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THE UNIT BALL OF $\mathcal{L}(^2\mathbb{R}^2_{h(w)})$

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Abstract. We classify the extreme bilinear forms of the unit ball of the space of bilinear forms on $\mathbb{R}^2$ with hexagonal norms.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $extB_E$ the set of extreme points of B_E. $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x, x)$ for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x, y)|$. $\mathcal{L}_s(^2E)$ denotes the closed subspace of $\mathcal{L}(^2E)$ consisting of all continuous symmetric bilinear forms on E. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

In 1998, Choi *et al.* [2, 3] characterized the extreme points of the unit ball of $\mathcal{P}(^2l_1^2)$ and $\mathcal{P}(^2l_2^2)$. In 2007, Kim [11] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2l_p^2)$ $(1 \leq p \leq \infty)$, where $l_p^2 = \mathbb{R}^2$ with the l_p-norm. Recently, Kim [13, 15, 19] classify the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$, where $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x, y)\|_{d_*} = \max\{|x|, |y|, \frac{|x|+|y|}{1+w}\}$. In 2009, Kim [12] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2l_\infty^2)$. Recently, Kim [14, 16–18, 21]

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classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s({}^2d_*(1,w)^2)$ and $\mathcal{L}({}^2d_*(1,w)^2)$.

We refer to ([1–6], [8–26] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. For $0<w<1$, $\mathbb{R}^2_{h(w)}$ denotes $\mathbb{R}^2$ endowed with a hexagonal norm $\|(x,y)\|_{h(w)} := \max\{|y|, |x|+(1-w)|y|\}$. In this paper, we classify the extreme bilinear forms of the unit ball of $\mathcal{L}({}^2\mathbb{R}^2_{h(w)})$.

2. Main results

Let $0<w<1$ and $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ for some reals a,b,c,d. For simplicity we will write

$$T((x_1,y_1),(x_2,y_2)) = (a,b,c,d).$$

Let

$$\begin{aligned}
T_1((x_1,y_1),(x_2,y_2)) &:= T((x_2,y_2),(x_1,y_1)) = (a,b,d,c),\\
T_2((x_1,y_1),(x_2,y_2)) &:= T((x_1,-y_1),(x_2,y_2)) = (a,-b,c,-d),\\
T_3((x_1,y_1),(x_2,y_2)) &:= T((x_1,y_1),(x_2,-y_2)) = (a,-b,-c,d),\\
T_4((x_1,y_1),(x_2,y_2)) &:= T((x_1,y_1),(-x_2,-y_2)) = (-a,-b,-c,-d).
\end{aligned}$$

Then $\|T_i\| = \|T\|$ $(i=1,\ldots,4)$. Hence, without loss of generality, we may assume that $a\geq 0$ and $c\geq d\geq 0$.

Theorem 2.1. *Let $0<w<1$ and $T((x_1,y_1),(x_2,y_2)) := (a,b,c,d) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ with $a\geq 0$ and $c\geq d\geq 0$. Then*

$$\|T\| = \max\{a, aw+c, |aw^2+b|+(c+d)w, |aw^2-b|+(c-d)w\}.$$

Proof. Note that $extB_{\mathbb{R}^2_{h(w)}} = \{(\pm1,0),(w,\pm1),(-w,\pm1)\}$. By the Krein-Milman theorem, $B_{\mathbb{R}^2_{h(w)}} = \overline{co}(extB_{\mathbb{R}^2_{h(w)}})$, where $\overline{co}(A)$ is the closed convex hull of the set A. By the bilinearity of T, it follows that

$$\begin{aligned}
\|T\| = &\max\{|T((\pm1,0),(\pm1,0))|, |T((\pm1,0),(w,\pm1))|,\\
&\quad |T((w,\pm1),(\pm1,0))|, |T((w,\pm1),(w,\pm1))|\}\\
= &\max\{|T((1,0),(1,0))|, |T((1,0),(w,1))|, |T((w,1),(1,0))|,\\
&\quad |T((1,0),(w,-1))|, |T((w,-1),(1,0))|, |T((w,1),(w,1))|,\\
&\quad |T((w,-1),(w,-1))|, |T((w,1),(w,-1))|, |T((w,-1),(w,1))|\}\\
= &\max\{|a|, |a|w+|c|, |a|w+|d|, |aw^2+b|+|c+d|w, |aw^2-b|+|c-d|w\}\\
= &\max\{a, aw+c, |aw^2+b|+(c+d)w, |aw^2-b|+(c-d)w\}.
\end{aligned}$$

□

Note that if $\|T\|=1$, then $|a|\leq 1, |b|\leq 1, |c|\leq 1$ and $|d|\leq 1$.

Theorem 2.2. *Let* $0 < w < 1$ *and* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$. *Then the followings are equivalent*:
(1) T *is extreme*;
(2) (a, b, d, c) *is extreme*;
(3) $(a, -b, c, -d)$ *is extreme*;
(4) $(a, -b, -c, d)$ *is extreme*;
(5) $(-a, -b, -c, -d)$ *is extreme.*

Proof. It follows from Theorem 2.1 and the remark above of Theorem 2.1. □

Theorem 2.3 ([20, Theorem 2.3])**.** *Let* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})$ *with* $a \geq 0, c \geq 0$.
(a) *Let* $0 < w < \frac{1}{2}$. *Then,* $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})}$ *if and only if*

$$T \in \{(0, \pm 1, 0, 0), (1, (1-w)^2, 1-w, 1-w), (1, 1-w^2, 0, 0),\\ (1, w^2-1, w, w), (0, 1-2w, 1, 1), (1, -3w^2+2w-1, 1-w, 1-w)\}.$$

(b) *Let* $w = \frac{1}{2}$. *Then,* $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(\frac{1}{2})})}$ *if and only if*

$$T \in \{(0, \pm 1, 0, 0), (1, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}), (1, \frac{3}{4}, 0, 0), (0, 0, 1, 1), (1, -\frac{3}{4}, \frac{1}{2}, \frac{1}{2})\}.$$

(c) *Let* $\frac{1}{2} < w < 1$. *Then,* $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})}$ *if and only if*

$$T \in \{(0, \pm 1, 0, 0), (1, (1-w)^2, 1-w, 1-w), (1, 1-w^2, 0, 0),\\ (1, w^2-1, 1-w, 1-w), (\frac{1}{2w}, \frac{w-2}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}.$$

Let

$$\begin{aligned} &Norm(T)\\ &= \{((x_1, y_1), (x_2, y_2)) \in \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (1, 0)),\\ &((1, 0), (w, -1)), ((w, -1), (1, 0)), ((w, 1), (w, 1)), ((w, -1), (w, -1)),\\ &((w, 1), (w, -1)), ((w, -1), (w, 1))\} : |T((x_1, y_1), (x_2, y_2))| = \|T\|\}. \end{aligned}$$

We call $Norm(T)$ *the norming set* of T.

Theorem 2.4. *Let* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ *with* $a \geq 0$ *and* $c \geq d \geq 0$.
(a) *Let* $0 < w < \frac{1}{2}$. *Then,* $T \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$ *if and only if*

$$T \in \{(0, \pm(1-w), 1, 0), (1, -(w^2-w+1), 1-w, w), (0, \pm 1, 0, 0),\\ (1, (1-w)^2, 1-w, 1-w), (1, w^2-1, w, w), (0, 1-2w, 1, 1),\\ (0, -1+2w, 1, 1), (1, -3w^2+2w-1, 1-w, 1-w)\}.$$

(b) *Let* $w = \frac{1}{2}$. *Then,* $T \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ *if and only if*

$$T \in \{(0, \pm\frac{1}{2}, 1, 0), (0, \pm 1, 0, 0), (1, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}), (0, 0, 1, 1), (1, -\frac{3}{4}, \frac{1}{2}, \frac{1}{2})\}.$$

(c) *Let* $\frac{1}{2} < w < 1$. *Then,* $T \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$ *if and only if*

$$\begin{aligned} T \in \{&(0, \pm(1-w), 1, 0), (0, \pm 1, 0, 0), (1, (1-w)^2, 1-w, 1-w), \\ &(1, w^2 - 1, 1-w, 1-w), (\frac{1}{2w}, \frac{w-2}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, 1, \frac{1}{w} - 1), \\ &(\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}. \end{aligned}$$

Proof. Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$ with $\|T\| = 1$. By the remark above of Theorem 2.1, we may assume that $a \geq 0$ and $c \geq d \geq 0$. Suppose that T is extreme. In what follows we will distinguish three cases: $(c > d = 0)$ or $(c > d > 0)$ or $(c = d \geq 0)$.

Case 1: $c > d = 0$.

By the remark above of Theorem 2.1, we may assume that $b \geq 0$. Note that $1 = \max\{a, aw + c, |aw^2 + b| + cw, |aw^2 - b| + cw\}$. We have the following subcases to consider:

$(a = 0, b = 0)$ or $(a = 0, b > 0)$ or $(a > 0, b = 0)$ or $(a > 0, b > 0)$.

Subcase 1: $a = 0, b = 0$.

Then, $1 = \max\{c, cw\}$, hence, $T = (0, 0, 1, 0)$, which is not extreme. Indeed, let $T_1 = (0, \frac{1}{n}, 1, 0), T_2 = (0, -\frac{1}{n}, 1, 0)$ for $n \in \mathbb{N}$ with $\frac{1}{n} + w < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Subcase 2: $a = 0, b > 0$.

Then, $1 = \max\{c, b + cw\}$, hence, $T = (0, 1 - w, 1, 0)$.

Claim: $T = (0, 1 - w, 1, 0)$ is extreme.

Let $T_1 = (\epsilon, (1 - w) + \delta, 1 + \gamma, \rho), T_2 = (-\epsilon, (1 - w) - \delta, 1 - \gamma, -\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1, 0), (w, 1))| \leq 1, |T_i((1, 0), (w, -1))| \leq 1, |T_i((w, 1), (w, 1))| \leq 1, |T_i((w, 1), (w, -1))| \leq 1$, we have

$$\begin{aligned} w\epsilon + \gamma &= 0, \\ w\epsilon - \gamma &= 0, \\ w^2\epsilon + \delta + w\gamma + w\rho &= 0, \\ w^2\epsilon - \delta - w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$. Hence, $T = (0, 1 - w, 1, 0)$ is extreme. By Theorem 2.2, $(0, -(1 - w), 1, 0)$ is extreme.

Subcase 3: $a > 0, b = 0$.

Then, $1 = \max\{a, aw + c, aw^2 + cw\}$. Note that $aw^2 + cw \leq aw^2 + (1 - aw)w = w$. Let $T_1 = (a, \frac{1}{n}, c, \frac{1}{n}), T_2 = (a, -\frac{1}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with

$aw^2 + cw + \frac{1+w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Subcase 4: $a > 0, b > 0$.

Then, $1 = \max\{a, aw + c, aw^2 + b + cw\}$. Note that

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,1),(w,1))\}.$$

Then, $|aw^2 - b| + cw < 1$. If $aw^2 - b > 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 = (a, b + \frac{w}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with $0 < b - \frac{w}{n}, b + \frac{1}{n} < 1, 0 < c - \frac{1}{n}, c + \frac{1}{n} < 1, aw^2 + b - \frac{w}{n} > 0, aw^2 - b - \frac{w}{n} > 0$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction. If $aw^2 - b = 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 = (a, b + \frac{w}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with $cw + \frac{2w}{n} < 1, 0 < b - \frac{w}{n}, b + \frac{1}{n} < 1, 0 < c - \frac{1}{n}, c + \frac{1}{n} < 1, aw^2 + b - \frac{w}{n} > 0$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

If $aw^2 - b < 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 = (a, b + \frac{w}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with $0 < b - \frac{w}{n}, b + \frac{1}{n} < 1, 0 < c - \frac{1}{n}, c + \frac{1}{n} < 1, aw^2 + b - \frac{w}{n} > 0, aw^2 - b + \frac{w}{n} < 0, |aw^2 - b| + cw + \frac{2w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

<u>Case 2</u>: $c > d > 0$.

We have the following subcases to consider:

$(0 \leq a < 1, b \geq 0)$ or $(a = 1, b \geq 0)$ or $(0 \leq a < 1, b < 0)$ or $(a = 1, b < 0)$.

Subcase 1: $0 \leq a < 1, b \geq 0$.

Note that $1 = \max\{aw + c, aw^2 + b + (c + d)w\}$. Calculation shows that if $b < 0$, then $Norm(T) \subseteq \{((1,0),(w,1)),((w,1),(w,1)),((w,-1),(w,1))\}$, and if $(a = 0, b > 0)$ or $(a \neq 0, b \geq 0)$, then $Norm(T) \subseteq \{((1,0),(w,1)),((w,1),(w,1))\}$. If $a \neq 0$ or $b \neq 0$, let $T_1 = (a + \frac{1}{n}, b - \frac{w}{n}, c - \frac{w}{n}, d + \frac{1}{n}), T_2 = (a - \frac{1}{n}, b + \frac{w}{n}, c + \frac{w}{n}, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $a + \frac{1}{n} < 1, b + \frac{w}{n} < 1, c + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Therefore, $a = 0 = b$, and $1 = \max\{c, (c + d)w\}$. Since T is extreme, $c = 1 = (c + d)w$. Hence, $T = (0, 0, 1, \frac{1}{w} - 1)$.

Claim: $T = (0, 0, 1, \frac{1}{w} - 1)$ is extreme for $\frac{1}{2} < w < 1$

Let $T_1 = (\epsilon, \delta, 1 + \gamma, \frac{1}{w} - 1 + \rho), T_2 = (-\epsilon, -\delta, 1 - \gamma, \frac{1}{w} - 1 - \rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(w,1))| \leq 1, |T_i((1,0),(w,-1))| \leq 1, |T_i((w,1),(w,1))| \leq 1, |T_i((w,-1),(w,-1))| \leq 1$, we have

$$\begin{aligned} w\epsilon + \gamma &= 0, \\ w\epsilon - \gamma &= 0, \\ w^2\epsilon + \delta + w\gamma + w\rho &= 0, \\ w^2\epsilon + \delta - w\gamma - w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Subcase 2: $a = 1, b \geq 0$.

Note that $1 = \max\{a, aw + c, aw^2 + b + (c + d)w\}$. Hence,

$$Norm(T) \subseteq \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (w, 1))\}.$$

Let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $b + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Subcase 3: $0 \leq a < 1, b < 0$.

Note that $1 = \max\{aw + c, |aw^2 + b| + (c + d)w, aw^2 - b + (c - d)w\}$. First, suppose that $aw^2 + b \geq 0$. Then, $1 = \max\{aw + c, aw^2 + b + (c + d)w, aw^2 - b + (c - d)w\}$. Hence,

$$Norm(T) \subseteq \{((1, 0), (w, 1)), ((w, 1), (w, 1)), ((w, -1), (w, 1))\}.$$

Let $T_1 = (a + \frac{1}{n}, b - \frac{w}{n}, c - \frac{w}{n}, d + \frac{1}{n}), T_2 = (a - \frac{1}{n}, b + \frac{w}{n}, c + \frac{w}{n}, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $a + \frac{1}{n} < 1, b + \frac{w}{n} < 1, c + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction. Next, suppose that $aw^2 + b < 0$. Then, $1 = \max\{aw + c, -(aw^2 + b) + (c + d)w, aw^2 - b + (c - d)w\}$. Hence,

$$Norm(T) \subseteq \{((1, 0), (w, 1)), ((w, -1), (w, -1)), ((w, -1), (w, 1))\}.$$

Let $T_1 = (a + \frac{1}{n}, b - \frac{w^2}{n}, c - \frac{w}{n}, d + \frac{w}{n}), T_2 = (a - \frac{1}{n}, b + \frac{w^2}{n}, c + \frac{w}{n}, d - \frac{w}{n})$ for some $n \in \mathbb{N}$ with $a + \frac{1}{n} < 1, b + \frac{w^2}{n} < 1, c + \frac{w}{n} < 1, d + \frac{w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Subcase 4: $a = 1, b < 0$.

Note that $1 = \max\{a, w + c, |w^2 + b| + (c + d)w, w^2 - b + (c - d)w\}$. First, suppose that $w^2 + b \geq 0$. Then, $1 = \max\{a, w + c, w^2 + b + (c + d)w, w^2 - b + (c - d)w\}$. Hence,

$$Norm(T) \subseteq \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (w, 1)), ((w, -1), (w, 1))\}.$$

If $w^2 + b > 0$, let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $w^2 + b - \frac{w}{n} > 0$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction. Suppose that $w^2 + b = 0$. We claim that $(c + d)w < 1$. If $(c + d)w = 1$, then $1 = cw + dw < c + w \leq 1$, which is impossible. Let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $(c + d)w + \frac{2w}{n} < 1, w^2 - b + \frac{w}{n} > 0, b + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $\|T_i\| = 1$ for $i = 1, 2$, which is a contradiction.

Next, suppose that $w^2 + b < 0$. Then, $1 = \max\{a, w + c, -(w^2 + b) + (c + d)w, w^2 - b + (c - d)w\}$. We will show that

$$Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, -1), (w, -1)), ((w, -1), (w, 1))\}.$$

Otherwise, we have the following subcases to consider:

$$Norm(T) \subseteq \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, -1), (w, -1))\}$$

or

$$Norm(T) \subseteq \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, -1), (w, 1))\}$$

or
$$Norm(T) \subseteq \{((1,0),(1,0)),((w,-1),(w,-1)),((w,-1),(w,1))\}.$$
First, suppose that
$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,-1))\}.$$
Let $T_1 = (1, b+\frac{w}{n}, c, d+\frac{1}{n}), T_2 = (1, b-\frac{w}{n}, c, d-\frac{1}{n})$ for some $n \in \mathbb{N}$ with $w^2+b+\frac{w}{n} < 0, w^2-b-\frac{w}{n} > 0, |b|+\frac{w}{n} < 1, d+\frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1+T_2)$ and $\|T_i\| = 1$ for $i = 1,2$, which is a contradiction.
Suppose that
$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,1))\}.$$
Then, $-(w^2+b)+(c+d)w < 1$. Let $T_1 = (1, b-\frac{w}{n}, c, d+\frac{1}{n}), T_2 = (1, b+\frac{w}{n}, c, d-\frac{1}{n})$ for some $n \in \mathbb{N}$ with $-(w^2+b)+(c+d)w+\frac{2w}{n} < 1, |b|+\frac{w}{n} < 1, d+\frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1+T_2)$ and $\|T_i\| = 1$ for $i = 1,2$, which is a contradiction.
Suppose that
$$Norm(T) \subseteq \{((1,0),(1,0)),((w,-1),(w,-1)),((w,-1),(w,1))\}.$$
Then, $w+c < 1$. Let $T_1 = (1, b+\frac{w}{n}, c+\frac{1}{n}, d), T_2 = (1, b-\frac{w}{n}, c-\frac{1}{n}, d)$ for some $n \in \mathbb{N}$ with $w+c+\frac{1}{n} < 1, w^2+b+\frac{w}{n} < 0, w^2-b-\frac{w}{n} > 0, |b|+\frac{w}{n} < 1$. Then $T = \frac{1}{2}(T_1+T_2)$ and $\|T_i\| = 1$ for $i = 1,2$, which is a contradiction. Therefore, we conclude the claim. By a calculation, $T = (1, -(w^2-w+1), 1-w, w)$ for $w < \frac{1}{2}$.

Claim: $T = (1, -(1-w+w^2), 1-w, w)$ is extreme for $w < \frac{1}{2}$.

Let $T_1 = (1+\epsilon, -(1-w+w^2)+\delta, 1-w+\gamma, w+\rho), T_2 = (1-\epsilon, -(1-w+w^2)-\delta, 1-w-\gamma, w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1, |T_i((1,0),(w,1))| \leq 1, |T_i((w,-1),(w,1))| \leq 1, |T_i((w,-1),(w,-1))| \leq 1$, we have
$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w^2\epsilon - \delta + w\gamma - w\rho &= 0, \\ w^2\epsilon + \delta - w\gamma - w\rho &= 0, \end{aligned}$$
which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Case 3: $c = d \geq 0$.

Since $T = (a,b,c,c) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$, then $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})}$. By Theorem 2.3, for $0 < w \leq \frac{1}{2}$,
$$\begin{aligned} T \in \{&(0,\pm1,0,0),(1,(1-w)^2,1-w,1-w),(1,1-w^2,0,0), \\ &(1,w^2-1,w,w),(0,1-2w,1,1),(1,-3w^2+2w-1,1-w,1-w)\} \end{aligned}$$
and for $\frac{1}{2} < w < 1$, $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})}$ if and only if
$$T \in \{(0,\pm1,0,0),(1,(1-w)^2,1-w,1-w),(1,1-w^2,0,0),$$

$$(1, w^2-1, 1-w, 1-w), (\frac{1}{2w}, \frac{w-2}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}.$$

Claim: $(1, 1-w^2, 0, 0) \notin B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$ for $0 < w < 1$.

Let $T_1 = (1, 1-w^2, -\frac{1}{n}, \frac{1}{n})$ and $T_2 = (1, 1-w^2, \frac{1}{n}, -\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ such that $\|T_i\| = 1$ for $i = 1, 2$. Since $(1, 1-w^2, 0, 0) = \frac{1}{2}(T_1 + T_2)$, $(1, 1-w^2, 0, 0)$ is not extreme.

Claim: $T = (0, \pm 1, 0, 0) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$ for $0 < w < 1$.

Note that

$$\begin{aligned} Norm(T) = \{&((w,1),(w,1)), ((w,-1),(w,1)), ((w,1),(w,-1)), \\ &((w,-1),(w,-1))\}. \end{aligned}$$

Let $T_1 = (\epsilon, 1+\delta, \gamma, \rho)$ and $T_2 = (\epsilon, 1-\delta, -\gamma, -\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((w,1),(w,1))| \leq 1$, $|T_i((w,-1),(w,1))| \leq 1$, $|T_i((w,1),(w,-1))| \leq 1$, $|T_i((w,-1),(w,-1))| \leq 1$, we have

$$\begin{aligned} w^2\epsilon + \delta + w\gamma + w\rho &= 0, \\ w^2\epsilon - \delta + w\gamma - w\rho &= 0, \\ w^2\epsilon + \delta - w\gamma - w\rho &= 0, \\ w^2\epsilon - \delta - w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

First, suppose that $0 < w \leq \frac{1}{2}$.

Claim: $T = (1, (1-w)^2, 1-w, 1-w) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$.

Note that

$$Norm(T) = \{((1,0),(1,0)), ((1,0),(w,1)), ((w,1),(1,0)), ((w,1),(w,1))\}.$$

Let $T_1 = (1+\epsilon, (1-w)^2+\delta, 1-w+\gamma, 1-w+\rho)$ and $T_2 = (1-\epsilon, (1-w)^2-\delta, 1-w-\gamma, 1-w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1$, $|T_i((1,0),(w,1))| \leq 1$, $|T_i((w,1),(1,0))| \leq 1$, $|T_i((w,1),(w,1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon + \delta + w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (1, w^2-1, w, w) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$.

Note that

$$\begin{aligned} Norm(T) = \{&((1,0),(1,0)), ((w,-1),(w,-1)), ((w,1),(w,-1)), \\ &((w,-1),(w,1))\}. \end{aligned}$$

Let $T_1 = (1+\epsilon, w^2-1+\delta, w+\gamma, w+\rho)$ and $T_2 = (1-\epsilon, w^2-1-\delta, w-\gamma, w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \le 1$, $|T_i((w,-1),(w,-1))| \le 1$, $|T_i((w,1),(w,-1))| \le 1$, $|T_i((w,-1),(w,1))| \le 1$, we have

$$\begin{aligned}\epsilon &= 0,\\ w^2\epsilon + \delta - w\gamma - w\rho &= 0,\\ w^2\epsilon - \delta - w\gamma + w\rho &= 0,\\ w^2\epsilon - \delta + w\gamma - w\rho &= 0,\end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (0, 1-2w, 1, 1) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$\begin{aligned}Norm(T) = \{&((1,0),(w,1)),((w,1),(1,0)),((1,0),(w,-1)),((w,-1),(1,0)),\\ &((w,1),(w,1))\}.\end{aligned}$$

Let $T_1 = (\epsilon, 1-2w+\delta, 1+\gamma, 1+\rho)$ and $T_2 = (-\epsilon, 1-2w-\delta, 1-\gamma, 1-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(w,1))| \le 1$, $|T_i((w,1),(1,0))| \le 1$, $|T_i((1,0),(w,-1))| \le 1$, $|T_i((w,1),(w,1))| \le 1$, we have

$$\begin{aligned}w\epsilon + \gamma &= 0,\\ w\epsilon + \rho &= 0,\\ w\epsilon - \gamma &= 0,\\ w^2\epsilon + \delta + w\gamma + w\rho &= 0,\end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (1, -3w^2+2w-1, 1-w, 1-w) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$Norm(T) = \{((1,0),(1,0)),((1,0),(w,1)),((w,1),(1,0)),((w,-1),(w,-1))\}.$$

Let $T_1 = (1+\epsilon, -3w^2+2w-1+\delta, 1-w+\gamma, 1-w+\rho)$ and $T_2 = (1-\epsilon, -3w^2+2w-1-\delta, 1-w-\gamma, 1-w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \le 1$, $|T_i((1,0),(w,1))| \le 1$, $|T_i((w,1),(1,0))| \le 1$, $|T_i((w,-1),(w,-1))| \le 1$, we have

$$\begin{aligned}\epsilon &= 0,\\ w\epsilon + \gamma &= 0,\\ w\epsilon + \rho &= 0,\\ w^2\epsilon + \delta - w\gamma - w\rho &= 0,\end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Next, suppose that $\frac{1}{2} < w < 1$.

Claim: $T = (1, (1-w)^2, 1-w, 1-w) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$Norm(T) = \{((1,0),(1,0)),((1,0),(w,1)),((w,1),(1,0)),((w,1),(w,1))\}.$$

Let $T_1 = (1+\epsilon,(1-w)^2+\delta,1-w+\gamma,1-w+\rho)$ and $T_2 = (1-\epsilon,(1-w)^2-\delta,1-w-\gamma,1-w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon,\delta,\gamma,\rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1$, $|T_i((1,0),(w,1))| \leq 1$, $|T_i((w,1),(1,0))| \leq 1$, $|T_i((w,1),(w,1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon + \delta + w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (1, w^2-1, 1-w, 1-w) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$\begin{aligned} Norm(T) = \{&((1,0),(1,0)),((1,0),(w,1)),((w,1),(1,0)),((w,1),(w,-1)), \\ &((w,-1),(w,1))\}. \end{aligned}$$

Let $T_1 = (1+\epsilon,w^2-1+\delta,1-w+\gamma,1-w+\rho)$ and $T_2 = (1-\epsilon,w^2-1-\delta,1-w-\gamma,1-w-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon,\delta,\gamma,\rho \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1$, $|T_i((1,0),(w,1))| \leq 1$, $|T_i((w,1),(1,0))| \leq 1$, $|T_i((w,1),(w,-1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon - \delta - w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (\frac{1}{2w}, \frac{2-w}{2}, \frac{1}{2}, \frac{1}{2}) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$Norm(T)=\{((1,0),(w,1)),((w,1),(1,0)),((w,-1),(w,-1)),((w,1),(w,-1))\}.$$

Let $T_1 = (\frac{1}{2w}+\epsilon, \frac{2-w}{2}+\delta, \frac{1}{2}+\gamma, \frac{1}{2}+\rho)$ and $T_2 = (\frac{1}{2w}-\epsilon, \frac{2-w}{2}-\delta, \frac{1}{2}-\gamma, \frac{1}{2}-\rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon,\delta,\gamma,\rho \in \mathbb{R}$. Since $|T_i((1,0),(w,1))| \leq 1$, $|T_i((w,1),(1,0))| \leq 1$, $|T_i((w,-1),(w,-1))| \leq 1$, $|T_i((w,1),(w,-1))| \leq 1$, we have

$$\begin{aligned} w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon + \delta - w\gamma - w\rho &= 0, \\ w^2\epsilon - \delta - w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \frac{1}{2w}, \frac{1}{2w}) \in extB_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$.

Note that

$$Norm(T) = \{((1,0),(w,1)),((w,1),(1,0)),((w,1),(w,1)),((w,-1),(w,-1))\}.$$

Let $T_1 = (\frac{2w-1}{2w^2} + \epsilon, \frac{1-2w}{2} + \delta, \frac{1}{2w} + \gamma, \frac{1}{2w} + \rho)$ and $T_2 = (\frac{2w-1}{2w^2} - \epsilon, \frac{1-2w}{2} - \delta, \frac{1}{2w} - \gamma, \frac{1}{2w} - \rho)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0),(w,1))| \leq 1$, $|T_i((w,1),(1,0))| \leq 1$, $|T_i((w,1),(w,1))| \leq 1$, $|T_i((w,-1),(w,-1))| \leq 1$, we have

$$w\epsilon + \gamma = 0,$$
$$w\epsilon + \rho = 0,$$
$$w^2\epsilon + \delta + w\gamma + w\rho = 0,$$
$$w^2\epsilon + \delta - w\gamma - w\rho = 0,$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$. Therefore, we complete the proof. □

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