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GLOBAL STABILITY OF THE POSITIVE EQUILIBRIUM OF A MATHEMATICAL MODEL FOR UNSTIRRED MEMBRANE REACTORS

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Abstract. This paper devotes to the study of a diffusive model for unstirred membrane reactors with maintenance energy subject to a homogeneous Neumann boundary condition. It shows that the unique constant steady state is globally asymptotically stable when it exists. This result further implies the non-existence of the non-uniform steady state solution.

1. Introduction

Consider the following system

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{D_1}{V}\Delta S + \frac{F}{V}(S_0 - S) - \frac{\mu_m S X}{\alpha(K_S + S)} - m_S X, \\ \frac{\partial X}{\partial t} &= \frac{D_2}{V}\Delta X + \frac{\mu_m S X}{K_S + S} - k_d X, \end{aligned} \tag{1}$$

which, when $D_i = 0$ $(i = 1, 2)$ is the model for continuous flow membrane bioreactor with Monod growth rate and was investigated in [6, 10]. In this paper, we consider a spatially generalised version of the model, namely, the case when $D_i \neq 0$. To simplify the discussion, we first introduce

$$u = k_1 S,\ v = k_2 X,\ t^* = k_3 t,\ \tau = \frac{V}{F}$$

with

$$k_1 = \frac{1}{K_S},\ k_2 = \frac{1}{\alpha K_S},\ k_3 = \mu_m$$

and then let

$$d_i = \frac{D_i}{k_3 V},\ \tau^* = k_3 \tau,\ s_0 = K_1 S_0,\ m_S^* = \frac{k_1 m_S}{k_2 k_3},\ k_d^* = \frac{k_d}{k_3}$$

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so that we can reduce the number of parameters in the model. Then after dropping the asterisks for notational simplicity we reach the nondimensional model

$$(2)\qquad \begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + \frac{1}{\tau}(s_0 - u) - \frac{uv}{1+u} - m_S v, \\ \frac{\partial v}{\partial t} = d_2\Delta v + \frac{uv}{1+u} - k_d v, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0,\ x \in \Omega, \end{cases}$$

where $u(x,t), v(x,t)$ are concentrations of the substrate and microorganisms in the reactor, respectively. All parameters are positive and more precisely, $d_i, i = 1,2$ are diffusive coefficients, which may result in much richer dynamics [4, 5, 7, 9, 11]. s_0 is known as input density, k_d the death rate of the microorganisms and m_S denotes the maintenance energy; if $m_S = 0$, model (2) is the conventional chemostat model with Monod growth kinetics [8]. Inspired by reference [5], we assume model (2) is subject to homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,\ x \in \partial\Omega.$$

It is easy to verify [6, 10] or directly from system

$$\begin{cases} -d_1\Delta u = \frac{1}{\tau}(s_0 - u) - \frac{uv}{1+u} - m_S v \\ -d_2\Delta v = \frac{uv}{1+u} - k_d v \end{cases}$$

that system (2) always has a washout equilibrium $(s_0, 0)$, which is semistable and implies that the microorganisms eventually go extinction. Furthermore, (2) has a unique, uniformly positive equilibrium $E^*(u^*, v^*)$ where

$$v^* = \frac{s_0 - u^*}{\tau(k_d + m_s)}, u^* = \frac{k_d}{1 - k_d}$$

if and only if $0 < k_d < \frac{s_0}{1+s_0}$. Since, in practice we are only interested in positive equilibrium, in the rest of this letter we always assume $k_d < \frac{s_0}{1+s_0}$.

The rest of this letter devotes to the study of local and global stability of E^*, which biologically implies the coexistence of two species. Mathematically a stable positive equilibrium implies the non-existence of spatial patterns.

2. Local stability analysis of E^*

Notice that the Jacobian at E^* is

$$(3)\qquad J = \begin{pmatrix} d_1\Delta + J_{11} & J_{12} \\ J_{21} & d_2\Delta + J_{22} \end{pmatrix},$$

where

$$J_{11} = -\frac{1}{\tau} - \frac{v}{(1+u)^2}, J_{12} = -\frac{u}{1+u} - m_S, J_{21} = \frac{v}{(1+u)^2}, J_{22} = \frac{u}{1+u} - k_d.$$

Then we can prove the following theorem.

Theorem 2.1. *When $0 < k_d < \frac{s_0}{1+s_0}$ system* (2) *has a unique positive equilibrium E^*. And when it exists, E^* is uniformly asymptotically stable in the sense of* [2].

Proof. The existence of the uniform steady state has been discussed in previous section. Next, we prove the stability by verifying that all eigenvalues of the linear operator associated with (2) have negative real part. To this end, we first revisit some notations in [7]. Assume $\lambda_{i+1} > \lambda_i > \lambda_0 = 0$, $i = 1, 2, \dots$ are eigenvalues of $-\Delta$ on its domain Ω with Neumann boundary condition and $E(\lambda_i)$ are the associated eigenspaces. Furthermore we denote the orthonormal basis of $E(\lambda_i)$ by X_i. Then the solution space, $\mathbf{X} = \{(u, v)\}$, of (2) can be decomposed as

$$\mathbf{X} = \bigoplus_{i=0}^{\infty} \mathbf{X_i}.$$

It is easy to see that $\mathbf{X_i}$ is an invariant set under the Jacobian J defined in (3). As pointed out by Peng and Wang [7], eigenvalues of J on $\mathbf{X_i}$ are equivalent to that of matrix

$$M_i = \begin{pmatrix} -d_1\lambda_i + J_{11} & J_{12} \\ J_{21} & -d_2\lambda_i + J_{22} \end{pmatrix}.$$

Since at the positive equilibrium we have

$$J_{11} < 0, J_{12} < 0, J_{21} > 0 \text{ and } J_{22} = 0,$$

the determinant and trace of M_i satisfy

$$\det M_i = \begin{vmatrix} -d_1\lambda_i + J_{11} & J_{12} \\ J_{21} & -d_2\lambda_i + J_{22} \end{vmatrix} = d_1 d_2 \lambda_i^2 - d_2 J_{11}\lambda_i - J_{12}J_{21} > 0$$

and

$$\operatorname{tr} M_i = -(d_1 + d_2)\lambda_i + J_{11} < 0$$

for all $i = 0, 1, 2, \dots$, respectively. Then we obtain that E_1 is uniformly asymptotically stable. □

Remark 2.2. Theorem 2.1 shows the local stability of the positive equilibrium of system (2), which implies (2) does not have non-constant positive steady state in a neighbourhood of E^*.

Remark 2.3. Notice that at the washout equilibrium point $E_0(s_0, 0)$,

$$J_{11} = -\frac{1}{\tau},\ J_{22} = \frac{s_0}{1+s_0} - k_d,\ J_{12} = -\frac{s_0}{1+s_0} - m_S \text{ and } J_{21} = 0.$$

Then it is asymptotically stable if (2) does not have a positive equilibrium. When E^* exists, the Jacobian at E_0 has two eigenvalues

$$\eta_{1i} = -d_1\lambda_i + J_{11} < 0$$

and

$$\eta_{2i} = -d_2\lambda_i + J_{22}$$

on each X_i. Noticing $\eta_{20} = J_{22} > 0$ yields that the washout equilibrium is unstable, which is the same as the case without diffusion.

3. Global stability of the positive equilibrium E^*

In previous section, we have proven the local stability of E^*. This section dedicates the proof of the global stability. We start with proving the following lemma.

Lemma 3.1. *System* (2) *has a positively invariant set* Γ, *which attracts all solutions of* (2) *and includes* E^*.

Proof. First, we can easily verify that $u(\cdot,t)$ and $v(\cdot,t)$ remain positive for t large enough and $u(\cdot,t_0) > 0, v(\cdot,t_0) > 0$. Next, we prove that $v(\cdot,t)$ is uniformly bounded on Ω by contradiction. Otherwise, there are some $x^* \in \Omega$ such that $v(x^*,t) \to +\infty$ as $t \to \infty$. Then for any $M > 0$ there exists $t_1 > 0$ such that $v(x^*,t) > M$ for all $t > t_1$. From the first equation of (2) and for the above x^*, M we have

$$\frac{\partial u}{\partial t} - d_1\Delta u = \frac{1}{\tau}(s_0 - u) - \frac{uv}{1+u} - m_S v < \frac{1}{\tau}(s_0 - u) - m_S v < \frac{1}{\tau}(s_0 - u) - m_S M.$$

Then $w(t)$, the solution of

$$\begin{cases} \frac{dw(t)}{dt} = \frac{1}{\tau}(s_0 - w(t)) - m_S M, \\ w(t_0) = \max_\Omega u(\cdot, t_0), \end{cases}$$

is an upper solution of

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u = \frac{1}{\tau}(s_0 - u) - m_S M, \\ u_0 = \ u(x^*, t_0) > 0. \end{cases}$$

Then $\lim_{t\to\infty} \sup(u(x^*,t)) \le \lim_{t\to\infty} w(t)$. Notice that $w(t) \to (s_0 - \tau m_S M)$ as $t \to +\infty$. Then for any $\epsilon > 0$ there is $t_2 > t_1$ such that

$$u(x^*,t) \le w(t) < s_0 - \tau m_S M + \epsilon \text{ for all } t > t_2.$$

Then for $0 < \epsilon < \frac{k_d}{2-k_d}$ and

$$M = \frac{(1 - k_d + \frac{\epsilon}{2})(s_0 + \epsilon) - k_d + \frac{\epsilon}{2}}{(1 - k_d + \frac{\epsilon}{2})\tau m_S} > \frac{(1 - k_d)(s_0 + \epsilon) - k_d}{(1 - k_d)\tau m_S},$$

we have

$$\frac{u(x^*,t)}{1 + u(x^*,t)} - k_d < -\frac{\epsilon}{2} < 0 \text{ for } t > t_2.$$

Then from the second equation of the model we have $v(x^*,t) \to 0$ as $t \to \infty$, which is a contradiction. Hence, there exists $M_1 > 0$ such that $v(x,t) < M_1$ uniformly for all $t > t_1$ and $x \in \Omega$.

Since we are interested in the asymptotical behaviour of system (2), in this sense region Γ enclosed by the positive axes, $u = s_0$ and $v = M_1$ is a positively invariant set. And obviously, it attracts all solutions of (2). □

Theorem 3.2. *The positive equilibrium, E^* of system* (2) *is globally asymptotically stable when it exists.*

Proof. Denote the solution of (2) by $(u(x,t), v(x,t))$ with positive initial values. Inspired by the work of Hsu [3] and of Hattaf and Yousfi [1], we construct a Lyapunov function as follows.

Define

$$Q(u) = \frac{\tau v^*(f(u) - k)}{s_0 - u},$$

where $f(u) = \frac{u}{1+u} + m_S$ and $k = m_S + k_d$, and let

$$W(u, v) = \int_{u^*}^{u} Q(\xi) d\xi + \int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta.$$

Then

$$E(t) = \int_\Omega W dx$$

is the Lyapunov function we need. Notice for any function $h(u)$ and u satisfying the Neumann boundary condition on $\partial\Omega$ we have

$$\begin{aligned} \int_\Omega h(u)\Delta u dx = \int_\Omega h(u)\nabla^2 u dx &= -\int_\Omega \nabla h(u) \cdot \nabla u dx + \int_{\partial\Omega} h(u)\frac{\partial u}{\partial n} \\ &= -\int_\Omega \nabla h(u) \cdot \nabla u dx = -\int_\Omega h'(u)|\nabla u|^2 dx. \end{aligned}$$

The straightforward calculation along the trajectory of (2) yields

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_\Omega (W_u u_t + W_v v_t) dx \\ &= \int_\Omega \Big\{ Q(u)\Big(d_1\Delta u + \frac{1}{\tau}(s_0 - u) - \frac{uv}{1+u} - m_S v\Big) \\ &\quad + \frac{v - v^*}{v}\Big(d_2 \Delta v + \frac{uv}{1+u} - k_d v\Big)\Big\} dx \\ &= -\int_\Omega \Big(d_1 Q'(u)|\nabla u|^2 + \frac{d_2 v^*}{v^2}|\nabla v|^2\Big) dx \qquad (4) \\ &\quad + \int_\Omega \Big\{ Q(u)\Big(\frac{1}{\tau}(s_0 - u) - \frac{uv}{1+u} - m_S v\Big) \qquad (5) \\ &\quad + \frac{v - v^*}{v}\Big(\frac{uv}{1+u} - k_d v\Big)\Big\} dx. \qquad (6) \end{aligned}$$

Next, we show that $\frac{dE(t)}{dt} < 0$, which together with Lemma 3.1 implies the globally asymptotical stability of E^*.

Since $Q(u)$ can be written as

$$Q(u) = \frac{\tau v^*(\frac{u}{1+u} - k_d)}{s_0 - u},$$

the derivative of Q with respective to u is

$$Q'(u) = \frac{\tau v^* Q_1(u)}{(1+u)^2(s_0-u)^2}, \; Q_1(u) = s_0 + u^2 - k_d(1+u)^2.$$

Obviously, Q_1 is a quadratic polynomial in terms of u, with the coefficient of the leading term $1-k_d>0$. Then $Q_1(u)$, at $u=\frac{k_d}{1-k_d}$, has a minimal value

$$Q_{1,\min} = Q_1|_{u=k_d/(1-k_d)} = s_0 - \frac{k_d}{1-k_d} > 0.$$

Hence,

$$Q_1(u) \geq Q_{1,\min} > 0 \text{ and } Q'(u) > 0,$$

which implies the integral over Ω in (4) is strictly less than zero. Furthermore, we claim that

$$\begin{aligned} f_1 &= Q(u)\Big(\frac{1}{\tau}(s_0-u) - \frac{uv}{1+u} - m_S v\Big) + \frac{v-v^*}{v}\Big(\frac{uv}{1+u} - k_d v\Big) \\ &= \frac{\tau v\big(f(u)-k\big)}{(s_0-u)}\Big(\frac{1}{\tau}(s_0-u) - v^* f(u)\Big) \leq 0. \end{aligned} \tag{7}$$

If this claim is not true, then we have two subcases

$$\begin{cases} f(u) - k > 0, \\ \frac{1}{\tau}(s_0-u) - v^* f(u) > 0, \end{cases} \tag{8}$$

or

$$\begin{cases} f(u) - k < 0, \\ \frac{1}{\tau}(s_0-u) - v^* f(u) < 0, \end{cases} \tag{9}$$

since $u<s_0$ and $v>0$. In what follows, we prove the case of (8) can not happen. Notice that

$$f(u) = \frac{u}{1+u} + m_S$$

is increasing about u and $f(u^*) - k = 0$. Then $f(u)-k>0$ implies that $u > u^* = \frac{k_d}{1-k_d}$. From the second equation of (8), we have

$$v^* < \frac{s_0-u}{\tau f(u)} < \frac{s_0-u}{\tau f(u^*)} < \frac{s_0-u^*}{\tau(k_d+m_S)} = v^*.$$

This contradiction implies that case (8) can not happen. We then show (9) can not happen either. Otherwise, from the first equation we have $f(u) < k = f(u^*)$, which implies that $0<u<u^*$. From the second equation, we have

$$v^* > \frac{s_0-u}{\tau f(u)} = \frac{s_0-u}{\tau(\frac{u}{1+u}+m_S)} > \frac{s_0-u^*}{\tau(k_d+m_S)} = v^*.$$

Again, this is a contradiction implying that (9) is not true. Hence, $f_1 \leq 0$. Therefore the integral in (5) and (6) is nonpositive. Then from the above analysis, we know that

$$\frac{dE(t)}{dt} < 0$$

which implies that (u^*, v^*) is globally asymptotically stable. □

Remark 3.3. Theorem 3.2 shows the global non-existence of the non-constant positive solution, namely globally system (2) has no spatial patterns.

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