ into itself.

Let $x \in X_0$ and $y \in Y$ be given. Since the series $\sum_{n=0}^{k} T^{k^q-(k-n)^q}(x)$ is unconditionally convergent, uniformly in $k \geq 0$, there exists a finite subset $F \subset \mathbb{N}$ such that for any $\epsilon > 0$,

$$\|y\|\left\|\sum_{n \in F \cap \{1,\ldots,k\}} T^{k^q-(k-n)^q}(x)\right\| < \epsilon.$$

Then we have that

$$\begin{aligned}\Bigg\|\sum_{n\in F\cap\{1,\ldots,k\}}(T\otimes I_Y)^{k^q-(k-n)^q}(x\otimes y)\Bigg\| &= \Bigg\|\sum_{n\in F\cap\{1,\ldots,k\}}T^{k^q-(k-n)^q}(x)\otimes y\Bigg\|\\ &\le \|y\|\Bigg\|\sum_{n\in F\cap\{1,\ldots,k\}}T^{k^q-(k-n)^q}(x)\Bigg\|\\ &< \epsilon,\end{aligned}$$

which implies that $\sum_{n=0}^{k}(T\otimes I_Y)^{k^q-(k-n)^q}(x\otimes y)$ unconditionally converges. We can also see that $\sum_{n=0}^{\infty}(A\otimes I_Y)^{(n+k)^q-k^q}(x\otimes y)$ is unconditionally convergent since the series $\sum_{n=0}^{\infty}A^{(n+k)^q-k^q}(x)$ is unconditionally convergent. Since $(T\otimes I_Y)(A\otimes I_Y)(x\otimes y)=x\otimes y$, $T\otimes I_Y$ is q-frequently hypercyclic. □

An operator T in $\mathcal{B}(X)$ is called *quasi-conjugate* to $S\in\mathcal{B}(Y)$ if there is a continuous map $\phi: Y\longrightarrow X$ with a dense range such that $T\circ\phi=\phi\circ S$. That is, the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\phi"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "T"'] & X \end{tikzcd}$$

By Proposition 9.4 of [10], frequent hypercyclicity is preserved under quasi-conjugacy. Similarly, we can see that the q-frequent hypercyclicity is also preserved under quasi-conjugacy, i.e., T is q-frequently hypercyclic whenever S is q-frequently hypercyclic and quasi-conjugate to S.

Proposition 4.2. *Let X be a separable infinite dimensional Banach space. If $T\in\mathcal{B}(X)$ satisfies the q-frequent hypercyclicity criterion, then $T\oplus T: X\oplus X\longrightarrow X\oplus X$ is q-frequently hypercyclic.*

Proof. By Proposition 4.1, if T satisfies the q-frequent hypercyclicity criterion, then $T\otimes I_X$ on $X\otimes_\gamma X$ is a q-frequently hypercyclic operator. Let $\{x_i\}$ and $\{y_i\}$ be two basis for X and let $y_i^*\in X^*$ be the dual pairing, where X^* is the dual space of X. That is, $y_j^*(y_i)=\delta_{ij}$. Define $\phi: X\otimes_\gamma X\longrightarrow X\oplus X$ by

$$\phi\bigg(\sum_i x_i\otimes y_i\bigg)=\sum_i y_1^*(y_i)x_i\oplus\sum_i y_2^*(y_i)x_i.$$

Then the map ϕ is well-defined and for any $a,b\in X$, we have that

$$\phi(a\otimes y_1+b\otimes y_2)=a\oplus b,$$

which implies that ϕ is surjective. One can easily verify the following equation

$$T\oplus T\left(\phi\bigg(\sum_i x_i\otimes y_i\bigg)\right)=\sum_i y_1^*(y_i)Tx_i\oplus\sum_i y_2^*(y_i)Tx_i$$

$$= \phi\left((T \otimes I_X)\left(\sum_i x_i \otimes y_i\right)\right).$$

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} X \otimes_\gamma X & \xrightarrow{T\otimes I_X} & X \otimes_\gamma X \\ \phi\big\downarrow & & \big\downarrow\phi \\ X \oplus X & \xrightarrow[T\oplus T]{} & X \oplus X \end{array}$$

Since $T \otimes I_X$ is q-frequently hypercyclic, $T \oplus T$ is also q-frequently hypercyclic. □

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