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ON A CLASS OF FINSLER METRICS WITH ISOTROPIC BERWALD CURVATURE

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Abstract. In this paper, we study a class of Finsler metrics called general (α, β)-metrics, which are defined by a Riemannian metric α and a 1-form β. We show that every general (α, β)-metric with isotropic Berwald curvature is either a Berwald metric or a Randers metric. Moreover, a lot of new isotropic Berwald general (α, β)-metrics are constructed explicitly.

1. Introduction

In Finsler geometry, the Berwald curvature is an important non-Riemannian quantity. A Finsler metric F on a manifold M is said to be of *isotropic Berwald curvature* if its Berwald curvature ${B_j}^i{}_{kl}$ satisfies

$$B_j{}^i{}_{kl} = \tau(x)(F_{y^jy^k}\delta^i{}_l + F_{y^jy^l}\delta^i{}_k + F_{y^ly^k}\delta^i{}_j + F_{y^jy^ky^l}y^i), \tag{1.1}$$

where $\tau(x)$ is a scalar function on M. A Finsler metric is called a *Berwald metric* if $\tau(x) = 0$. Berwald metrics are just a bit more general than Riemannian and locally Minkowskian metrics. A Berwald space is that all tangent spaces are linearly isometric to a common Minkowski space.

Chen-Shen showed that a Finsler metric F is of isotropic Berwald curvature if and only if it is a Douglas metric with isotropic mean Berwald curvature [6]. Tayebi-Rafie's result tells us that every isotropic Berwald metric is of isotropic S-curvature [16]. In [15], Tayebi-Najafi proved that isotropic Berwald metrics of scalar flag curvature are of Randers type. Recently, Guo-Liu-Mo have shown that every spherically symmetric Finsler metric of isotropic Berwald curvature is a Randers metric [10]. Hence, isotropic Berwald metrics form a rich class of Finsler metrics.

Let us look at two important examples.

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(a) In [2], Bao-Robles-Shen perturb the Euclidean metric. The resulting Randers metric $F = \alpha + \beta$, where

$$\alpha := \frac{\sqrt{\varepsilon^2(xu + yv + zw)^2) + (u^2 + v^2 + w^2)[1 - \varepsilon^2(x^2 + y^2 + z^2)]}}{1 - \varepsilon^2(x^2 + y^2 + z^2)},$$

$$\beta := \frac{-\varepsilon(xu + yv + zw)}{1 - \varepsilon^2(x^2 + y^2 + z^2)},$$

is of constant S-curvature and satisfies $d\beta = 0$. By using Chen-Shen and Bacso-Matsumoto results, we obtain that F is a Douglas metric with isotropic mean Berwald curvature [1, 6]. Hence, F is of isotropic Berwald curvature.

(b) Consider a Minkowski norm $\psi : \mathbf{R}^n \to \mathbf{R}$ on $\mathbf{R}^n$. By using the Minkowski metric $F(x, y) = \psi(y)$ and the homothetic field $V_x = x$, we produce the Funk metric $\bar{F}$ on the strongly convex domain $\Omega := \{v \in \mathbf{R}^n \mid \psi(v) < 1\}$ [9]. Then $\bar{F}$ has isotropic Berwald curvature with $\tau = \frac{1}{2}$ [6]. In particular, when $\psi(y) = |y|$, then

$$\bar{F} = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2}. \tag{1.2}$$

are the well-known Funk metrics on the unit ball $\mathbf{B}^n(1)$.

Randers metrics, which are introduced by a physicist G. Randers in 1941 when he studied general relativity, are an important class of Finsler metrics. Generally, a Randers metric is of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form. But it can also be expressed in the following famous navigation form

$$F = \frac{\sqrt{(1 - \bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2}}{1 - \bar{b}^2} + \frac{\bar{\beta}}{1 - \bar{b}^2}, \tag{1.3}$$

where $\bar{b} := \|\bar{\beta}_x\|_{\bar{\alpha}}$. $(\bar{\alpha}, \bar{\beta})$ is called *the navigation data* of the Randers metric F. In [16], Tayebi-Rafie showed that if a Randers metric (1.3) is a non-trivial isotropic Berwald metric, then $\bar{\beta}$ is a conformal 1-form with respect to $\bar{\alpha}$, namely, $\bar{\beta}$ satisfies

$$\bar{b}_{i|j} + \bar{b}_{j|i} = c(x)\bar{a}_{ij},$$

where $c(x) \neq 0$ and $\bar{b}_{i|j}$ is the covariant derivation of $\bar{\beta}$ with respect to $\bar{\alpha}$.

In fact, the navigation expression (1.3) of Randers metrics is also given in the form

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big), \tag{1.4}$$

where α is a Riemannian metric, β is a 1-form, $b := \|\beta_x\|_\alpha$ and $\phi(b^2, s)$ is a smooth function. Such kind of Finsler metrics are called *general* (α, β)-*metrics* [17, 18, 20]. If $\phi = \phi(s)$ is independent of b^2, then $F = \alpha\phi(\frac{\beta}{\alpha})$ is an (α, β)-metric. If $\alpha = |y|$, $\beta = \langle x, y\rangle$, then $F = |y|\phi(|x|^2, \frac{\langle x,y\rangle}{|y|})$ is the so-called spherically symmetric Finsler metrics [12, 19]. Moreover, general (α, β)-metrics include part of Bryant's metrics [3, 17] and part of generalized fourth root metrics [11]. That is to say, general (α, β)-metrics make up of a much large class of Finsler metrics, which makes it possible to find out more Finsler metrics to be of great properties. For example, in (α, β)-metric we can't find out any non-Ricci flat Einstein metric unless it is of Randers type [8]. The main reason is that the category of (α, β) metrics is a little small. If we search Einstein metrics in general (α, β)-metrics, then it is not hard to find out metrics with positive and negative Ricci constant [14]. The classification of projective general (α, β)-metrics with constant flag curvature has just been completed recently by the author and C. Yu [18]. In this paper, we will show the following classification theorem:

Theorem 1.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a regular general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies*

$$b_{i|j} = ca_{ij}, \tag{1.5}$$

where $c = c(x) \neq 0$ is a scalar function on M. If F is of isotropic Berwald curvature, then one of the following holds:

(1) *F is a Berwald metric which can be expressed by*

$$F = \alpha\varphi\Big(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2} + s^2\int t_2 e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2}\Big)e^{\int(\frac{1}{2}b^2t_2-\frac{1}{b^2})db^2}s, \tag{1.6}$$

where $\varphi(\cdot)$ is any positive continuously differentiable function and t_2 is a smooth function of b^2.

(2) *F is a Randers metric which can be expressed by*

$$F = \sqrt{f(b^2)\alpha^2 + g(b^2)\beta^2} + h(b^2)\beta, \tag{1.7}$$

where the smooth functions $f(b^2)$, $g(b^2)$ and $h(b^2)$ satisfy

$$f[2(f + gb^2)h' - (2f' + g'b^2)h] = h(f + f'b^2)(g - h^2), \tag{1.8}$$

where $b^2 = \|\beta\|_\alpha$, $f' = \frac{\partial f}{\partial b^2}$, $g' = \frac{\partial g}{\partial b^2}$ and $h' = \frac{\partial h}{\partial b^2}$.

Remark. (1) we assume that β is closed and conformal with respect to α, i.e., (1.5) holds. According to the relate discussions for isotropic Berwald metrics [6, 10, 15, 16], we believe that the assumption here is reasonable and appropriate.

(2) It should be pointed out that if the scalar function $c(x) = 0$, then according to Proposition 3.1, $B_j{}^i{}_{kl} = 0$, namely, $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ is a Berwald metric for any function $\phi(b^2, s)$. So it will be regarded as a trivial case. Moreover, in the singular case, general (α, β)-metrics with isotropic Berwald curvature also contain Kropina metric.

By analyzing Eq. (1.8), we explicitly manufacture new Finsler metrics of isotropic Berwald curvature and determine their isotropic S-curvature. Precisely, we prove the following theorem:

Theorem 1.2. *The following Randers metric*

$$F=\sqrt{f(b^2)\alpha^2+\varepsilon^2 f^2(b^2)\beta^2}+\varepsilon f(b^2)\beta \tag{1.9}$$

has isotropic Berwald curvature, where β satisfies (1.5). *Furthermore, its S-curvature is given by*

$$S=\frac{(n+1)\varepsilon}{2}\frac{c(f+f'b^2)}{f(1+\varepsilon^2 fb^2)}F, \tag{1.10}$$

where $b^2:=\|\beta\|^2_\alpha$, $c=c(x)$ is a scalar function on M. $f=f(b^2)$ is an any differentiable function and ε is a constant.

Remark. (1) Take $\alpha=|y|$ and $\beta=\langle x,y\rangle$, then the corresponding general (α,β)-metrics of (1.9) have isotropic Berwald curvature. They are just spherically symmetric Finsler metrics ([10, Theorem 1.2]).

(2) Take $\alpha=|y|$ and $\beta=\langle x,y\rangle+\langle a,y\rangle$, where a is a constant vector. When $f(b^2)=\frac{1}{1+\xi b^2}$, where ξ is a constant. Then the corresponding general (α,β)-metrics of (1.9)

$$F=\frac{\sqrt{[1+\xi(|x|^2+2\langle a,x\rangle+|a|^2)]|y|^2+\varepsilon^2(\langle x,y\rangle+\langle a,x\rangle)^2}}{1+\xi(|x|^2+2\langle a,x\rangle+|a|^2)} + \frac{\varepsilon(\langle x,y\rangle+\langle a,x\rangle)}{1+\xi(|x|^2+2\langle a,x\rangle+|a|^2)}$$

are of isotropic Berwald curvature and their S-curvature is given by $S=\frac{n+1}{2}\frac{\varepsilon}{1+(\varepsilon^2+\xi)|x|^2}F$. In particular, when $\xi=-1$ and $\varepsilon=\pm1$, they are just the generalized Funk metrics expressed in some other local coordinate system. Furthermore, if $a=0$, $\xi=-1$ and $\varepsilon=\pm1$, then F is just the Funk metric (1.2).

Finally, it is worth mentioning that the S-curvature is an important non-Riemannian quantity in Finsler geometry [4, 7, 16]. It interacts with the flag curvature in a mysterious way. Recently, Cheng-Shen have characterized (α,β)-metrics with isotropic S-curvature [7].

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, which are defined by

$$G^i=\frac{1}{4}g^{il}\left\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\right\},$$

where $(g^{ij}):=\left(\frac{1}{2}[F^2]_{y^iy^j}\right)^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$.

Lemma 2.1 ([17])**.** *Let* $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ *be a general* (α, β)*-metric on an* n*-dimensional manifold* M*. Then the function* F *is a regular Finsler metric for any Riemannian metric* α *and any* 1*-form* β *if and only if* $\phi(b^2, s)$ *is a positive smooth function defined on the domain* $|s| \le b < b_o$ *for some positive number* (*maybe infinity*) b_o *satisfying*

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.1}$$

when $n \ge 3$ *or*

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.2}$$

when $n = 2$*.*

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Denote the coefficients of the covariant derivative of β with respect to α by $b_{i|j}$, and let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^iy^j,\ s^i{}_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^jr_{ji},\ s_i = b^js_{ji},\ r_0 = r_iy^i,\ s_0 = s_iy^i,\ r^i = a^{ij}r_j,\ s^i = a^{ij}s_j,\ r = b^ir_i,$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. It is easy to see that β is closed if and only if $s_{ij} = 0$.

Lemma 2.2 ([17])**.** *The spray coefficients* G^i *of a general* (α, β)*-metric* $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ *are related to the spray coefficients* ${}^\alpha G^i$ *of* α *and given by*

$$\begin{aligned} G^i = {}^\alpha G^i &+ \alpha Q s^i{}_0 + \big\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\big\}\frac{y^i}{\alpha} \\ &+ \big\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\big\}b^i - \alpha^2R(r^i + s^i), \end{aligned} \tag{2.3}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2},$$
$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Psi = \frac{\phi_{22}}{2\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)},$$
$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2. In the following, we will introduce an important non-Riemannian quantity.

Definition 2.3 ([13])**.** Let

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \tag{2.4}$$

where G^i are the spray coefficients of F. The tensor $B := B_j{}^i{}_{kl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called *Berwald tensor*. A Finsler metric is called *a Berwald metric* if the Berwald tensor vanishes, i.e., the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_xM$ at every point $x \in M$.

Let $\gamma(t)$ be the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. Let

$$S(x, y) = \frac{d}{dt}[\tau(\gamma(t)), \dot{\gamma}(t)]|_{t=0},$$

where $\tau(x, y)$ is the distortion of F. $S(x, y)$ is called the *S-curvature* [7, 9, 16]. F is said to have *isotropic S-curvature* if there is a scalar function $\kappa(x)$ on M such that

$$S = (n+1)\kappa(x)F. \tag{2.5}$$

The following result is given in [5]. It is required in Section 5.

Lemma 2.4. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Suppose that β is closed. Then F has isotropic S-curvature, i.e.,* (2.5) *holds, if and only if*

$$b_{i|j} = 2\kappa(x)(a_{ij} - b_i b_j). \tag{2.6}$$

By a straightforward computation, we have the following lemma.

Lemma 2.5. *Let*

$$\bar{\alpha} := \sqrt{f(b^2)\alpha^2 + g(b^2)\beta^2} \tag{2.7}$$

and

$$\bar{\beta} := h(b^2)\beta. \tag{2.8}$$

Then

$$\begin{aligned} \bar{b}_{i|j} = {} & hb_{i|j} + 2h'(r_j + s_j)b_i - \lambda(f' + g'b^2)[b_i(r_j + s_j) + b_j(r_i + s_i)] \\ & + \lambda(f'a_{ij} + g'b_i b_j)r - \lambda g(b^2 r_{ij} + s_j b_i + s_i b_j), \end{aligned} \tag{2.9}$$

where $\lambda := \frac{h}{f+gb^2}$.

Proof. It follows from (2.7) and (2.8) that

$$\bar{a}_{ij} = fa_{ij} + gb_i b_j, \ \bar{b}_i = hb_i. \tag{2.10}$$

By (2.10) and Chern-Shen's Lemma 1.1.1 [9], we obtain

$$\bar{a}^{ij} = \frac{1}{f}(a^{ij} - \frac{g}{f + gb^2}b^i b^j), \tag{2.11}$$

where $(a^{ij}) = (a_{ij}^{-1})$. Note that

$$(b^2)_{x^i} = 2(r_i + s_i). \tag{2.12}$$

Differentiating the first equality with respect to x^k yields

$$\begin{aligned} \frac{\partial \bar{a}_{ij}}{\partial x^k} &= f\frac{\partial a_{ij}}{\partial x^k} + 2f'(r_k + s_k)a_{ij} + 2g'(r_k + s_k)b_i b_j + g\big(\frac{\partial b_i}{\partial x^k}b_j + \frac{\partial b_j}{\partial x^k}b_i\big) \\ &= f\frac{\partial a_{ij}}{\partial x^k} + 2f'(r_k + s_k)a_{ij} + 2g'(r_k + s_k)b_i b_j \\ &\quad + g\big[(b_{i|k} + b_t\gamma^t_{ik})b_j + (b_{i|k} + b_t\gamma^t_{ik})b_j\big], \end{aligned} \tag{2.13}$$

where we have used (2.12). This implies that

$$\begin{aligned}\bar{\gamma}_{kij} &:= \frac{1}{2}\Big(\frac{\partial \bar{a}_{ik}}{\partial x^j} + \frac{\partial \bar{a}_{jk}}{\partial x^i} - \frac{\partial \bar{a}_{ij}}{\partial x^k}\Big)\\ &= f\gamma_{kij} + f'\big[a_{ik}(r_j+s_j) + a_{jk}(r_i+s_i) - a_{ij}(r_k+s_k)\big]\\ &\quad + g'\big[(r_j+s_j)b_ib_k + (r_i+s_i)b_jb_k - (r_k+s_k)b_ib_j\big]\\ &\quad + g(r_{ij}b_k + s_{kj}b_i + s_{ki}b_j + b_t\gamma^t_{ij}b_k).\end{aligned} \tag{2.14}$$

By (2.11) and (2.14), we have

$$\begin{aligned}\bar{\gamma}^l_{ij} &= \bar{a}^{lk}\bar{\gamma}_{kij}\\ &= \gamma^l_{ij} + \frac{f'}{f}[\delta^l_i(r_j+s_j) + \delta^l_j(r_i+s_i) - a_{ij}(r^l+s^l)]\\ &\quad + \frac{g'}{f}\Big\{\frac{f}{f+gb^2}[(r_j+s_j)b_ib^l + (r_i+s_i)b_jb^l]\\ &\qquad\qquad - (r^l+s^l)b_ib_j + \frac{rg}{f+gb^2}b^lb_jb_i\Big\}\\ &\quad + \frac{g}{f}\Big[\frac{f}{f+gb^2}r_{ij}b^l + s^l_jb_i + s^l_ib_j - \frac{g}{f+gb^2}(s_jb_ib^l + s_ib_jb^l)\Big]\\ &\quad - \frac{f'g}{f(f+gb^2)}[b^lb_i(r_j+s_j) + b^lb_j(r_i+s_i) - a_{ij}b^lr],\end{aligned} \tag{2.15}$$

where $r := b^ir_i$. Together with the second equality of (2.10), we have

$$\frac{\partial \bar{b}_i}{\partial x^j} = \frac{\partial b_i}{\partial x^j}h + 2h'(r_j+s_j)b_i = h(b_{i|j} + b_l\gamma^l_{ij}) + 2h'(r_j+s_j)b_i. \tag{2.16}$$

By using (2.15) and (2.16), we have

$$\begin{aligned}\bar{b}_{i|j} &= \frac{\partial \bar{b}_i}{\partial x^j} - \bar{b}_l\bar{\gamma}^l_{ij}\\ &= hb_{i|j} + 2h'(r_j+s_j)b_i - hb_l(\bar{\gamma}^l_{ij} - \gamma^l_{ij})\\ &= hb_{i|j} + 2h'(r_j+s_j)b_i - \lambda(f'+g'b^2)[b_i(r_j+s_j) + b_j(r_i+s_i)]\\ &\quad + \lambda(f'a_{ij} + g'b_ib_j)r - \lambda g(b^2r_{ij} + s_jb_i + s_ib_j),\end{aligned} \tag{2.17}$$

where $\lambda := \frac{h}{f+gb^2}$. □

3. Berwald curvature of general (α, β)-metrics

In this section, we will compute the Berwald curvature of a general (α,β)-metric.

Proposition 3.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β satisfies* (1.5)*, then the Berwald curvature of F is given by*

$$B_j{}^i{}_{kl} = \frac{c}{\alpha}\left\{[(E - sE_2)a_{kl} + E_{22}b_lb_k]\,\delta^i{}_j\right.$$

$$+\frac{1}{\alpha^2}\left[\frac{s}{\alpha}(3E_{22}+sE_{222})y_ly_j\right.$$
$$\left.-(E_{22}+sE_{222})b_ly_j\right]b_ky^i\Big\}\,(k\to l\to j\to k)$$
$$-\frac{c}{\alpha^2}\Big\{sE_{22}\left[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i\right]$$
$$+\frac{1}{\alpha}(E-sE_2-s^2E_{22})(y_l\delta^i{}_j+a_{lj}y^i)y_k\Big\}\,(k\to l\to j\to k)$$
$$+\frac{c}{\alpha^2}\left[\frac{1}{\alpha^3}(3E-3sE_2-6s^2E_{22}-s^3E_{222})y_ky_jy_l+E_{222}b_lb_kb_j\right]y^i$$
$$+\frac{c}{\alpha}\left[(H_2-sH_{22})(b_j-\frac{s}{\alpha}y_j)a_{kl}-\frac{1}{\alpha^2}(H_2-sH_{22}-s^2H_{222})b_ly_jy_k\right.$$
$$\left.-\frac{sH_{222}}{\alpha}b_kb_ly_j\right]b^i(k\to l\to j\to k)$$
$$+\frac{c}{\alpha}\left[\frac{s}{\alpha^3}(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_l+H_{222}b_lb_kb_j\right]b^i, \tag{3.1}$$

where $y_i := a_{ij}y^j$ *and* $b^i := a^{ij}b_j$, $c = c(x) \neq 0$ *is a scalar function on* M.

$$E := \frac{\phi_2+2s\phi_1}{2\phi} - H\frac{s\phi+(b^2-s^2)\phi_2}{\phi}, \tag{3.2}$$

$$H := \frac{\phi_{22}-2(\phi_1-s\phi_{12})}{2\big[\phi-s\phi_2+(b^2-s^2)\phi_{22}\big]}. \tag{3.3}$$

Proof. By (1.5), we have

$$r_{00}=c\alpha^2, r_0=c\beta, r=cb^2, r^i=cb^i, s^i{}_0=0, s_0=0, s^i=0. \tag{3.4}$$

Substituting (3.4) into (2.3) yields

$$G^i = {}^\alpha G^i + c\alpha\left\{\Theta(1+2Rb^2)+s\Omega\right\}y^i + c\alpha^2\left\{\Psi(1+2Rb^2)+s\Pi-R\right\}b^i$$
$$= {}^\alpha G^i + c\alpha E y^i + c\alpha^2 H b^i, \tag{3.5}$$

where E and H are given by (3.2) and (3.3) respectively. Note that

$$\alpha_{y^i} = \frac{y_i}{\alpha},\ \ s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2}, \tag{3.6}$$

where $y_i := a_{ij}y^j$. Put

$$W^i := \alpha E y^i + \alpha^2 H b^i. \tag{3.7}$$

Differentiating (3.7) with respect to y^j yields

$$\frac{\partial W^i}{\partial y^j} = \alpha E\delta^i{}_j + (E\alpha_{y^j}+\alpha E_2 s_{y^j})y^i + \left\{[\alpha^2]_{y^j}H + \alpha^2 H_2 s_{y^j}\right\}b^i. \tag{3.8}$$

Differentiating (3.8) with respect to y^k yields

$$\frac{\partial^2 W^i}{\partial y^j \partial y^k} = \left[(E\alpha_{y^k}+\alpha E_2 s_{y^k})\delta^i{}_j + E_2 s_{y^k}\alpha_{y^j}y^i + H_2[\alpha^2]_{y^j}s_{y^k}b^i\right](k\leftrightarrow j)$$

$$+\big(E\alpha_{y^jy^k}+\alpha E_{22}s_{y^k}s_{y^j}+\alpha E_2 s_{y^jy^k}\big)y^i$$
$$+\big\{[\alpha^2]_{y^jy^k}H+\alpha^2H_{22}s_{y^k}s_{y^j}+\alpha^2H_2s_{y^jy^k}\big\}b^i, \tag{3.9}$$

where $k\leftrightarrow j$ denotes symmetrization. Therefore, it follows from (3.9) that

$$\begin{aligned}\frac{\partial^3W^i}{\partial y^j\partial y^k\partial y^l} = &\big[E_2(\alpha_{y^k}s_{y^l}+\alpha_{y^l}s_{y^k}+\alpha s_{y^ky^l})\\ &+E\alpha_{y^ky^l}+\alpha E_{22}s_{y^l}s_{y^k}\big]\delta^i{}_j(k\to l\to j\to k)\\ &+\big[E_2(s_{y^k}\alpha_{y^jy^l}+\alpha_{y^k}s_{y^jy^l})\\ &+E_{22}(\alpha_{y^k}s_{y^j}+\alpha s_{y^ky^j})s_{y^l}\big]y^i(k\to l\to j\to k)\\ &+\{H_2\big([\alpha^2]_{y^ky^l}s_{y^j}+[\alpha^2]_{y^k}s_{y^jy^l}\big)\\ &+H_{22}\big([\alpha^2]_{y^k}s_{y^l}s_{y^j}+\alpha^2s_{y^ky^l}s_{y^j}\big)\}b^i(k\to l\to j\to k)\\ &+\big(E\alpha_{y^jy^ky^l}+\alpha E_{222}s_{y^j}s_{y^k}s_{y^l}+\alpha E_2s_{y^jy^ky^l}\big)y^i\\ &+\big\{H[\alpha^2]_{y^jy^ky^l}+\alpha^2H_{222}s_{y^j}s_{y^k}s_{y^l}+\alpha^2H_2s_{y^jy^ky^l}\big\}b^i,\end{aligned} \tag{3.10}$$

where $k\to l\to j\to k$ denotes cyclic permutation. It follows from (3.6) that

(3.11)
$$[\alpha^2]_{y^l}=2y_l,\ [\alpha^2]_{y^ly^j}=2a_{lj},\ [\alpha^2]_{y^ly^jy^k}=0,$$

(3.12)
$$\alpha_{y^ly^j}=\frac{1}{\alpha}\big(a_{lj}-\frac{y_l}{\alpha}\frac{y_j}{\alpha}\big),\ \alpha_{y^ly^jy^k}=-\frac{1}{\alpha^3}[a_{kl}y_j(k\to l\to j\to k)-\frac{3}{\alpha^2}y_ly_jy_k],$$

(3.13)
$$s_{y^ly^j}=-\frac{1}{\alpha^2}\big[sa_{lj}+\frac{1}{\alpha}(b_ly_j+b_jy_l)-\frac{3s}{\alpha^2}y_ly_j\big],$$

(3.14)
$$s_{y^ly^jy^k}=\frac{1}{\alpha^5}\big\{[\alpha(3sy_j-\alpha b_j)a_{lk}+3b_ky_ly_j](k\to l\to j\to k)-\frac{15s}{\alpha}y_ky_ly_j\big\}.$$

Plugging (3.11)-(3.14) into (3.10) yields

$$\begin{aligned}\frac{\partial^3W^i}{\partial y^j\partial y^k\partial y^l} = \frac{1}{\alpha}&\,\big\{\big[(E-sE_2)a_{kl}+E_{22}b_lb_k\big]\delta^i{}_j\\ &+\frac{1}{\alpha^2}\big[\frac{s}{\alpha}(3E_{22}+sE_{222})y_l\\ &-(E_{22}+sE_{222})b_l\big]y_jb_ky^i\big\}\,(k\to l\to j\to k)\\ &-\frac{1}{\alpha^2}\,\big\{sE_{22}\big[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i\big]\\ &+\frac{1}{\alpha}(E-sE_2-s^2E_{22})(y_l\delta^i_j+a_{jl}y^i)y_k\}(k\to l\to j\to k)\\ +\frac{1}{\alpha^2}&\left[\frac{1}{\alpha^3}(3E-3sE_2-6s^2E_{22}-s^3E_{222})y_ky_jy_l+E_{222}b_lb_kb_j\right]y^i\end{aligned}$$

$$+\frac{1}{\alpha}\Big[(H_2-sH_{22})(b_j-\frac{s}{\alpha}y_j)a_{kl} -\frac{1}{\alpha^2}(H_2-sH_{22}-s^2H_{222})b_ly_jy_k -\frac{sH_{222}}{\alpha}b_kb_ly_j\Big]b^i(k\to l\to j\to k)$$

$$+\frac{1}{\alpha}\Big[\frac{s}{\alpha^3}(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_l+H_{222}b_lb_kb_j\Big]b^i. \tag{3.15}$$

It follows from ${}^{\alpha}G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ that

$$\frac{\partial^{3\alpha}G^i}{\partial y^j\partial y^k\partial y^l}=0. \tag{3.16}$$

By (2.4), (3.5), (3.7), (3.15) and (3.16), we obtain (3.1). □

4. General (α,β)-metrics with isotropic Berwald curvature

In this section, we will classify general (α,β)-metrics with isotropic Berwald curvature under certain condition. Firstly, we show the following:

Lemma 4.1. *Suppose that β satisfies* (1.5)*, then a general (α,β)-metric $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ is of isotropic Berwald curvature if and only if*

$$E-sE_2-\rho(x)(\phi-s\phi_2)=0, \tag{4.1}$$

$$H_2-sH_{22}=0, \tag{4.2}$$

where $\rho(x)=\frac{\tau(x)}{c(x)}$, E and H are given by (3.2) *and* (3.3) *respectively. In particular, F is a Berwald metric if and only if $E-sE_2=H_2-sH_{22}=0$.*

Proof. For a general (α,β)-metric $F=\alpha\phi(b^2,s)$, a direct computation yields

$$F_{y^j}=\alpha_{y^j}\phi+\alpha\phi_2s_{y^j},$$

$$F_{y^jy^k}=\alpha_{y^jy^k}\phi+(\alpha_{y^j}s_{y^k}+\alpha_{y^k}s_{y^j})\phi_2+\alpha\phi_{22}s_{y^k}s_{y^j}+\alpha\phi_2s_{y^jy^k}, \tag{4.3}$$

$$F_{y^jy^ky^l}=[(\alpha_{y^jy^k}s_{y^l}+\alpha_{y^j}s_{y^ky^l})\phi_2 +(\alpha_{y^j}s_{y^l}+\alpha s_{y^jy^l})s_{y^k}\phi_{22}](j\to k\to l\to j) +\alpha_{y^jy^ky^l}\phi+\alpha\phi_{222}s_{y^l}s_{y^k}s_{y^j}+\alpha\phi_2s_{y^jy^ky^l}. \tag{4.4}$$

Plugging (3.6) and (3.12)-(3.14) into (4.3) and (4.4) yields

$$F_{y^jy^k}=\frac{1}{\alpha}(\phi-s\phi_2)a_{jk}-\frac{s\phi_{22}}{\alpha^2}(b_ky_j+b_jy_k)+\frac{\phi_{22}}{\alpha}b_jb_k -\frac{1}{\alpha^3}(\phi-s\phi_2-s^2\phi_{22})y_jy_k, \tag{4.5}$$

$$F_{y^jy^ky^l}=\frac{1}{\alpha^2}\big[\frac{1}{\alpha}(s\phi_2+s^2\phi_{22}-\phi)a_{kl}y_j+\frac{s}{\alpha^2}(3\phi_{22}+s\phi_{222})b_ky_ly_j-s\phi_{22}a_{jl}b_k -\frac{1}{\alpha}(\phi_{22}+s\phi_{222})b_lb_jy_k\big](j\to k\to l\to j)$$

$$+\frac{1}{\alpha^5}(\phi-3s\phi_2-6s^2\phi_{22}-s^3\phi_{222})y_jy_ky_l+\frac{1}{\alpha^2}\phi_{222}b_lb_kb_j. \tag{4.6}$$

Suppose that F is of isotropic Berwald curvature, by (1.1), (4.5) and (4.6), we obtain

$$\begin{aligned}B_j{}^i{}_{kl}=&\frac{\tau(x)}{\alpha}\big[(\phi-s\phi_2)a_{jk}-\frac{s\phi_{22}}{\alpha}(b_ky_j+b_jy_k)+\phi_{22}b_jb_k\\&-\frac{1}{\alpha^2}(\phi-s\phi_2-s^2\phi_{22})y_jy_k\big]\delta^i{}_l(j\to k\to l\to j)\\&+\frac{\tau(x)}{\alpha^2}\big[\frac{1}{\alpha}(s\phi_2+s^2\phi_{22}-\phi)a_{kl}y_j+\frac{s}{\alpha^2}(3\phi_{22}+s\phi_{222})b_ky_ly_j\\&-s\phi_{22}a_{jl}b_k-\frac{1}{\alpha}(\phi_{22}+s\phi_{222})b_lb_jy_k\big]y^i(j\to k\to l\to j)\\&+\frac{\tau(x)}{\alpha^2}\big[\frac{1}{\alpha^3}(3\phi-3s\phi_2-6s^2\phi_{22}-s^3\phi_{222})y_jy_ky_l+\phi_{222}b_lb_kb_j\big]y^i.\end{aligned} \tag{4.7}$$

By (3.1) and (4.7), we obtain

$$T_1+\alpha T_2=0, \tag{4.8}$$

where

$$\begin{aligned}T_1:=&\alpha^4\big\{[E-sE_2-\rho(\phi-s\phi_2)]a_{kl}\delta^i_j+(E_{22}-\rho\phi_{22})b_lb_k\delta^i_j\\&+(H_2-sH_{22})a_{kl}b_j\big\}(j\to k\to l\to j)\\&-\alpha^2[E-sE_2-s^2E_{22}-\rho(\phi-s\phi_2-s^2\phi_{22})]\\&(y_l\delta^i_j+a_{lj}y^i)y_k(j\to k\to l\to j)+\alpha^4H_{222}b_lb_kb_jb^i\\&-\alpha^2\big\{[E_{22}+sE_{222}-\rho(\phi_{22}+s\phi_{222})]b_ky^i\\&+(H_2-sH_{22}-s^2H_{222})y_kb^i)\big\}b_ly_j(j\to k\to l\to j)\\&+[3E-3sE_2-6s^2E_{22}-s^3E_{222}-\rho(3\phi-3s\phi_2-6s^2\phi_{22}-s^3\phi_{222})]\\&y_jy_ky_ly^i,\end{aligned} \tag{4.9}$$

$$\begin{aligned}T_2:=&-\alpha^2s\big\{(E_{22}-\rho\phi_{22})[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i]\\&+[(H_2-sH_{22})y_ja_{kl}+H_{222}b_kb_ly_j]b^i\big\}(j\to k\to l\to j)\\&+s\big[3E_{22}+sE_{222}-\rho(3\phi_{22}+s\phi_{222})\big]y_ly_jb_ky^i(j\to k\to l\to j)\\&+\alpha^2(E_{222}-\rho\phi_{222})b_lb_kb_jy^i+s(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_lb^i.\end{aligned} \tag{4.10}$$

By (4.8), we know that

$$T_1=0,\ T_2=0.$$

For $s\neq 0$, it follows from $T_2y^jy^ky^l=0$ that

$$(E_{222}-\rho\phi_{222})y^i-\alpha H_{222}b^i=0. \tag{4.11}$$

Both rational part and irrational part of (4.11) equal zero, namely

$$E_{222}-\rho\phi_{222}=0,\ H_{222}=0. \tag{4.12}$$

Plugging (4.12) into $T_2 = 0$ yields

$$\begin{aligned}&-\alpha^2\big\{(E_{22}-\rho\phi_{22})[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i]\\&+(H_2-sH_{22})a_{kl}y_jb^i\big\}(j\to k\to l\to j)\\&+3(E_{22}-\rho\phi_{22})y_ly_jb_ky^i(j\to k\to l\to j)\\&+3(H_2-sH_{22})y_jy_ky_lb^i=0.\end{aligned}\tag{4.13}$$

Contracting (4.13) by $b^jb^kb^l$ yields

$$(E_{22}-\rho\phi_{22})b^2(b^2-3s^2)y^i+\alpha s[2b^2(E_{22}-\rho\phi_{22})+(H_2-sH_{22})(b^2-s^2)]b^i=0.\tag{4.14}$$

Hence, it follows from (4.14) that

$$E_{22}-\rho\phi_{22}=0,\ \ H_2-sH_{22}=0.\tag{4.15}$$

Inserting (4.12) and (4.15) into $T_1 = 0$ yields

$$\begin{aligned}&[E-sE_2-\rho(\phi-s\phi_2)]\big\{\alpha^2[\alpha^2a_{kl}\delta^i_j-(y_l\delta^i_j+a_{lj}y^i)y_k](j\to k\to l\to j)\\&+3y_jy_ky_ly^i\big\}=0.\end{aligned}\tag{4.16}$$

Multiplying (4.16) by $b^jb^kb^l$ yields

$$[E-sE_2-\rho(\phi-s\phi_2)](b^2-s^2)(\alpha b^i-sy^i)=0.\tag{4.17}$$

Hence, it is easy to see from (4.17) that

$$E-sE_2-\rho(\phi-s\phi_2)=0.\tag{4.18}$$

Note that

$$\begin{aligned}E_{222}-\rho\phi_{222}&=(E_{22}-\rho\phi_{22})_2,\\ s(E_{22}-\rho\phi_{22})&=-\big[E-sE_2-\rho(\phi-s\phi_2)\big]_2,\\ sH_{222}&=-(H_2-sH_{22})_2.\end{aligned}$$

Therefore, (4.18) implies that the first equalities of (4.12) and (4.15) hold. The second equality of (4.15) implies that the second equality of (4.12) holds. Moreover, if a general (α,β)-metric $F=\alpha\phi(b^2,s)$ is of isotropic Berwald curvature, then (4.1) and (4.2) hold.

Conversely, suppose that (4.1) and (4.2) hold. Setting $\psi := E-\rho(x)\phi$, then (4.1) is equivalent to

$$\psi-s\psi_2=0.\tag{4.19}$$

By solving Eq. (4.19), we obtain $\psi=\frac{1}{2}\sigma(b^2)s$. Hence,

$$E-\rho(x)\phi=\frac{1}{2}\sigma(b^2)s.\tag{4.20}$$

By (4.2), there exist two functions $t_1(b^2)$ and $t_2(b^2)$ such that

$$H=\frac{1}{2}\big[t_1(b^2)+t_2(b^2)s^2\big].\tag{4.21}$$

By (3.5), (4.20) and (4.21),

$$G^i - \tau(x)Fy^i = {}^{\alpha}G^i + c(x)\alpha E y^i + c(x)\alpha^2 H b^i - \tau(x)Fy^i$$
$$= {}^{\alpha}G^i + \alpha\big[c(x)E - \tau(x)\phi\big]y^i + \frac{1}{2}c(x)\alpha^2\big[t_1(b^2) + t_2(b^2)s^2\big]b^i$$
$$= {}^{\alpha}G^i + \frac{1}{2}c(x)\big\{\sigma(b^2)\beta y^i + [t_1(b^2)\alpha^2 + t_2(b^2)\beta^2]b^i\big\}. \tag{4.22}$$

Hence, G^i are quadratic in $y = y^i\frac{\partial}{\partial x^i}|_x$. On the other hand, it follows from (2.4) that

$$(G^i - \tau(x)Fy^i)_{y^j y^k y^l} = B_j{}^i{}_{kl} - \tau(x)(F_{y^j y^k}\delta^i{}_l + F_{y^j y^l}\delta^i{}_k + F_{y^l y^k}\delta^i{}_j + F_{y^j y^k y^l}y^i). \tag{4.23}$$

Using (4.22) and (4.23), we obtain that $F = \alpha\phi(b^2, s)$ is of isotropic Berwald curvature.

Observe that F is a Berwald metric if and only if its Berwald curvature $B_j{}^i{}_{kl} = 0$. By (1.1), we obtain that F is a Berwald metric if and only if $\tau(x) = 0$, i.e., $\rho(x) = 0$. Hence, by the above process of proof, we get that F is a Berwald metric if and only if $E - sE_2 = H_2 - sH_{22} = 0$. □

Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a general (α,β)-metric with isotropic Berwald curvature. From Lemma 4.1 and its proof, we see that there exist three functions $\sigma(b^2)$, $t_1(b^2)$ and $t_2(b^2)$ such that (4.20) and (4.21) hold. Plugging (4.20)-(4.21) into (3.2) and (3.3) yields

$$\frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{1}{2}(t_1 + t_2 s^2)\frac{s\phi + (b^2 - s^2)\phi_2}{\phi} = \rho\phi + \frac{1}{2}\sigma s, \tag{4.24}$$

$$\frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\big[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big]} = \frac{1}{2}\big(t_1 + t_2 s^2\big), \tag{4.25}$$

where we use ρ, σ, t_1 and t_2 instead of $\rho(x)$, $\sigma(b^2)$, $t_1(b^2)$ and $t_2(b^2)$, respectively. (4.24) and (4.25) are equivalent to

$$\big[1 - (t_1 + t_2 s^2)(b^2 - s^2)\big]\phi_2 + 2s\phi_1 - s\big[(t_1 + t_2 s^2) + \sigma\big]\phi - 2\rho\phi^2 = 0, \tag{4.26}$$

$$\big[1 - (b^2 - s^2)(t_1 + t_2 s^2)\big]\phi_{22} - 2\phi_1 + 2s\phi_{12} + s(t_1 + t_2 s^2)\phi_2 - (t_1 + t_2 s^2)\phi = 0. \tag{4.27}$$

Differentiating (4.26) with respect to s yields

$$\big[1 - (t_1 + t_2 s^2)(b^2 - s^2)\big]\phi_{22} + 2\phi_1 + 2s\phi_{12} + s\big(t_1 - \sigma - 2b^2 t_2 + 3t_2 s^2\big)\phi_2 - \big(t_1 + \sigma + 3t_2 s^2\big)\phi - 4\rho\phi\phi_2 = 0. \tag{4.28}$$

From (4.28) – (4.27), we obtain

$$4\phi_1 - s\big[2t_2(b^2 - s^2) + \sigma\big]\phi_2 - (\sigma + 2t_2 s^2)\phi - 4\rho\phi\phi_2 = 0. \tag{4.29}$$

From (4.26) × 2 – (4.29) × s, we obtain

$$\big[2 - 2t_1(b^2 - s^2) + \sigma s^2\big]\phi_2 - (2t_1 + \sigma)s\phi - 4\rho\phi^2 + 4\rho s\phi\phi_2 = 0. \tag{4.30}$$

Note that (4.30) is equivalent to

$$(4.31)\qquad \Big(\frac{2-2t_1(b^2-s^2)+\sigma s^2}{\phi^2}\Big)_2+\Big(\frac{8\rho s}{\phi}\Big)_2=0.$$

Case 1. $\rho\neq 0$

1) $2-2t_1(b^2-s^2)+\sigma s^2\neq 0$.

Integrating (4.31) yields

$$(4.32)\qquad \phi=\frac{4\rho s+\sqrt{2k(1-t_1b^2)+(16\rho^2+2kt_1+k\sigma)s^2}}{k},$$

where $k=k(b^2)$ is any non-zero smooth function. Then the corresponding general (α,β)-metric $F=\alpha\phi(b^2,s)$ is a Randers metric.

2) $2-2t_1(b^2-s^2)+\sigma s^2=0$.

In this case, (4.31) is reduced to $(\frac{s}{\phi})_2=0$, it is easy to obtain that $\phi=\frac{s}{a(b^2)}$. Hence, the corresponding general (α,β)-metric $F=\alpha\phi(b^2,s)$ is a Kropina metric. This metric is singular. We will omit it because the Finsler metric discussed is assumed to be regular.

Case 2. $\rho=0$

In this case, it follows from (1.1) that F is a Berwald metric. (4.30) is reduced to

$$(4.33)\qquad \big[2-2t_1(b^2-s^2)+\sigma s^2\big]\phi_2-(2t_1+\sigma)s\phi=0.$$

i) $2-2t_1(b^2-s^2)+\sigma s^2\neq 0$.

By (4.33), we obtain

$$(4.34)\qquad \phi=t_3(b^2)\sqrt{2(1-b^2t_1)+(\sigma+2t_1)s^2},$$

where $t_3(b^2)$ is any positive smooth function. Hence, in this case, the corresponding general (α,β)-metric $F=\alpha\phi(b^2,s)$ is a Riemannian metric.

ii) $2-2t_1(b^2-s^2)+\sigma s^2=0$.

Note that $\phi>0$ and $s\neq 0$. In this case, (4.33) is equivalent to

$$(4.35)\qquad \sigma+2t_1=0,\ 2-2(b^2-s^2)t_1+\sigma s^2=0.$$

By (4.35), we have

$$(4.36)\qquad \sigma=-\frac{2}{b^2},\ t_1=\frac{1}{b^2}.$$

In this case, (4.29) imply that (4.26) holds. By the above calculations, it is easy to see that (4.26) and (4.29) imply (4.27). Therefore, we only need to solve (4.29). Plugging (4.36) into (4.29) yields

$$(4.37)\qquad \phi_1+\frac{1}{2}s[\frac{1}{b^2}-(b^2-s^2)t_2]\phi_2=\frac{1}{2}(-\frac{1}{b^2}+t_2s^2)\phi.$$

The characteristic equation of PDE (4.37) is

$$(4.38)\qquad \frac{db^2}{1}=\frac{ds}{\frac{1}{2}s[\frac{1}{b^2}-(b^2-s^2)t_2]}=\frac{d\phi}{\frac{1}{2}(-\frac{1}{b^2}+t_2s^2)\phi}.$$

Firstly, we solve

$$\frac{db^2}{1} = \frac{ds}{\frac{1}{2}s[\frac{1}{b^2} - (b^2 - s^2)t_2]}. \tag{4.39}$$

(4.39) is equivalent to

$$\frac{ds}{db^2} = \frac{1}{2}(\frac{1}{b^2} - b^2 t_2)s + \frac{1}{2}t_2 s^3.$$

This is a Bernoulli equation which can be rewritten as

$$\frac{d}{db^2}\left(\frac{1}{s^2}\right) = \left(b^2 t_2 - \frac{1}{b^2}\right)\frac{1}{s^2} - t_2.$$

This is a linear 1-order ODE of $\frac{1}{s^2}$. One can easily get its solution

$$\frac{1}{s^2} = e^{\int (b^2 t_2 - \frac{1}{b^2})db^2}\left[\tilde{c}_1 - \int t_2 e^{\int(\frac{1}{b^2} - b^2 t_2)db^2} db^2\right], \tag{4.40}$$

where $\tilde{c}_1$ is a constant. Hence, by (4.40), one independent integral of Eq. (4.38) is

$$\frac{s^2}{e^{\int(\frac{1}{b^2} - b^2 t_2))db^2} + s^2 \int t_2 e^{\int(\frac{1}{b^2} - b^2 t_2)db^2} db^2} = \frac{1}{\tilde{c}_1}. \tag{4.41}$$

Note that the characteristic equation (4.38) is equivalent to

$$\frac{db^2}{1} = \frac{d\ln s}{\frac{1}{2}[\frac{1}{b^2} - (b^2 - s^2)t_2]} = \frac{d\ln\phi}{\frac{1}{2}(-\frac{1}{b^2} + t_2 s^2)}. \tag{4.42}$$

Eq. (4.42) implies

$$\frac{db^2}{1} = \frac{d\ln s - d\ln\phi}{\frac{1}{b^2} - \frac{1}{2}b^2 t_2}. \tag{4.43}$$

By integrating Eq. (4.43), we obtain another independent integral of Eq. (4.38)

$$\ln\frac{s}{\phi} - \int(\frac{1}{b^2} - \frac{1}{2}b^2 t_2)db^2 = \tilde{c}_2, \tag{4.44}$$

where $\tilde{c}_2$ is a constant. Hence, the general solution of Eq. (4.37) is

$$\Phi\left(\frac{s^2}{e^{\int(\frac{1}{b^2} - b^2 t_2))db^2} + s^2 \int t_2 e^{\int(\frac{1}{b^2} - b^2 t_2)db^2} db^2}, \ln\frac{s}{\phi} - \int(\frac{1}{b^2} - \frac{1}{2}b^2 t_2)db^2\right) = 0, \tag{4.45}$$

where $\Phi(\xi, \eta)$ is any continuously differentiable function. Suppose $\Phi'_\eta \neq 0$, then we can solve from (4.45) that

$$\phi = \varphi(\frac{s^2}{e^{\int(\frac{1}{b^2} - b^2 t_2))db^2} + s^2 \int t_2 e^{\int(\frac{1}{b^2} - b^2 t_2)db^2} db^2}) e^{\int(\frac{1}{2}b^2 t_2 - \frac{1}{b^2})db^2} s, \tag{4.46}$$

where $\varphi(\cdot)$ is any positive continuously differentiable function. Hence, the corresponding general (α,β)-metric is

$$F=\alpha\varphi\Big(\frac{s^2}{e^{\int(\frac{1}{b^2}-b^2t_2))db^2}+s^2\int t_2e^{\int(\frac{1}{b^2}-b^2t_2)db^2}db^2}\Big)e^{\int(\frac{1}{2}b^2t_2-\frac{1}{b^2})db^2}s.$$

Therefore, by the above discussion, we have the following theorem.

Theorem 4.2. *Let $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ be a regular general (α,β)-metric on an n-dimensional manifold M. Suppose that β satisfies* (1.5). *If F is of isotropic Berwald curvature, then F is either a Berwald metric or a Randers metric.*

5. Isotropic Berwald metrics of Randers type

In this section, we will prove Theorems 1.1 and 1.2. Suppose that β satisfies (1.5), we have the following.

Lemma 5.1. *Let $F:=\bar{\alpha}+\bar{\beta}$ be any function defined by* (2.7) *and* (2.8). *Then* 1*-form $\bar{\beta}$ is closed. Moreover, F is a Randers metric if and only if*

$$f(b^2)>\max\{0,b^2(h^2-g)(b^2)\},\tag{5.1}$$

where $b:=\|\beta\|_\alpha$.

Proof. Since β satisfies (1.5), it is easy to see that

$$s_i=0,\ r_i=c(x)b_i,\ r:=b^ir_i=c(x)b^2.\tag{5.2}$$

Plugging (5.2) into (2.9) yields

$$\bar{b}_{i|j}=c\lambda(f+f'b^2)a_{ij}+c\big[2h'-\lambda(2f'+g'b^2)\big]b_ib_j,\tag{5.3}$$

where $\lambda:=\frac{h}{f+gb^2}$. It follows from (5.3) that $\bar{s}_{ij}=0$. Hence, the 1-form $\bar{\beta}$ is closed.

By (2.10) and (2.11), we have

$$\det(\bar{a}_{ij})=(f+gb^2)f^{n-1}\det(a_{ij}),\ \bar{b}^2:=\|\bar{\beta}\|^2_{\bar{\alpha}}=\bar{a}^{ij}\bar{b}_i\bar{b}_j=\frac{h^2b^2}{f+gb^2}.\tag{5.4}$$

Because $F=\alpha+\beta$ is a Randers metric, it implies that α is positive definite and $\|\bar{\beta}\|_{\bar{\alpha}}<1$. Hence,

$$f>0,\ f+gb^2>0,\ f+(g-h^2)b^2>0.\tag{5.5}$$

From (5.5), we will obtain (5.1) easily. □

From (2.10), we have

$$\bar{a}_{ij}-\bar{b}_i\bar{b}_j=fa_{ij}+(g-h^2)b_ib_j.\tag{5.6}$$

By Lemma 5.1, we know that $\bar{\beta}$ is closed. Together with Lemma 2.4, F has isotropic S-curvature if and only if (2.6) holds. By (5.3) and (5.6), (2.6) holds if and only if

$$c\lambda(f+f'b^2)a_{ij}+c\big[2h'-\lambda(2f'+g'b^2)\big]b_ib_j=2\kappa(x)[fa_{ij}+(g-h^2)b_ib_j].\tag{5.7}$$

Eq. (5.7) is equivalent to (1.8). In this case, we have

$$\kappa(x) = \kappa(b^2) = \frac{ch}{2f}\frac{f + f'b^2}{f + gb^2}.$$

Thus, we obtain the following theorem.

Theorem 5.2. *Let $F = \bar{\alpha} + \bar{\beta}$ be a Randers metric on an n-dimensional manifold M defined by* (2.7) *and* (2.8). *Then F has isotropic S-curvature if and only if* (1.8) *holds. In this case, the S-curvature is given by*

$$S = \frac{(n+1)ch}{2f}\frac{f + f'b^2}{f + gb^2}F. \tag{5.8}$$

Proof of Theorem 1.2. Let us take a look at the special case. When $g = h^2$, the Randers metric is given by $F = \sqrt{f(b^2)\alpha^2 + h^2(b^2)\beta^2} + h(b^2)\beta$. By using (1.8), F has isotropic S-curvature if and only if

$$0 = 2(f + h^2b^2)h' - (2f' + (h^2)'b^2)h = 2(fh' - f'h) = 2\Big(\frac{h}{f}\Big)', \tag{5.9}$$

where we have used $f(b^2) > 0$ in Lemma 5.1. It is easy to see from (5.9) that $\frac{h}{f} = \varepsilon$, where ε is a constant. Plugging this into (5.8), we obtain (1.10). □

Proof of Theorem 1.1. Let $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric with isotropic Berwald curvature on an n-dimensional manifold M. By Theorem 4.2, we obtain that F is either a Berwald metric given by (1.6) or a Randers metric expressed by (1.7). By Theorem 1.1 in [16], F is of isotropic S-curvature. From Theorem 5.2, f, g and h in (1.7) satisfy (1.8). Thus, we complete the proof of the theorem. □

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