Bull. Korean Math. Soc. **54** (2017), No. 2, pp. 429–442
https://doi.org/10.4134/BKMS.b151004
pISSN: 1015-8634 / eISSN: 2234-3016

SCREEN ISOTROPIC LEAVES ON LIGHTLIKE HYPERSURFACES OF A LORENTZIAN MANIFOLD

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Abstract. In the present paper, screen isotropic leaves on lightlike hypersurfaces of a Lorentzian manifold are introduced and studied which are inspired by the definition of isotropic immersions in the Riemannian context. Some examples of such leaves are mentioned. Furthermore, some relations involving curvature invariants are obtained.

1. Introduction

The notion of isotropic immersions in Riemannian geometry was firstly introduced by B. O'Neill [28] in 1965 as follows:

Let $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion between Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$. The immersion φ is called λ-*isotropic* if there exists a real valued function λ such that at any point $p \in M$, the second fundamental form h satisfies

$$\|h(X, X)\| = \lambda \tag{1.1}$$

for all unit vector $X \in T_pM$. If the function λ is constant at every point of M, then M is called a (*constant*) *isotropic submanifold*.

Later, the isotropic immersions between non-degenerate manifolds have been studied by many authors in [1, 7, 8, 9, 14, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31] etc.

The main purpose of the present paper is to continue this frame of works for degenerate immersions, especially for lightlike hypersurfaces of a Lorentzian manifold. But there are some difficulties about studying isotropy for these submanifolds. The fundamental problems are that the second fundamental form of a lightlike hypersurface is a null vector and a screen distribution on lightlike submanifolds isn't canonical. Thus, the notion of isotropy in a lightlike hypersurface can be studied only on any leaf of a screen distribution which must be canonical and integrable.

Received December 5, 2015; Revised September 7, 2016.
2010 *Mathematics Subject Classification.* Primary 53C42, 53C50.
Key words and phrases. isotropic immersion, lightlike hypersurface, Lorentzian manifold.

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2. Preliminaries

In this section, we recall some basic facts about lightlike hypersurfaces by following the notations and formulas used in [10, 12, 13].

Let (M, g) be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ with the induced degenerate metric g from $\widetilde{g}$. The *radical space* Rad T_pM at $p \in M$, is a vector bundle of rank 1, defined by

$$\text{Rad } T_pM = \{\xi \in T_pM : g_p(\xi, X) = 0, \ \forall X \in T_pM\}. \tag{2.1}$$

The complementary non-degenerate vector bundle $S(TM)$ of Rad TM in tangent bundle TM is called *screen distribution* of M. Thus, we have

$$TM = \text{Rad } TM \oplus_{\text{ort}} S(TM), \tag{2.2}$$

where $\oplus_{\text{ort}}$ denotes the orthogonal direct sum. From (2.2), there exits a field of frame $\{\xi, e_1, \ldots, e_n\}$ on a coordinate neighborhood $\mathcal{U}$ on M such that Rad $TM|_{\mathcal{U}} = \text{Span}\{\xi\}$ and $S(TM)|_{\mathcal{U}} = \text{Span}\{e_1, \ldots, e_n\}$. It is known that there exists a unique smooth section $\{N\}$ for any basis $\{\xi\}$ on Rad TM satisfying

$$\widetilde{g}(N, X) = \widetilde{g}(N, N) = 0, \ \widetilde{g}(N, \xi) = 1 \tag{2.3}$$

for all $X \in \Gamma(S(TM))$. The bundle $\text{tr}(TM) = \text{Span}\{N\}$ is called the *lightlike transversal bundle* of M. From (2.2) and (2.3), we have the following decomposition:

$$T\widetilde{M} = TM \oplus \text{tr}(TM) = S(TM) \oplus_{\text{ort}} (\text{Rad}TM \oplus \text{tr}(TM)), \tag{2.4}$$

where $\oplus$ denotes the direct sum, but it is not orthogonal.

Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\widetilde{M}$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X, Y), \nabla_X N \in \Gamma(\text{tr}(TM))$. Here, the tensors h and A_N are called the *second fundamental form* and the *shape operator* of M, respectively. If we put

$$B(X, Y) = \widetilde{g}(h(X, Y), \xi), \ w(x) = \widetilde{g}(\nabla^{\perp}_X N, \xi) \tag{2.7}$$

in (2.5) and (2.6), respectively, we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.8}$$

$$\widetilde{\nabla}_X N = -A_N X + w(X)N. \tag{2.9}$$

Taking into consideration (2.3) and (2.8), it is clear that B is symmetric, independent of choosing screen distribution and it vanishes on the radical space.

Let P be the projection morphism of $\Gamma(TM)$ onto $\Gamma(S(TM))$. From (2.2), we can write

$$\nabla_X PY = \nabla^*_X Y + C(X,Y)\xi, \tag{2.10}$$

$$\nabla_X \xi = -A^*_\xi X + w(X)\xi \tag{2.11}$$

for all $X, Y \in TM$, where $\nabla^*_X Y, A^*_\xi X \in \Gamma(S(TM))$. Using equations (2.3) and (2.8)-(2.11), we have

$$B(X,Y) = g(A^*_\xi X, Y), \quad C(X,Y) = g(A_N X, Y) \tag{2.12}$$

and

$$\widetilde{g}(A_N X, N) = \widetilde{g}(A^*_\xi X, N) = 0. \tag{2.13}$$

A lightlike hypersurface $(M, g, S(TM))$ is called *totally geodesic* if $h = 0$. If there exists a smooth function μ on $\mathrm{tr}(TM)$ satisfying

$$B(X,Y) = g(X,Y)\mu \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$, then M is called *totally umbilical* [11]. Furthermore, M is called *minimal* if

$$\mathrm{trace}|_{S(TM)}(h) = 0, \tag{2.15}$$

where $\mathrm{trace}|_{S(TM)}(h)$ denotes the trace of h restricted to $S(TM)$ with respect to the degenerate metric g [5].

A lightlike hypersurface $(M, g, S(TM))$ is called *screen locally conformal* if the shape operators A_N and A^*_ξ are related by

$$A_N = \varphi A^*_\xi, \tag{2.16}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M [3].

Let us denote the curvature tensors of the ambient manifold and the lightlike hypersurface by $\widetilde{R}$ and R, respectively. Then the following relation between these tensors holds:

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, PU) &= g(R(X,Y)Z, PU) + B(X,Z)C(Y,PU) \\ &\quad - B(Y,Z)C(X,PU) \end{aligned} \tag{2.17}$$

for any $X, Y, Z, U \in TM$.

Let $\Pi = \mathrm{Span}\{e_i, e_j\}$ be a 2-dimensional non-degenerate plane in T_pM. Then the sectional curvature at p is expressed by [4]

$$K(\Pi) = \frac{g(R_p(e_j, e_i)e_i, e_j)}{g_p(e_i, e_i)g_p(e_j, e_j) - g_p(e_i, e_j)^2}. \tag{2.18}$$

Now, we recall the following result [6]:

Theorem 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent*:

i) *$S(TM)$ is integrable.*

ii) *h^* is symmetric on $\Gamma(S(TM))$.*

iii) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

As a consequence of Theorem 2.1, we obtain the following:

Corollary 2.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface. The sectional curvature is symmetric if and only if $S(TM)$ is integrable.*

3. Screen isotropy in a lightlike hypersurface

Let $(M, g, S(TM))$ be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Suppose $S(TM)$ is integrable and (M', g') is an n-dimensional leaf of $S(TM)$ immersed in $\widetilde{M}$ of codimension 2 with the non-degenerate metric g'. From (2.8) and (2.10), we have

$$\widetilde{\nabla}_X Y = \nabla'_X Y + C(X, Y)\xi + B(X, Y)N \tag{3.1}$$

for all $X, Y \in \Gamma(S(TM))$. Here, ∇' denotes the induced connection of M' from ∇. It follows that the second fundamental form of M', denoted by h', is given by

$$h'(X, Y) = C(X, Y)\xi + B(X, Y)N. \tag{3.2}$$

Hence, the squared norm of the vector $h'(X, Y)$ is given by

$$\|h'(X, Y)\|^2 = 2C(X, Y)B(X, Y). \tag{3.3}$$

Let $\Pi = \text{Span}\{X, Y\}$ be a non-degenerate plane section in T_pM. The discriminant of the tensor h', denoted by $\Delta(\Pi)$, is a real valued function on the plane section Π which is defined by

$$\Delta(\Pi) = \frac{\widetilde{g}(h'(X, X), h'(Y, Y)) - \|h'(X, Y)\|^2}{Q(X, Y)}, \tag{3.4}$$

where $Q(X, Y)$ is the area of the parallelogram with sides X and Y such that

$$Q(X, Y) = g(X, X)g(Y, Y) - g(X, Y)^2. \tag{3.5}$$

From equations (2.17), (2.18) and (3.4), we have the following lemma:

Lemma 3.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Then, we have*

$$2K(\Pi) = 2\widetilde{K}(\Pi) + \Delta(\Pi) \tag{3.6}$$

for any non-degenerate plane section Π. Here, $\widetilde{K}$ denotes the sectional curvature map of ambient manifold $\widetilde{M}$.

Considering Corollary 2.2 and Lemma 3.1, we get the followings immediately:

Theorem 3.2. *The discriminant Δ is well defined and invariant for any non-degenerate plane section in T_pM if and only if $S(TM)$ is integrable.*

Theorem 3.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian space form $R_1^m(c)$ of constant curvature c. The discriminant Δ is constant for any non-degenerate plane section in T_pM, $p \in M$, if and only if M is of constant curvature.*

Now, we state the following definition:

Definition. Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a leaf of $S(TM)$ which is integrable. The manifold M' is called λ-*screen isotropic* and the tensor h' is called λ-*isotropic* if there exists a real valued function λ such that $\|h'(X, X)\| = 2\lambda$ for all unit vectors $X \in \Gamma(S(TM))$. If λ is constant for all points, then M' is called a *isotropic leaf.*

Example 3.4 (Lightlike Cone in the Minkowski space)**.** Let $\mathbb{R}_1^4$ be the Minkowski space with signature $(-, +, +, +)$ of the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4\}$ and M be a submanifold of $\mathbb{R}_1^4$ given by

$$\{(t, t\cos u \cos v, t\cos u \sin v, t\sin u) : \ t > 0, \ u \in (0, \frac{\pi}{2}), \ v \in [0, 2\pi]\}.$$

Then we have

$$\begin{aligned} &\xi = \partial x_1 + \cos u \cos v \partial x_2 + \cos u \sin v \partial x_3 + \sin u \partial x_4, \\ &N = \frac{1}{2}\left(-\partial x_1 + \cos u \cos v \partial x_2 + \cos u \sin v \partial x_3 + \sin u \partial x_4\right), \\ &w_1 = t(-\sin u \cos v \partial x_2 - \sin u \sin v \partial x_3 + \cos u \partial x_4), \\ &w_2 = t(-\cos u \sin v \partial x_2 + \cos u \cos v \partial x_3). \end{aligned}$$

If we put $e_1 = \frac{w_1}{\|w_1\|}$ and $e_2 = \frac{w_2}{\|w_2\|}$, then it follows that $\{e_1, e_2\}$ is a set of orthonormal vectors. Furthermore, it can be considered a 2-dimensional leaf M' of $S(TM)$ such that

$$TM' = \mathrm{Span}\{e_1, e_2\}.$$

By a straightforward computation, we obtain

$$B(e_1, e_1)C(e_1, e_1) = B(e_2, e_2)C(e_2, e_2) = \frac{1}{2t^2},$$

which implies that M' is screen isotropic with $\lambda = \mp\frac{1}{2t}$.

Example 3.5 (Lightlike Monge hypersurfaces of $\mathbb{R}_1^{n+2}$)**.** Let Ω be an open set of $\mathbb{R}_1^{n+2}$ and $F : \Omega \to \mathbb{R}$ be a smooth function. A Monge hypersurface is defined by

$$M = \{(x_0, \ldots, x_{n+1}) \in \mathbb{R}_1^{n+2} : \ x_0 = F(x_1, \ldots, x_{n+1})\}.$$

Such a hypersurface is lightlike if and only if F is a solution of the partial differential equation

$$\sum_{\alpha=1}^{n+1} (F'_{x_\alpha})^2 = 1.$$

It is known these types of hypersurfaces are screen locally conformal with the conformal function $\varphi = \frac{1}{2}$ and the tensor B is given by [10]

$$B(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}) = -\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}.$$

Therefore, a leaf of a lightlike Monge hypersurface is λ-screen isotropic if and only if

$$F(x_1, \ldots, x_{n+1}) = c \mp \lambda \sum_{\alpha}^{n+1} x_\alpha^2 + b_\alpha x_\alpha,$$

where b_α, c are any real numbers for $\alpha = 1, \ldots, n+1$.

Theorem 3.6. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional $(n \geq 2)$ lightlike hypersurface of a Lorentzian manifold and M' be a leaf of $S(TM)$. For any non-degenerate plane section Π spanned by unit vector fields X and Y, we have the following statements*:

i) $\Delta(\Pi) = 4\lambda^2$ *if and only if* $h'(X, X) = h'(Y, Y)$ *and* $B(X, Y) = 0$ *or* $C(X, Y) = 0$.

ii) $\Delta(\Pi) = -8\lambda^2$ *if and only if* $\|h'(X, Y)\|^2 = 4\lambda^2$ *and* h' *is minimal, that is,* $h'(X, X) + h'(Y, Y) = 0$.

Proof. Let us consider a quadrilinear function L defined by

$$L(X, Y, U, V) = \widetilde{g}\,(h'(X, Y), h'(U, V)) - 4\lambda^2 \widetilde{g}(X, Y)\widetilde{g}(U, V) \tag{3.7}$$

for any $X, Y, U, V \in \Gamma(S(TM))$. From Theorem 2.1, it is clear that this function is symmetric. Since h' is λ-isotropic, we obtain

$$F(X) = L(X, X, X, X) = 0 \tag{3.8}$$

for all unit vector fields $X \in \Gamma(S(TM))$. Thus, we have also

$$F(X + Y) + F(X - Y) = 0 \tag{3.9}$$

and it follows that

$$L(X, X, Y, Y) + 2L(X, Y, X, Y) = 0, \tag{3.10}$$

which implies that

$$\widetilde{g}(h'(X, X), h'(Y, Y)) + 2\|h'(X, Y)\|^2 = 4\lambda^2 \tag{3.11}$$

for all unit vector fields X and Y. From (3.4) and (3.11), we obtain

$$\Delta(\Pi) + 3\|h'(X, Y)\|^2 = 4\lambda^2 \tag{3.12}$$

and

$$2\Delta(\Pi) + 4\lambda^2 = 3\widetilde{g}(h'(X, X), h'(Y, Y)). \tag{3.13}$$

From (3.12) and (3.13), the proof of theorem is straightforward. □

Definition. A leaf M' of a integrable screen distribution $S(TM)$ on a lightlike hypersurface is called *totally umbilical* if

$$h'(X,Y) = g(X,Y)H', \quad \forall X,Y \in \Gamma(S(TM)), \tag{3.14}$$

where H' is the mean curvature vector of M'.

Theorem 3.7. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian manifold and M' be a λ-screen isotropic leaf of $S(TM)$. Then the conformal factor φ can't be negative.*

Proof. If $(M, g, S(TM))$ is screen locally conformal, then we can write

$$h'(X,X) = B(X,X)N + \varphi B(X,X)\xi \tag{3.15}$$

for any orthonormal vector field X on $\Gamma(S(TM))$. Since M' is a λ-screen isotropic leaf of $S(TM)$, we have

$$\varphi[B(X,X)]^2 = 2\lambda^2, \tag{3.16}$$

which shows that φ can't be negative. □

Corollary 3.8. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian manifold and M' be a λ-screen isotropic leaf of $S(TM)$. Then we have*

$$\Delta(\Pi) \leq 4\lambda^2. \tag{3.17}$$

$\Delta(\Pi) = 4\lambda^2$ *for all non-degenerate plane section* Π *if and only if M' is totally umbilical.*

Proof. From (3.11), we have

$$\widetilde{g}(h'(X,X), h'(Y,Y)) = 4(\lambda^2 - B(X,Y)C(X,Y)) \tag{3.18}$$

for all orthonormal vectors $X, Y \in \Gamma(S(TM))$. Since (M, g) is screen conformal, we get from Theorem 3.7 that

$$\widetilde{g}(h'(X,X), h'(Y,Y)) \leq 4\lambda^2. \tag{3.19}$$

Putting (3.19) in (3.11), (3.12) and (3.13), we get (3.17).

If $\Delta(\Pi) = 4\lambda^2$ for all non-degenerate plane section, then we obtain

$$\varphi[B(X,Y)]^2 = 0, \tag{3.20}$$

which implies that $h'(X,Y) = 0$ since φ is a non-vanishing function. From the statement (i) of Theorem 3.6, it is clear that M' is totally umbilical. The proof of the converse part is straightforward. □

Theorem 3.9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a totally umbilical leaf of $S(TM)$. If M' is a λ-screen isotropic leaf, then the following assertions hold*:

i) *The mean curvature vector H' of M' is space-like.*
ii) *M is screen conformal.*

Proof. From (3.1), we write

$$H' = \alpha\xi + \rho N, \tag{3.21}$$

where ρ and α are two smooth functions. Since M' is totally umbilical, we have

$$h'(X,Y) = g(X,Y)(\alpha\xi + \rho N) \tag{3.22}$$

for all $X, Y \in \Gamma(S(TM))$. Using (3.22), it follows that

$$B(X,Y) = \rho g(X,Y) \quad \text{and} \quad C(X,Y) = \alpha g(X,Y). \tag{3.23}$$

Therefore, $\widetilde{g}(H', H') = 2\alpha\rho > 0$, which implies that H' is space-like. Also, we have

$$C(X,Y) = \frac{\alpha}{\rho} B(X,Y) \tag{3.24}$$

for all $X, Y \in \Gamma(S(TM))$, which shows that M is screen conformal with the conformal factor $\frac{\alpha}{\rho}$. □

Theorem 3.10. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a totally umbilical leaf of $S(TM)$. If M' is λ-screen isotropic, then we have*

$$trace(A^*_\xi) trace(A_N) \geq 0. \tag{3.25}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Using (3.14), we have

$$B(e_1, e_1)C(e_1, e_1) = \cdots = B(e_n, e_n)C(e_n, e_n) = 2\lambda^2. \tag{3.26}$$

Furthermore, we also have from (3.14) that

$$B(e_1, e_1)N + C(e_1, e_1)\xi = \frac{1}{n}\left(\text{trace}A^*_\xi N + \text{trace}A_N \xi\right). \tag{3.27}$$

Therefore, we obtain

$$\text{trace}(A^*_\xi)\text{trace}(A_N) = 2n^2\lambda^2, \tag{3.28}$$

which implies (3.25). □

Corollary 3.11. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian manifold and M' be a totally umbilical leaf of $S(TM)$. If M' is λ-screen isotropic, then*

$$trace(A^*_\xi) = \mp \frac{\sqrt{2} n\lambda}{\sqrt{\varphi}}. \tag{3.29}$$

We now recall the following theorem of D. H. Jin in [18]:

Theorem 3.12. *Let (M, g) be an $(n+1)$-dimensional $(n \geq 3)$ lightlike hypersurface of a semi-Riemannian space form $(\widetilde{M}(c), \widetilde{g})$ such that $S(TM)$ is totally umbilical. Then $B = 0$ or $C = 0$.*

Considering Theorem 3.12, we get the following corollary:

Corollary 3.13. *If M' be any leaf in a lightlike hypersurface which satisfies the assumptions of Theorem 3.12. Then M' isn't λ-isotropic.*

Definition. Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be an n-dimensional leaf of $S(TM)$. The manifold M' is called *minimal* if, for every point $p \in M$,

$$H'(p) = \sum_{i=1}^{n} h'(e_i, e_i) = 0, \tag{3.30}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis on $\Gamma(S(TM))$.

From the above definition, we have the followings immediately:

Theorem 3.14. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a leaf of $S(TM)$. The manifold M' is minimal if and only if (M, g) is minimal and trace$(A_N) = 0$.*

Corollary 3.15. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian manifold and M' be a leaf of $S(TM)$. Then M' is minimal if and only if $(M, g, S(TM))$ is minimal.*

Corollary 3.16. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a 2-dimensional λ-screen isotropic leaf of $S(TM)$. If M' is minimal, then*

$$\widetilde{g}(h'(X, X), h'(Y, Y)) < 0 \tag{3.31}$$

for all orthonormal vector fields $X, Y \in \Gamma(S(TM))$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis $\Gamma(S(TM))$. Then we have

$$h'(e_1, e_1) + h'(e_2, e_2) = 0, \tag{3.32}$$

which is equivalent to

$$[B(e_1, e_1) + B(e_2 + e_2)]N + [C(e_1, e_1) + C(e_2 + e_2)]\xi = 0. \tag{3.33}$$

Therefore, we get

$$\begin{aligned} &B(e_1, e_1)C(e_1, e_1) + B(e_1, e_1)C(e_2, e_2) + B(e_2, e_2)C(e_1, e_1) \\ &+ B(e_2, e_2)C(e_2, e_2) = 0. \end{aligned} \tag{3.34}$$

Since M' is λ-screen isotropic, we have

$$B(e_1, e_1)C(e_2, e_2) + B(e_2, e_2)C(e_1, e_1) = -4\lambda^2. \tag{3.35}$$

Since we can choose $X = e_1$ and $Y = e_2$, therefore the above equation implies (3.31). □

Proposition 3.17. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and M' be a screen isotropic leaf of $S(TM)$. If M is totally umbilical, then we have the following statements*:

i) *M' is totally umbilical.*

ii) *M' isn't minimal.*

Proof. Under the assumption, we have from equation (2.14) that

$$B(X,X)=\mu \tag{3.36}$$

for all unit vector fields $X\in\Gamma(S(TM))$. Since M' is λ-screen isotropic, we also have from equation (3.36) that

$$C(X,X)=\frac{\lambda^2}{2\mu} \tag{3.37}$$

for all unit vector fields $X\in\Gamma(S(TM))$. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Then, for any point $p\in M$, we get

$$\begin{aligned} H'(p) &= \frac{1}{n}\left(h'(e_1,e_1)+\cdots+h'(e_n,e_n)\right)\\ &= \mu N+\frac{\lambda^2}{2\mu}\xi, \end{aligned} \tag{3.38}$$

which implies that

$$H'(p)=h'(e_i,e_i) \tag{3.39}$$

for all $i\in\{1,\dots,n\}$. Thus, it is clear from (3.39) that M' is totally umbilical and it isn't minimal. □

4. Some results on lightlike hypersurfaces of a semi-Euclidean space

We begin this section with recalling the following definition given in [15].

Definition. Let $(M,g,S(TM))$ be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold and $S(TM)$ be an integrable distribution. Then the *screen Ricci tensor*, denoted by $\mathrm{Ric}_{S(TM)}$, is defined by

$$\mathrm{Ric}_{S(TM)}(X,Y)=\mathrm{trace}|_{S(TM)}\{Z\to R(X,Z)Y\} \tag{4.1}$$

for any $X,Y,Z\in\Gamma(S(TM))$. Here, $\mathrm{trace}|_{S(TM)}$ denotes the trace restricted to $S(TM)$ with respect to the degenerate metric g.

Suppose that $\{e_1,\dots,e_{n-1},X\}$ to be an orthonormal basis of $\Gamma(S(TM))$. The screen Ricci curvature at a unit vector $X\in\Gamma(S(TM))$ is given by

$$\mathrm{Ric}_{S(TM)}(X)=\sum_{j=1}^{n-1} g(R(X,e_j)e_j,X). \tag{4.2}$$

Let us consider the following functions f_1 and f_2 defined by

$$f_1(X)=\|h'(X,X)\|^2 \tag{4.3}$$

and

$$f_2(X)=\sum_{j=1}^{n-1}\|h'(e_i,X)\|^2+f_1(X), \tag{4.4}$$

respectively. Then we have the following lemma:

Lemma 4.1. *Let $(M, g, S(TM))$ be $(n+1)$-dimensional lightlike hypersurface of a semi-Euclidean space. Then we have*

$$Ric_{S(TM)}(X) = ng(A_{H'}X, X) - f_2(X) \tag{4.5}$$

for any unit vector X in $\Gamma(S(TM))$.

Proof. Using (2.17), (3.15), (4.2) and (4.4), we have

$$\mathrm{Ric}_{S(TM)}(X) = \sum_{j=1}^{n-1}[\widetilde{g}(h'(e_j, e_j), h'(X, X)) - f_1(X) - \|h'(e_j, X)\|^2],$$

which is equivalent to (4.5). □

Let M' be a leaf of $S(TM)$ on a lightlike hypersurface of R_1^{n+2}. Denote the set of all unit vectors in T_pM' by T_p^1M', that is,

$$T_p^1M' = \{X \in T_pM' : g_p(X, X) = 1\}. \tag{4.6}$$

From Lemma 4.1 and equation (4.6), we have the following corollaries:

Corollary 4.2. *Let M' be a leaf of $S(TM)$ on a screen conformal lightlike hypersurface of a semi-Euclidean space. Then we have the followings*:

i) *Let the conformal function φ be non-negative. The screen Ricci curvature $Ric_{S(TM)}$ is non-negative for all vector $X \in T_p^1M'$ if and only if $\{X \in T_p^1M' : \ g(A_HX, X) < 0\}$ is a empty set for all points $p \in M$.*

ii) *Let the conformal function φ be non-positive. The screen Ricci curvature $Ric_{S(TM)}$ is non-positive for all vector $X \in T_p^1M'$ if and only if $\{X \in T_p^1M' : \ g(A_HX, X) > 0\}$ is a empty set for all points $p \in M$.*

Corollary 4.3. *Let M' be an n-dimensional isotropic leaf of $S(TM)$. For any $X \in T_p^1M'$, we have*

$$Ric_{S(TM)}(X) \leq ng(A_{H'}X, X) - \lambda^2. \tag{4.7}$$

The equality case of (4.7) holds at a point $p \in M$ if and only if M' is totally umbilical.

Corollary 4.4. *If M' is an n-dimensional isotropic leaf of $S(TM)$, then*

$$Ric_{S(TM)}(X) < ng(A_{H'}X, X) \tag{4.8}$$

for any $X \in T_p^1M'$.

Definition ([15])**.** Let $(M, g, S(TM))$ be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold. Suppose $S(TM)$ is integrable and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. The *screen scalar curvature* at a point $p \in M$, denoted by $r_{S(TM)}(p)$, is defined by

$$r_{S(TM)}(p) = \frac{1}{2}\sum_{i,j=1}^{n} K(\Pi_{ij}), \tag{4.9}$$

where $\Pi_{ij} = \mathrm{Span}\{e_i, e_j\}$ is a plane section on $\Gamma(S(TM))$.

Now, we recall the following lemmas for future uses:

Lemma 4.5 ([2]). *Let $\bar{M}$ be an n-dimensional non-degenerate submanifold of a real space form $R^m(c)$ and p be a point of M. Then, we have*

$$\int_{T_p^1\bar{M}} \langle \bar{A}_H X, X\rangle dx_p = \|\bar{H}(p)\|^2 vol(T_p^1\bar{M}), \tag{4.10}$$

where dx_p is the canonical volume element of $T_p^1\bar{M}$ and $vol(T_p^1\bar{M})$ denotes the volume of $T_p^1\bar{M}$.

Lemma 4.6 ([19]). *Let $\bar{M}$ be an n-dimensional non-degenerate submanifold of a real space form $R^m(c)$ and p be a point of M. Then we have*

$$\int_{T_p^1\bar{M}} \bar{S}(X,X)dx_p = \frac{\bar{r}}{n}\ vol(T_p^1\bar{M}), \tag{4.11}$$

where $\bar{S}$ and $\bar{r}$ are the Ricci tensor and the scalar curvature defined on $\bar{M}$, respectively.

From the above facts, we have the following:

Theorem 4.7. *Let M' be an n-dimensional leaf of $S(TM)$ on a lightlike hypersurface $(M, g, S(TM))$ of a semi-Euclidean space. Then we have*

$$\int_{T_p^1 M'} f_2(X)dx_p = \left(n\|H'(p)\|^2 - \frac{r_{S(TM)}(p)}{n}\right) vol(T_p^1 M') \tag{4.12}$$

for any $X \in T_p^1 M'$.

Proof. If we integrate in (4.5) over the range $T_p^1 M'$, we get

$$\int_{T_p^1 M'} \mathrm{Ric}_{S(TM)}(X) = \int_{T_p^1 M'} [n\ g(A_{H'}X, X) - f_2(X)]\,\mathrm{vol}(T_p^1 M'). \tag{4.13}$$

From Lemma 4.5, Lemma 4.6 and (4.13), equation (4.12) is straightforward. □

Taking into account of Corollary 4.4 and Theorem 4.7, we obtain the following corollaries immediately:

Corollary 4.8. *If M' is an n-dimensional isotropic leaf of $S(TM)$, then we have*

$$r_{S(TM)}(p) < n^2\|H'(p)\|^2. \tag{4.14}$$

Corollary 4.9. *If the leaf M' is a minimal isotropic leaf, then the screen scalar curvature is negative.*

With similar arguments to the proof of Theorem 3.4 in [16], we have the following:

Theorem 4.10. *Let* $(M, g, S(TM))$ *be a screen locally conformal lightlike hypersurface of a semi-Euclidean space and* M' *be an* n*-dimensional leaf of* $S(TM)$*. Then we have*

$$r_{S(TM}(p) \leq n(n-1)\|H'(p)\|^2. \tag{4.15}$$

The equality case of (4.15) *holds at every point of* M' *if and only if* M' *is totally umbilical.*

From Theorem 4.7 and Theorem 4.10, we obtain the following corollary:

Corollary 4.11. *If* M' *be an* n*-dimensional totally umbilical leaf in an* $(n+1)$*-dimensional screen locally conformal lightlike hypersurface of a semi-Euclidean space, then we have*

$$\int_{T_p^1 M'} f_2(X)dx_p = \|H'(p)\|^2 vol(T_p^1 M'). \tag{4.16}$$

Acknowledgements. The author is thankful to the referee for his/her constructive suggestions which improved the content and presentation of the paper.

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