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ON CERTAIN HYPERPLANE ARRANGEMENTS AND COLORED GRAPHS

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Abstract. We exhibit a one-to-one correspondence between 3-colored graphs and subarrangements of certain hyperplane arrangements denoted $\mathcal{J}_n$, $n \in \mathbb{N}$. We define the notion of centrality of 3-colored graphs, which corresponds to the centrality of hyperplane arrangements. Via the correspondence, the characteristic polynomial $\chi_{\mathcal{J}_n}$ of $\mathcal{J}_n$ can be expressed in terms of the number of central 3-colored graphs, and we compute $\chi_{\mathcal{J}_n}$ for $n = 2, 3$.

1. Introduction

A hyperplane arrangement is a finite set of affine hyperplanes in a real affine space. In this article, we shall consider a hyperplane arrangement problem of a specific type: Given a positive integer n, let $[n]$ denote $\{1, 2, 3, \ldots, n\}$. For each $1 \le \alpha < \beta \le n$, we define

$$H_{\alpha\beta} := \{\mathbf{x} \in \mathbb{R}^n \mid x_\alpha + x_\beta = 1\} = H_{\beta\alpha}$$

which are said to be walls or hyperplanes of **type I**. For each $i \in [n]$, define

$$0_i := \{\mathbf{x} \in \mathbb{R}^n \mid x_i = 0\}, \text{ and } 1_i := \{\mathbf{x} \in \mathbb{R}^n \mid x_i = 1\}$$

which are said to be walls of **type II**. Let $\mathcal{J}_n$ denote the hyperplane arrangement consisting of all hyperplanes of type I or type II. We are interested in the number of the *regions*, i.e., the connected components of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{J}_n} H$.

Our particular hyperplane arrangement has its origin in algebraic geometry, namely the number of certain moduli spaces [1, Problem 5.2]. But as we were researching for a solution, we discovered a neat relation with graph theory and soon this interplay has taken the center stage. The problem was then modified to better suit the graph theoretic approach. The arrangement $\mathcal{J}_n$ should remind experts of (deformations of) the braid arrangement, especially the well known *Shi arrangement* [3] which consists of walls of the forms $x_i - x_j = 0$ and $x_i - x_j = 1$. We hope to explore the relation to Shi arrangements in the future.

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We refer to Lecture 1 of Stanley's chapter [5] on hyperplane arrangements for many fundamental results, but recall a few key notions here. Let $\mathcal{B} = \{H_i \mid i \in I\}$ be a hyperplane arrangement where

$$H_i = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} a_{ij} x_j = b_i \right\}$$

and I is an index set. The arrangement $\mathcal{B}$ is said to be *central* if the intersection of all hyperplanes in $\mathcal{B}$ is nonempty. The *rank* of a hyperplane arrangement is the dimension of the space spanned by the normal vectors to the hyperplanes in the arrangement.

Definition 1. Let A and b denote the matrices (a_{ij}) and $b = (b_i)$, $i = 1, \dots, |\mathcal{B}|$, respectively. The augmented matrix $B = [A|b]$ will be called the *matrix associated with the hyperplane arrangement* $\mathcal{B}$ or simply the *associated matrix of* $\mathcal{B}$.

Since the row vectors of A are the normal vectors to the hyperplanes, rank of $\mathcal{B}$ equals $\operatorname{rank}(A)$. In central hyperplane arrangements, this also equals $\operatorname{rank}(B)$, since the associated matrix is consistent. Note that $[A|b]$ completely determines $\mathcal{B}$, and giving a hyperplane arrangement is equivalent to giving its associated matrix.

Let $\mathcal{A}$ be a hyperplane arrangement. Then the *characteristic polynomial* of $\mathcal{A}$ is defined

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}} (-1)^{|\mathcal{B}|} t^{n - \operatorname{rank}(\mathcal{B})},$$

where $\mathcal{B}$ runs through all central sub-arrangements of $\mathcal{A}$: In fact, the characteristic polynomial is defined using the Möbius function [5, Definition 1.3] and the equivalence is a theorem due to H. Whitney [2, Lemma 2.3], but this form suits our purpose just fine. The following is perhaps the most fundamental theorem when it comes to counting the number of regions.

Theorem ([6]). *Let $\mathcal{A}$ be a hyperplane arrangement in an n-dimensional real vector space. Let $r(\mathcal{A})$ be the number of chambers and $b(\mathcal{A})$ be the number of relatively bounded chambers. Then we have*

(1) $b(\mathcal{A}) = (-1)^n \chi(+1)$.
(2) $r(\mathcal{A}) = (-1)^n \chi(-1)$.

In our case, we have $\binom{n}{2}$ walls of type I and $2n$ walls of type II. All together there are $N = \binom{n}{2} + 2n$ walls and a simple case by case analysis would require centrality examination and rank computation of 2^N subarrangements. The main theme of this paper is that, by a systematic use of symmetry and geometry, we can reduce that number significantly. Enumeration is further enhanced by using the notion of *associated graphs* (Definition 4) and associated matrices.

The graph theory approach is for the specific hyperplane arrangement problem studied in this paper. We associate a graph to each hyperplane subarrangement of $\mathcal{J}_n$, and we translate the centrality of hyperplane arrangements in terms of graph properties (Definition 2 and Theorem 1). This makes enumeration of central subarrangements much more systematic and efficient: We work out the basic examples of two and three dimensional cases in Section 3 but without borrowing any significant results from graph theory. We employ more substantial graph theory results to attack the higher dimensional cases in the forthcoming manuscript [4]. Also, we believe that the method can be generalized to treat other hyperplane arrangements by considering the *signed graphs* [7], and this will be taken up in a future work.

2. Associated colored graphs

Let (V, E) be a graph with vertices $V = [n]$ and the set of edges E. Let v and v' be vertices (which may be equal). A path is said to be *even* (resp., odd) if its length is even (resp., odd).

For the purpose of this paper, we shall consider $\{0, 1, *\}$-colored graphs on $V = [n]$ ($*$ indicating no numeric value assigned). We shall let $\gamma : V \to \{0, 1, *\}$ denote the color function. A vertex v with $\gamma(v) = *$ will be called *not colored.* By a 3-colored graph, we shall always mean a $\{0, 1, *\}$-colored graph.

Definition 2. A 3-colored graph $([n], E)$ is said to be *central* if

1. if v is colored, then it is not on a closed walk of odd length, and
2. $\gamma(v) = \gamma(v')$ (resp., $\gamma(v) \neq \gamma(v')$) for any pair of colored vertices v, v' such that there is a $v - v'$ path of even (resp., odd) length.

Remark 1.

1. Note that the centrality condition determines the parity of the paths between any two given colored vertices: In a central graph, if a $v - v'$ path is even (resp. odd), so are all other $v - v'$ paths.
2. Condition (1) of Definition 2 is equivalent to that every component C with a colored vertex is bipartite. Suppose C has a closed walk γ. Since C is connected, there exists a path τ from a colored vertex v to γ. Then traversing τ, γ, and τ^{-1} back to v is an odd cycle which contains v, violating Condition (1). Converse is obvious.

Definition 3. For a subarrangement $\mathcal{A} \subseteq \mathcal{J}_n$, $I(\mathcal{A})$ denotes the set of indices $\tau \in [n]$ that appear in $\mathcal{A}$. That is,

$$I(\mathcal{A}) = \left(\bigcup_{H_{\alpha\beta} \in \mathcal{A}} \{\alpha, \beta\} \right) \cup \left(\bigcup_{0_\alpha \in \mathcal{A}} \{\alpha\} \right) \cup \left(\bigcup_{1_\beta \in \mathcal{A}} \{\beta\} \right).$$

Definition 4. Let $\mathcal{A}$ be a subarrangement of $\mathcal{J}_n$ which does not contain both 0_i and 1_i, $\forall i$. The *associated graph* $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$ is a 3-colored graph with the vertex set $V(\Gamma_{\mathcal{A}}) = I(\mathcal{A})$, and edge set $E(\Gamma_{\mathcal{A}}) = \{\{\alpha, \beta\} : H_{\alpha\beta} \in \mathcal{A}\}$ where the vertices are assigned **exactly** one of $\{0, 1, *\}$ in the obvious fashion: given

$i \in I(\mathcal{A})$, we assign 0 (resp., 1) to i if $0_i \in \mathcal{A}$ (resp., $1_i \in \mathcal{A}$). If neither 0_i nor 1_i is in $\mathcal{A}$, we assign $*$ to i, i.e., the vertex i is not colored.

Let $\mathcal{S}_n$ be the set of all subarrangements of the hyperplane arrangements $\mathcal{J}_n$ satisfying the conditions of the Definition 4, in particular, those arrangements which do not contain both 0_i and 1_i, $\forall i$. Then we have the following lemma:

Lemma 1. *There is a one-to-one correspondence between $\mathcal{S}_n$ and the set of all 3-colored graphs on $[n]$.*

Proof. Given any 3-colored graph $\Gamma = ([n], E)$ with the color function $\gamma : [n] \to \{0, 1, *\}$, we associate the hyperplane arrangement

$$\mathcal{A}_\Gamma := \big\{\{x_i = \gamma(i)\}_{\gamma(i)\neq *}\big\} \cup \{H_{ij}\}_{\{i,j\}\in E}.$$

This clearly is inverse to the association $\mathcal{A} \mapsto \Gamma_\mathcal{A}$ defined in Definition 4. □

Note that for the graph Γ associated with a central arrangement $\mathcal{A}$, there are no isolated non-colored vertices and two adjacent colored vertices are of different colors.

Theorem 1. *Let $\mathcal{A} \in \mathcal{S}_n$. Then $\mathcal{A}$ is central if and only if $\Gamma_\mathcal{A}$ is central.*

Proof. The corresponding colored graph $\Gamma_\mathcal{A}$ can be decomposed into three subgraphs Γ', Γ'', Γ''':

(1) (graph of the first kind) Γ' is the union of colorless connected components;
(2) (graph of the second kind) Γ'' is the union of isolated colored vertices;
(3) (graph of the third kind) $\Gamma''' = \Gamma\backslash(\Gamma'\cup\Gamma'')$ is the union of the connected components with at least one colored vertex and at least one edge.

Accordingly, we can decompose $\mathcal{A}$ into $\mathcal{A}' \sqcup \mathcal{A}'' \sqcup \mathcal{A}'''$ such that the subarrangements $\mathcal{A}', \mathcal{A}'', \mathcal{A}'''$ correspond to Γ', Γ'', Γ''', respectively. For example, the hyperplane arrangement (Figure 1)

$$\mathcal{A} = \{H_{12}, H_{23}, 1_3, 0_5, H_{46}, 1_7, H_{8,9}\}$$

decomposes into

$$\{H_{46}, H_{89}\} \sqcup \{0_5, 1_7\} \sqcup \{H_{12}, H_{23}, 1_3\}$$

whose associated colored graph decomposition is

(1) $\Gamma' = (\{4, 6, 8, 9\}, \{\{4, 6\}, \{8, 9\}\})$;
(2) $\Gamma'' = (\{5, 7\}, \emptyset)$, $\gamma(5) = 0$, $\gamma(7) = +1$;
(3) $\Gamma''' = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$, $\gamma(3) = +1$.

Since the index sets $I(\mathcal{A}')$, $I(\mathcal{A}'')$ and $I(\mathcal{A}''')$ are all disjoint from each other, $\mathcal{A}$ is central if and only if each of them is separately central. It is also easy to see that $\mathcal{A}'$ and its associated graph $\Gamma_{\mathcal{A}'}$ are central: $\Gamma_{\mathcal{A}'}$ is trivially central since the centrality condition is vacuous for a non-colored graph. As for the hyperplane arrangement $\mathcal{A}'$, setting all variables x_i, $i \in I(\mathcal{A}')$, equal to $1/2$ satisfies all hyperplane equations in $\mathcal{A}'$. Note that $\mathcal{A}''$ is also always central

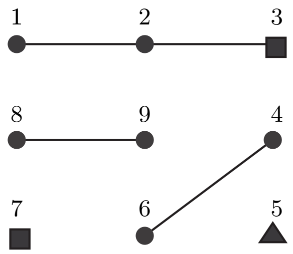

FIGURE 1. Disks denote $*$-colored vertices; squares, 1-colored; and triangles, 0-colored

except the obvious non-central ones containing both 0_i and 1_i for some i, which we have discarded in the beginning (definition of $\mathcal{S}_n$). Hence, without loss of generality, we may assume that $\mathcal{A} = \mathcal{A}'''$ so that its associated graph is connected and each vertex is connected to a distinguished colored vertex i_0 by a path, and that $\gamma(i_0) = 1$.

Now, we shall prove the assertion of the theorem for any such hyperplane arrangement. Suppose that the associated graph is central. We define a new coloring $\gamma^\star$ as follows. If j is colored, set $\gamma^\star(j) = \gamma(j)$. For a non-colored vertex j, choose a path σ from j to i_0 and assign $\gamma^\star(j) = \gamma(i_0) = 1$ if the path is even and $\gamma^\star(j) = 0$ otherwise. Note that all paths from j to i_0 have the same parity since two paths of different parity would give a closed walk of odd length from i_0 to i_0, and this violates the centrality of $\Gamma_{\mathcal{A}}$. Let $x^\star$ be a point whose coordinates satisfy $x^\star_j = \gamma^\star(j)$ for any $j \in I(\mathcal{A})$. We claim that $x^\star$ satisfies all hyperplane equations. Let $H_{\alpha\beta} \in \mathcal{A}$. Suppose there is an even path σ from α to i_0. By the defining property, $x^\star_\alpha = 1$. Then joining the edge $\{\beta, \alpha\}$ to σ creates an odd path from β to i_0 which implies that $x^\star_\beta = 0$. Thus $x^\star_\alpha + x^\star_\beta = 1$. The other case where $\gamma(\alpha) = 0$ is proved similarly.

Conversely, suppose that $\mathcal{A}$ is central. If there is an even path $\{v_0, v_1, \ldots, v_{2k}\}$ and $\gamma(v_0) = 0$, then the hyperplanes $x_{v_k} + x_{v_{k+1}} = 1$ corresponding to edges successively determine $x_{v_1} = 1$, $x_{v_2} = 0$, etc., so that $x_{v_{2k}} = 0 = \gamma(v_0)$. Likewise, if there is an odd path $\{v_0, v_1, \ldots, v_{2k+1}\}$, we have $\gamma(v_0) \neq \gamma(v_{2k+1})$. It follows that any two paths between two colored vertices have the same parity. $\square$

3. Arrangements in the plane and space

In this section, we use our main theorem and analyze the dimension 2 and 3 cases. We present them here while promising deeper results in a forthcoming paper where we use graph theoretic approach to give a formula for computing the characteristic polynomial in terms of the number of bipartite graphs of given rank and size.

Definition 5. (1) We let $r^n_{\epsilon,\nu}$ denote the number of central graphs on $[n]$ with precisely ϵ edges and ν colored vertices. When there is no danger of confusion, we drop the superscript n.
(2) A graph with ϵ edges and ν colored vertices will be called an (ϵ, ν) graph.

Note that ϵ, ν are the numbers of type I and of type II hyperplanes respectively, hence the order (ϵ, ν).

3.1. $n = 2$

$\mathcal{J}_2 = \{H_{12}, 0_1, 1_1, 0_2, 1_2\}$. The central subarrangements $\mathcal{B}$ are enumerated as follows, according to the cardinality.

(1) $|\mathcal{B}| = 1$: $\mathcal{B}$ contains either a single colored vertex, or an edge. There are five such cases.
(2) $|\mathcal{B}| = 2$: $\{H_{12}, 1_1\}$, $\{H_{12}, 0_1\}$, $\{H_{12}, 1_2\}$, $\{H_{12}, 0_2\}$, $\{0_1, 0_2\}$, $\{1_1, 0_2\}$, $\{0_1, 1_2\}$, $\{1_1, 1_2\}$. These are of rank 2.
(3) $|\mathcal{B}| = 3$: $\{H_{12}, 1_1, 0_2\}$, $\{H_{12}, 1_2, 0_1\}$. These are of rank 2.
(4) $|\mathcal{B}| \geq 4$: There are no central subarrangements with ≥ 4 hyperplanes.

With the trivial central arrangement with $|\mathcal{B}| = 0$, the characteristic polynomial is

$$\chi_{\mathcal{J}_2}(t) = ((-1)^2 8 + (-1)^3 2) \cdot t^{2-2} + (-1) \cdot 5t^{2-1} + t^{2-0} = 6 - 5t + t^2.$$

The number of chambers is $\chi(-1) = 12$ and the number of bounded chambers is $\chi(1) = 2$.

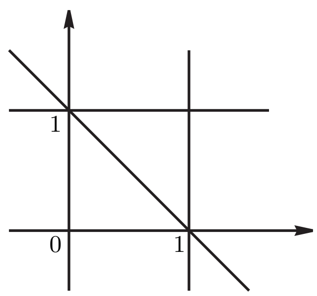

FIGURE 2. $\mathcal{J}_2$ has 12 regions and 2 bounded regions.

3.2. $n = 3$

In this case, $\mathcal{J}_3$ has three hyperplanes of type I and six hyperplanes of type II. Obviously, having both 0_i and 1_i for a given i makes the arrangement not central. Hence any central subarrangement has no more than three hyperplanes of type II, so a central subarrangement $\mathcal{B}$ satisfies $|\mathcal{B}| \leq 6$. We enumerate all central graphs, besides the trivial one with $|\mathcal{B}| = 0$.

(1) $|\mathcal{B}| = 1$: $\mathcal{B}$ either contains a single colored vertex, or an edge. There are nine such cases.
(2) $|\mathcal{B}| = 2$: $\mathcal{B}$ may have two edges and no colored vertices, or an edge and a colored vertex, or two colored vertices. These are all rank 2 arrangements.
   (a) ($\epsilon = 0$): $\binom{3}{2} \cdot 2 \cdot 2 = 12$ subarrangements;
   (b) ($\epsilon = 1$): There are $\binom{3}{1} \cdot (2+2) = 12$ central $(1,1)$ graphs with the colored vertex incident on the edge. There are six $(1,1)$ graphs with the colored vertex not incident on the edge.
   (c) ($\epsilon = 2$): These are three labeled trees on [3].
(3) $|\mathcal{B}| = 3$: These are all of rank 3 except the six $(1,2)$ graphs with an edge with both vertices colored (with opposing colors) which are of rank 2.
   (a) ($\epsilon = 0$): 2^3 cases, obviously;
   (b) ($\epsilon = 1$): Choose an edge and either color the two vertices incident upon the edge ($\binom{3}{2} \cdot 2 = 6$ cases, this is essentially the case of two vertices) or color one vertex incident on the edge and the other vertex not on the edge ($\binom{3}{2} \cdot 2^3 = 24$ cases);
   (c) ($\epsilon = 2$): Two edges and one colored vertex (of either color). There are $\binom{3}{2} \cdot 3 \cdot 2 = 18$ cases;
   (d) ($\epsilon = 3$): There is only 1 case.
(4) $|\mathcal{B}| = 4$: $\epsilon \geq 1$ since otherwise a vertex must have two colors. These are all of rank 3, and there are 30 central graphs corresponding to $\mathcal{B}$ with $|\mathcal{B}| = 4$.
   (a) ($\epsilon = 1$): There are two ways to color the two vertices incident on the one existing edge, and two ways to color the vertex not incident on the edge. Hence $\binom{3}{1} \cdot 2 \cdot 2 = 12$ central $(1,3)$ graphs.
   (b) ($\epsilon = 2$): There are two edges incident on two colored vertices and a non-colored vertex. The one non-colored vertex can be at an endpoint of the path or not. There are $\binom{3}{2} \cdot 2 \cdot 2 = 12$ and $\binom{3}{2} \cdot 2 = 6$ central $(2,2)$ graphs of each kind.
   (c) ($\epsilon = 3$): There are no central graphs of this type. A closed path from the colored vertex to itself is a cycle of length 3.
(5) $|\mathcal{B}| = 5$: There are no (3,2) central graphs. There are six central $(2,3)$ graphs.
(6) There are no central subarrangements with ≥ 6 hyperplanes since it would have to have either a cycle of length three with colored vertices or a vertex with two colors.

We summarize our findings in the table below.

rank $\backslash|\mathcal{B}|$	1	2	3	4	5
1	9	0	0	0	0
2	0	33	6	0	0
3	0	0	51	30	6

The characteristic polynomial is

$$\begin{aligned}\chi_{\mathcal{J}_3}(t) &= t^3 - 9t^{3-1} + (33-6)\,t^{3-2} + (-51+30-6)t^{3-3} \\ &= t^3 - 9t^2 + 27t - 27.\end{aligned}$$

Interestingly, the polynomials for $n = 2, 3$ factor into linear forms. In general, this turns out not to be the case $n \geq 4$ which will be illustrated in our forthcoming manuscript [4]. It would be interesting to understand when the characteristic polynomial factors.

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