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STANCU TYPE GENERALIZATION OF MODIFIED GAMMA OPERATORS BASED ON q-INTEGERS

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Abstract. In this paper, we propose the Stancu type generalization of a kind of modified q-Gamma operators. We estimate the moments of these operators and give the basic convergence theorem. We also obtain the Voronovskaja type theorem. Furthermore, we obtain the local approximation, rate of convergence and weighted approximation for these operators.

1. Introduction

In recent years, one of the most interesting areas of research in approximation theory is the application of q-calculus (see [1]). Phillips [15] first introduced the q-analogue of well-known Bernstein polynomials. After that, many other authors introduced modifications of the other important operators based on the q-integers, for example, q-Meyer-König operators [18], q-Bleimann, Butzer and Hahn operators [14], q-Szász-Mirakyan operators [13], q-Baskakov operators [5] and so on [6], [8], [9].

Now we mention certain definitions based on q-integers and the details can be found in [10]. For any fixed real number $q > 0$ and each non-negative integer n, we denote q-integers by $[n]_q$, where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & \text{if } q \neq 1; \\ n, & \text{if } q = 1. \end{cases} \tag{1.1}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q, & \text{if } n = 1, 2, \ldots; \\ 1, & \text{if } n = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}, \quad k = 0, 1, \ldots, n.$$

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It is obvious that q-binomial coefficients will reduce to the ordinary case when $q=1$. The q-improper integrals are defined as

$$\int_0^a f(x)\mathrm{d}_q x = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad a \in \mathbb{R},$$

and

$$\int_0^{\infty/A} f(x)\mathrm{d}_q x = (1-q)\sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A>0, \tag{1.2}$$

provided the sums converge absolutely.

Many researchers have studied approximation properties of the Gamma operators and their modifications (see [3], [11], [17], [20], [22] etc.) in some function spaces. Very recently, Cai and Zeng [2] proposed a kind of modified q-generalization of Gamma operators and studied their approximation properties. The modified q-Gamma operators were defined in [2] as follows:

$$\begin{aligned} G^*_{n,q}(f;x) = {} & \frac{[2n+3]_q!\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x\right)^{n+3} q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \\ & \times \int_0^{\infty/A} \frac{t^n}{\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x+t\right)_q^{2n+4}} f(t)\mathrm{d}_q t, \quad x>0. \end{aligned} \tag{1.3}$$

Recall that the explicit formulas for moments $G^*_{n,q}(t^m;x)$ were given in [2] as follows.

Lemma 1.1. *For any $k\in\mathbb{N}$, $k\le n+2$ and $q\in(0,1)$, we have*

$$G^*_{n,q}(t^m;x) = \frac{[n+k]_q![n-k+2]_q!}{[n]_q![n+2]_q!} q^{\frac{k-k^2}{2}} \left(\frac{[n+2]_q x}{[n+1]_q}\right)^k. \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2, we define the modified q-Gamma-Stancu operators and obtain the moments of them. In Section 3, we give the basic convergence theorem and then obtain their Voronovskaja type theorem. Also, we obtain the local approximation and rate of convergence for these operators.

2. Construction of the operators

Some authors have defined general sequences of linear positive operators where the classical sequences can be achieved as particular cases. For instance, Stancu [16] introduced the positive linear operators $P_n^{(\alpha,\beta)}: C[0,1]\to C[0,1]$ by

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and α, β be any two non-negative real numbers which satisfy the condition $0 \leq \alpha \leq \beta$.

All mentioned above motivate us to make this article. Now, we firstly give the definition of the modified q-Gamma-Stancu operators.

For fixed $0 \leq \alpha \leq \beta$, we introduce the modified q-Gamma-Stancu operators $\mathbb{G}_{n,q}(f;x)$: $C(0,\infty) \to C(0,\infty)$

$$\begin{aligned}\mathbb{G}_{n,q}(f;x) = {} & \frac{[2n+3]_q! \left(\frac{q^{n+1}[n+2]_q}{[n+1]_q} x\right)^{n+3} q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \\ & \times \int_0^{\infty/A} \frac{t^n}{\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q} x + t\right)_q^{2n+4}} f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) \mathrm{d}_q t\end{aligned} \tag{2.1}$$

for any real number $0 < q < 1$ and $f \in C(0,\infty)$. It is clear that $\mathbb{G}_{n,q}(f;x)$ is a linear and positive operator for $x \in (0,\infty)$.

The following two lemmas show five origin moments and three central moments which play a significant role in our work.

Lemma 2.1. *If we define the moments as*

$$\begin{aligned}T_{n,m}(x) &= \mathbb{G}_{n,q}(t^m;x) \\ &= \frac{[2n+3]_q! \left(\frac{q^{n+1}[n+2]_q}{[n+1]_q} x\right)^{n+3} q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \\ &\quad \times \int_0^{\infty/A} \frac{t^n}{\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q} x + t\right)_q^{2n+4}} \left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right)^k \mathrm{d}_q t,\end{aligned}$$

then we have

i) $T_{n,0}(x) = \mathbb{G}_{n,q}(1;x) = 1,$

ii) $T_{n,1}(x) = \mathbb{G}_{n,q}(t;x) = \dfrac{[n]_q x + \alpha}{[n]_q + \beta},$

iii) $T_{n,2}(x) = \mathbb{G}_{n,q}(t^2;x) = \dfrac{1}{([n]_q + \beta)^2} \left\{ \dfrac{[n]_q^2 [n+2]_q^2}{q[n+1]_q^2} x^2 + 2[n]_q \alpha x + \alpha^2 \right\},$

iv) $T_{n,3}(x) = \mathbb{G}_{n,q}(t^3;x)$

$$= \frac{1}{([n]_q + \beta)^3} \left\{ \frac{[n]_q^2 [n+2]_q^3 [n+3]_q}{[n+1]_q^3 q^3} x^3 + \frac{3\alpha [n]_q^2 [n+2]_q^2}{q[n+1]_q^2} x^2 + 3\alpha^2 [n]_q x + \alpha^3 \right\},$$

v) $T_{n,4}(x) = \mathbb{G}_{n,q}(t^4;x)$

$$= \frac{1}{([n]_q+\beta)^4}\left\{\frac{[n]_q^3[n+2]_q^4[n+3]_q[n+4]_q}{[n-1]_q[n+1]_q^4q^6}x^4 + \frac{4\alpha[n]_q^2[n+2]_q^3[n+3]_q}{[n+1]_q^3q^3}x^3 + \frac{6\alpha^2[n]_q^2[n+2]_q^2}{q[n+1]_q^2}x^2 + 4\alpha^3[n]_qx+\alpha^4\right\}$$

for $n>1$.

Proof. From Lemma 1.1, we have $T_{n,0}(x)=\mathbb{G}_{n,q}(1;x)=G^*_{n,q}(1;x)=1$. Next

$$\begin{aligned}
T_{n,1}(x) &= \frac{[2n+3]_q!\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x\right)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \\
&\quad\times\int_0^{\infty/A}\frac{t^n}{\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x+t\right)_q^{2n+4}}\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)\mathrm{d}_qt \\
&= \frac{[n]_q}{[n]_q+\beta}G^*_{n,q}(t;x)+\frac{\alpha}{[n]_q+\beta}G^*_{n,q}(1;x) \\
&= \frac{[n]_qx+\alpha}{[n]_q+\beta}.
\end{aligned}$$

Finally,

$$\begin{aligned}
T_{n,2}(x) &= \frac{[2n+3]_q!\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x\right)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \\
&\quad\times\int_0^{\infty/A}\frac{t^n}{\left(\frac{q^{n+1}[n+2]_q}{[n+1]_q}x+t\right)_q^{2n+4}}\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)^2\mathrm{d}_qt \\
&= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2G^*_{n,q}(t^2;x)+\frac{2[n]_q\alpha}{([n]_q+\beta)^2}G^*_{n,q}(t;x) \\
&\quad+\left(\frac{\alpha}{[n]_q+\beta}\right)^2G^*_{n,q}(1;x) \\
&= \frac{[n]_q^2[n+2]_q^2}{q[n+1]_q^2([n]_q+\beta)^2}x^2+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}x+\left(\frac{\alpha}{[n]_q+\beta}\right)^2 \\
&= \frac{1}{([n]_q+\beta)^2}\left\{\frac{[n]_q^2[n+2]_q^2}{q[n+1]_q^2}x^2+2[n]_q\alpha x+\alpha^2\right\}.
\end{aligned}$$

Similarly, we can obtain iv) and v). □

Remark 2.1. If $m \in \mathbb{N}$ and $0 \leq \alpha \leq \beta$, then we have the following recursive relation for the images of the monomials t^m under $\mathbb{G}_{n,q}(\cdot;x)$ in terms of $G^*_{n,q}(\cdot;x)$,

$$\mathbb{G}_{n,q}(t^m;x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]_q+\beta)^m} G^*_{n,q}(t^j;x).$$

Lemma 2.2. *Let $n > 1$ be a given natural number. For every $q \in (0,1)$, we have*

$$A_{n,q}(x) = \mathbb{G}_{n,q}(t-x;x) = \frac{\alpha - \beta x}{[n]_q + \beta}, \tag{2.2}$$

$$\begin{aligned} B_{n,q}(x) &= \mathbb{G}_{n,q}\left((t-x)^2;x\right) \\ &= \left\{ \frac{[n]_q^2[n+2]_q^2}{q[n+1]_q^2([n]_q+\beta)^2} - \frac{2[n]_q}{[n]_q+\beta} + 1 \right\} x^2 - \frac{2\alpha\beta}{([n]_q+\beta)^2} x \\ &\quad + \left(\frac{\alpha}{[n]_q+\beta} \right)^2, \end{aligned} \tag{2.3}$$

$$\begin{aligned} C_{n,q}(x) &= \mathbb{G}_{n,q}\left((t-x)^4;x\right) \\ &= \frac{1}{[n]_q+\beta} \left\{ \frac{[n]_q^3[n+2]_q^4[n+3]_q[n+4]_q}{([n]_q+\beta)^3[n+1]_q^4[n-1]_q q^6} - \frac{4[n]_q^2[n+2]_q^3[n+3]_q}{[n+1]_q^3([n]_q+\beta)^2 q^3} \right. \\ &\quad \left. + \frac{6[n]_q^2[n+2]_q^2}{([n]+\beta)[n+1]_q^2 q} - 3[n]_q + \beta \right\} x^4 \\ &\quad + \frac{4\alpha}{([n]_q+\beta)^2} \left\{ \frac{[n]_q^2[n+2]_q^3[n+3]_q}{([n]_q+\beta)^2[n+1]_q^3 q^3} - \frac{3[n]_q^2[n+2]_q^2}{q([n]_q+\beta)[n+1]_q^2} \right. \\ &\quad \left. + 2[n]_q - \beta \right\} x^3 \\ &\quad + \frac{6\alpha^2}{([n]_q+\beta)^4} \left\{ \frac{[n]_q^2}{q(1-q)[n+1]_q^2} + \beta^2 \right\} x^2 - \frac{4\alpha^3\beta x - \alpha^4}{([n]_q+\beta)^4}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 2.1, we calculate directly and get immediately Lemma 2.2. □

Remark 2.2. If we put $q = 1$ and $\alpha = \beta = 0$, we will get the relation: $L_n(f;x) = \mathbb{G}_{n,1}\left(f; \frac{n+1}{n+2}x\right)$, so we can get the moments of Gamma operators (see [11]) as

$$L_n(t^k;x) = \frac{(n+k)!(n-k+2)!}{n!(n+2)!} x^k, \ k \in \mathbb{N}.$$

3. Main results

First we give the following convergence theorem for the sequence $\{\mathbb{G}_{n,q_n}(f;x)\}$.

Theorem 3.1. *Let $q_n \in (0,1)$, the sequence $\{\mathbb{G}_{n,q_n}(f;x)\}$ converges uniformly to f on $[a,b] \subset (0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. Let $q_n \in (0,1)$ and $\lim_{n\to\infty} q_n = 1$, then we have $[n]_{q_n} \to \infty$ and for $s = 1,2,3$, $\lim_{n\to\infty} \frac{[n+s]_{q_n}}{[n]_{q_n}} = 1$ as $n \to \infty$ (see [19]). Thus, by Lemma 2.1, we have $\lim_{n\to\infty} \left\|\mathbb{G}_{n,q_n}(t^i;x) - x^i\right\|_{C[a,b]} = 0$, $j = 0,1,2$, where $\|f\|_{C[a,b]} = \max\{|f(x)| : x \in [a,b]\}$. According to the well-known Korovkin theorem [12], we get that the sequence $\{\mathbb{G}_{n,q_n}(f;x)\}$ converges uniformly to f on $[a,b]$.

We prove the converse result by contradiction. If $\{q_n\}$ does not tend to 1 as $n \to \infty$, then it must contain a subsequence $\{q_{n_k}\} \subset (0,1)$ with $n_k \geq k$, such that $\lim_{k\to\infty} q_{n_k} = q_0 \in [0,1)$. Thus

$$\lim_{k\to\infty} \frac{1}{[n_k]_{q_{n_k}}} = \lim_{k\to\infty} \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k}} = 1 - q_0.$$

Taking $n = n_k$, $q = q_{n_k}$ in $\mathbb{G}_{n,q}(t;x)$, by Lemma 2.1, we get

$$\mathbb{G}_{n_k,q_{n_k}}(t;x) = \frac{[n_k]_{q_{n_k}} x + \alpha}{[n_k]_{q_{n_k}} + \beta} \to \frac{x + \alpha(1-q_0)}{1+\beta(1-q_0)} \neq x \text{ as } k \to \infty.$$

This leads to a contradiction, hence $\lim_{n\to\infty} q_n = 1$. The proof is completed. □

3.1. Voronovskaja type theorem

Theorem 3.2. *Let f be a bounded and integrable function on the interval $[0,\infty)$ and $\{q_n\}_{n=1}^{\infty}$ be a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the second derivative $f''(x)$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{n\to\infty} [n+2]_{q_n} \left(\mathbb{G}_{n,q_n}(f;x) - f(x)\right) = (\alpha - \beta x) f'(x) + x^2 f''(x). \tag{3.1}$$

Proof. By the Taylor formula we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2,$$

where $r(t,x)$ is bounded and $\lim_{t\to x} r(t,x) = 0$. By applying the operator $\mathbb{G}_{n,q_n}(f;x)$ to the above equation, we obtain

$$\begin{aligned}
\mathbb{G}_{n,q_n}(f;x) - f(x) &= f'(x)\mathbb{G}_{n,q_n}((t-x);x) + \frac{1}{2}f''(x)\mathbb{G}_{n,q_n}\left((t-x)^2;x\right) \\
&\quad + \mathbb{G}_{n,q_n}\left(r(t,x)(t-x)^2;x\right) \\
&= f'(x)A_{n,q_n}(x) + \frac{1}{2}f''(x)B_{n,q_n}(x) \\
&\quad + \mathbb{G}_{n,q_n}\left(r(t,x)(t-x)^2;x\right).
\end{aligned}$$

By direct calculation, we obtain

$$[n+2]_{q_n} A_{n,q_n}(x) = \frac{[n+2]_{q_n}}{[n]_{q_n}+\beta}(\alpha-\beta x) \to \alpha - \beta x \qquad (n\to\infty),$$

and

$$\begin{aligned}
&[n+2]_{q_n} B_{n,q_n}(x)\\
&= [n+2]_{q_n}\left\{\frac{[n]_{q_n}^2[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2} - \frac{2[n]_{q_n}}{[n]_{q_n}+\beta}+1\right\}x^2\\
&\quad - \frac{2\alpha\beta[n+2]_{q_n}}{([n]_{q_n}+\beta)^2}x + \frac{\alpha^2[n+2]_{q_n}}{([n]_{q_n}+\beta)^2}\\
\to\quad & [n+2]_{q_n}\left\{\frac{[n]_{q_n}^2[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2} - \frac{2[n]_{q_n}}{[n]_{q_n}+\beta}+1\right\}x^2\\
&= [n+2]_{q_n}\left\{\frac{\left\{([n]_{q_n}+\beta)^2 - 2\beta([n]_{q_n}+\beta)+\beta^2\right\}[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2}\right.\\
&\qquad\qquad \left. + \frac{2\beta}{[n]_{q_n}+\beta} - 1\right\}x^2\\
&= [n+2]_{q_n}\left\{\frac{[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2} - \frac{2\beta[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)} + \frac{\beta^2[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2}\right.\\
&\qquad\qquad \left. + \frac{2\beta}{[n]_{q_n}+\beta} - 1\right\}x^2\\
\to\quad & \left\{\frac{[n+2]_{q_n}^3}{q_n[n+1]_{q_n}^2} - [n+2]_{q_n}\right\}x^2\\
\to\quad & 2x^2 \ (n\to\infty).
\end{aligned}$$

Similarly, we also get

$$\lim_{n\to\infty}[n+2]_{q_n} C_{n,q_n}(x) = 0.$$

Since $r(t,x)$ is bounded and $\lim_{t\to x} r(t,x) = 0$, then for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|r(t,x)| \le \epsilon + \frac{M}{\delta^2}(t-x)^2 \qquad (M \text{ is a positive constant}).$$

Thus

$$\begin{aligned}
[n+2]_{q_n}\left|\mathbb{G}_{n,q_n}\left(r(t,x)(t-x)^2;x\right)\right| &\le \epsilon[n+2]_{q_n}B_{n,q_n}(x) + \frac{M}{\delta^2}[n+2]_{q_n}C_{n,q_n}(x)\\
&\to 0 \quad (n\to\infty).
\end{aligned}$$

The proof is completed. □

3.2. Local approximation

In this subsection we establish direct and local approximation theorems in connection with the operators $\mathbb{G}_{n,q_n}$. Let $C_B(0,\infty)$ be the space of all real-valued continuous and bounded functions f on $(0,\infty)$ endowed with the norm $\|f\| = \sup\{|f(x)| : x \in (0,\infty)\}$. Further let us consider the following Peetre's K-functional:

$$K_2(f;\delta) = \inf_{g\in W^2} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B(0,+\infty) : g', g'' \in C_B(0,+\infty)\}$. From ([4], [7]), there exists an absolute $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \delta > 0, \tag{3.2}$$

where

$$\omega_2(f;\delta) := \sup_{0<h\le\delta} \sup_{x\in[0,+\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second-order modulus of smoothness of $f \in C_B(0,\infty)$. By

$$\omega(f;\delta) := \sup_{0<h\le\delta} \sup_{x\in(0,+\infty)} |f(x+h) - f(x)|$$

we denote the usual modulus of continuity of $f \in C_B(0,\infty)$.

In order to prove the theorems in this subsection, we need the following lemma.

Lemma 3.1. *Let $q \in (0,1)$, $x \in (0,\infty)$, $f \in C_B(0,\infty)$. Then, for all $g \in C_B^2(0,\infty)$, we have*

$$\left|\hat{\mathbb{G}}_{n,q}(g;x) - g(x)\right| \le \left(B_{n,q}(x) + D_{n,q}^2(x)\right)\|g''\|,$$

where

$$\begin{aligned} &\hat{\mathbb{G}}_{n,q}(f;x) = \mathbb{G}_{n,q}(f;x) + f(x) - f(T_{n,1}(x)), \\ &D_{n,q}(x) = T_{n,1}(x) - x. \end{aligned} \tag{3.3}$$

Proof. From (3.3) and Lemma 2.1, we have

$$\hat{\mathbb{G}}_{n,q}(t-x;x) = 0.$$

For $x \in (0,+\infty)$ and $g \in C_B^2[0,+\infty)$, using the Taylor's formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\mathrm{d}u,$$

we have

$$\begin{aligned} \hat{\mathbb{G}}_{n,q}(g;x) - g(x) &= \hat{\mathbb{G}}_{n,q}((t-x)g'(x);x) + \hat{\mathbb{G}}_{n,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\ &= g'(x)\hat{\mathbb{G}}_{n,q}((t-x);x) + \mathbb{G}_{n,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \end{aligned}$$

$$
\begin{aligned}
&- \int_x^{T_{n,1}(x)} \left(T_{n,1}(x) - u\right) g''(u)\mathrm{d}u \\
&= \mathbb{G}_{n,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u; x\right) \\
&- \int_x^{T_{n,1}(x)} \left(T_{n,1}(x) - u\right) g''(u)\mathrm{d}u.
\end{aligned}
$$

On the other hand, from

$$
\begin{aligned}
\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| &\le \left|\int_x^t |t-u||g''(u)|\mathrm{d}u\right| \\
&\le \|g''\| \left|\int_x^t |t-u|\mathrm{d}u\right| \le (t-x)^2\|g''\|,
\end{aligned}
$$

and

$$
\left|\int_x^{T_{n,1}(x)} \left(T_{n,1}(x) - u\right) g''(u)\mathrm{d}u\right| \le \left(T_{n,1}(x) - x\right)^2 \|g''\| = \left(D_{n,q}(x)\right)^2 \|g''\|,
$$

we conclude that

$$
\begin{aligned}
&\left|\hat{\mathbb{G}}_{n,q}(g;x) - g(x)\right| \\
&= \left|\mathbb{G}_{n,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u; x\right) - \int_x^{T_{n,1}(x)} \left(T_{n,1}(x) - u\right) g''(u)\mathrm{d}u\right| \\
&\le \mathbb{G}_{n,q}\left((t-x)^2\|g''\|; x\right) + \left(D_{n,q}(x)\right)^2 \|g''\| \\
&\le \left(B_{n,q}(x) + \left(D_{n,q}(x)\right)^2\right) \|g''\|.
\end{aligned}
$$

This completes the proof. □

Theorem 3.3. *Let $q \in (0,1)$, $f \in C_B[0,+\infty)$. Then, for every $x \in (0,+\infty)$, there exists a constant $C_1 > 0$ such that*

$$
|\mathbb{G}_{n,q}(f;x) - f(x)| \le C_1\omega_2\left(f; \sqrt{B_{n,q}(x) + \left(D_{n,q}(x)\right)^2}\right) + \omega\left(f; |D_{n,q}(x)|\right).
$$

Proof. By (3.3), we have

$$
|\hat{\mathbb{G}}_{n,q}^{\alpha,\beta}(f;x)| \le |\mathbb{G}_{n,q}^{\alpha,\beta}(f;x)| + 2\|f\| \le 3\|f\|. \tag{3.4}
$$

Using Lemma 3.1, for every $g \in C_B^2[0,+\infty)$, we obtain

$$
\begin{aligned}
&|\mathbb{G}_{n,q}(f;x) - f(x)| \\
&\le \left|\hat{\mathbb{G}}_{n,q}(f;x) - f(x)\right| + |f(x) - f(T_{n,1}(x))| \\
&\le \left|\hat{\mathbb{G}}_{n,q}(f-g;x) - (f-g)(x)\right| + |f(x) - f(T_{n,1}(x))| + |\hat{\mathbb{G}}_{n,q}(g;x) - g(x)| \\
&\le 4\|f-g\| + |f(x) - f(T_{n,1}(x))| + \left(B_{n,q}(x) + \left(D_{n,q}(x)\right)^2\right) \|g''\|
\end{aligned}
$$

$$\leq 4\|f-g\|+\omega\left(f;|D_{n,q}(x)|\right)+\left(B_{n,q}(x)+\left(D_{n,q}(x)\right)^2\right)\|g''\|.$$

Now, by taking infimum on the right-hand side for all $g \in C_B^2[0,\infty)$ and using (3.2), we get the following result:

$$\begin{aligned}|\mathbb{G}_{n,q}(f;x)-f(x)| &\leq 4K_2\left(f;B_{n,q}(x)+\left(D_{n,q}(x)\right)^2\right)+\omega\left(f;|D_{n,q}(x)|\right)\\ &\leq 4C_0\omega_2\left(f;\sqrt{B_{n,q}(x)+\left(D_{n,q}(x)\right)^2}\right)+\omega\left(f;|D_{n,q}(x)|\right)\\ &= C_1\omega_2\left(f;\sqrt{B_{n,q}(x)+\left(D_{n,q}(x)\right)^2}\right)+\omega\left(f;|D_{n,q}(x)|\right).\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $0<\gamma\leq 1$ and E be any bounded subset of the interval $(0,+\infty)$. If $f\in C_B(0,+\infty)\bigcap Lip_{C_2}(\gamma)$, then we have*

$$|\mathbb{G}_{n,q}(f;x)-f(x)|\leq C_2\left\{\left(B_{n,q}(x)\right)^{\frac{\gamma}{2}}+2(d(x;E))^{\gamma}\right\},$$

where M_3 is a constant depending only on α, $\mathrm{d}(x;E)$ *is the distance between x and E defined as*

$$\mathrm{d}(x;E)=\inf\left\{|t-x|: t\in E \text{ and } x\in(0,+\infty)\right\}.$$

Proof. From the properties of the infimum, there is at least one point t_0 in the closure of E such that

$$\mathrm{d}(x;E)=|t_0-x|.$$

By the triangle inequality, we have

$$|f(t)-f(x)|\leq|f(t)-f(t_0)|+|f(t_0)-f(x)|.$$

Thus

$$\begin{aligned}|\mathbb{G}_{n,q}(f;x)-f(x)| &\leq \mathbb{G}_{n,q}\left(|f(t)-f(x)|;x\right)\\ &\leq \mathbb{G}_{n,q}\left(|f(t)-f(t_0)|;x\right)+\mathbb{G}_{n,q}\left(|f(t_0)-f(x)|;x\right)\\ &\leq C_2\left\{\mathbb{G}_{n,q}\left(|t-t_0|^{\gamma};x\right)+|t_0-x|^{\gamma}\right\}\\ &\leq C_2\left\{\mathbb{G}_{n,q}\left(|t-x|^{\gamma};x\right)+2|t_0-x|^{\gamma}\right\}\end{aligned}$$

holds. Now we choose $p_1=\frac{2}{\gamma}$ and $p_2=\frac{2}{2-\gamma}$ such that $\frac{1}{p_1}+\frac{1}{p_2}=1$, then by Hölder inequality we have

$$\begin{aligned}|\mathbb{G}_{n,q}(f;x)-f(x)| &\leq C_2\left\{\left[\mathbb{G}_{n,q}(|t-x|^{\gamma p_1};x)\right]^{\frac{1}{p_1}}\left[\mathbb{G}_{n,q}(1^{p_2};x)\right]^{\frac{1}{p_2}}+2|t_0-x|^{\gamma}\right\}\\ &= C_2\left\{\mathbb{G}_{n,q}\left(|t-x|^2;x\right)^{\frac{\gamma}{2}}+2|t_0-x|^{\gamma}\right\}\\ &= C_2\left\{\left(\mathbb{G}_{n,q}(x)\right)^{\frac{\gamma}{2}}+2(\mathrm{d}(x;E))^{\gamma}\right\}.\end{aligned}$$

This completes the proof. □

3.3. Rate of convergence

Let $B_{x^2}(0,+\infty)$ be the set of all functions f defined on $(0,+\infty)$ satisfying the condition $|f(x)| \leq C_f(1+x^2)$, where C_f is a constant depending only on f. By $C_{x^2}(0,+\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}(0,+\infty)$. Also, let $C^*_{x^2}(0,+\infty)$ be the subspace of all functions $f \in C_{x^2}(0,+\infty)$, for which $\lim\limits_{x\to+\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}(0,+\infty)$ is $\|f\|_{x^2} = \sup\limits_{x\in(0,+\infty)} \frac{|f(x)|}{1+x^2}$. For any positive a, by

$$\omega_a(f;\delta) = \sup_{|t-x|<\delta}\ \sup_{x,t\in(0,a)} |f(t)-f(x)|$$

we denote the usual modulus of continuity of f on the interval $(0,a)$. We know that for a function $f \in C_{x^2}[0,+\infty)$, the modulus of continuity $\omega_a(f;\delta)$ tends to zero as $\delta \to 0$.

Now we give a rate of convergence theorem for the operators $\mathbb{G}_{n,q}$.

Theorem 3.5. *Let $f \in C_{x^2}(0,+\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f;\delta)$ be modulus of the continuity of f on the finite interval $(0,a+1] \subset (0,+\infty)$, where $a > 0$. Then for $n > 3$,*

$$|\mathbb{G}_{n,q}(f;x)-f(x)| \leq 5C_f(1+a^2)B_{n,q}(x)+\omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\left[B_{n,q}(x)\right]^{\frac{1}{2}}\right). \tag{3.5}$$

Proof. For $x \in (0,a]$ and $t > a+1$, since $t-x>1$, we have

$$\begin{aligned} |f(t)-f(x)| &\leq C_f(2+x^2+t^2) \\ &\leq C_f\left(2+x^2+t^2+(2x-t)^2\right) \\ &\leq C_f\left(2+3x^2+2(x-t)^2\right) \\ &\leq C_f(1+a^2)(t-x)^2. \end{aligned} \tag{3.6}$$

For $x \in (0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f;|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;|t-x|) \tag{3.7}$$

with $\delta > 0$.

From (3.6) and (3.7) we get

$$|f(t)-f(x)| \leq 5M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta) \tag{3.8}$$

for $x \in (0,a]$ and $t>0$. Thus

$$\begin{aligned} &|\mathbb{G}_{n,q}(f;x)-f(x)| \\ &\leq \mathbb{G}_{n,q}(|f(t)-f(x)|;x) \\ &\leq 5C_f(1+a^2)\mathbb{G}_{n,q}\left((t-x)^2;x\right)+\omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\left[\mathbb{G}_{n,q}((t-x)^2;x)\right]^{\frac{1}{2}}\right) \end{aligned}$$

$$\leq 5C_f(1+a^2)B_{n,q}(x) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\left[B_{n,q}(x)\right]^{\frac{1}{2}}\right).$$

The proof is completed. □

As is known, if f is not uniformly continuous on the interval $(0,\infty)$, the usual first modulus of continuity $\omega(f;\delta)$ does not tend to zero as $\delta\to 0$. For every $f\in C^*_{x^2}(0,\infty)$, we would like to take a weighted modulus of continuity $\Omega(f;\delta)$ which tends to zero as $\delta\to 0$.

Let

$$\Omega(f;\delta) = \sup_{0<h\leq\delta, x\geq 0}\frac{|f(x+h)-f(x)|}{1+(x+h)^2} \quad \text{for every } f\in C^*_{x^2}[0,\infty).$$

The weighted modulus of continuity $\Omega(f;\delta)$ was defined by Yuksel and Ispir in [21]. It is known that $\Omega(f;\delta)$ has the following properties.

Lemma 3.2 ([13])**.** *Let* $f\in C^*_{x^2}[0,\infty)$. *Then*:
i) $\Omega(f;\delta)$ *is a monotone increasing function of* δ,
ii) *For each* $f\in C^*_{x^2}[0,\infty)$, $\lim\limits_{\delta\to 0^+}\Omega(f;\delta)=0$,
iii) *For each* $m\in\mathbb{N}\setminus\{0\}$, $\Omega(f;m\delta)\leq m\Omega(f;\delta)$,
iv) *For each* $\lambda\in\mathbb{R}^+$, $\Omega(f;\lambda\delta)\leq(1+\lambda)\Omega(f;\delta)$.

Theorem 3.6. *Let* $f\in C^*_{x^2}[0,\infty)$ *and* $q=q_n\in(0,1)$ *such that* $q_n\to 1$ *as* $n\to\infty$. *Then there exists a positive constant* C_3 *such that the inequality*

$$\sup_{x\in(0,\infty)}\frac{|\mathbb{G}_{n,q_n}(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq C_3\Omega\left(f;\frac{1}{\sqrt{[n+2]_{q_n}}}\right) \tag{3.9}$$

holds.

Proof. For $t>0$, $x\in(0,\infty)$ and $\delta>0$, by the definition of $\Omega(f;\delta)$ and Lemma 3.2, we get

$$\begin{aligned}|f(t)-f(x)| &\leq (1+(x+|x-t|))^2\,\Omega(f;|t-x|)\\ &\leq 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Since $\mathbb{G}_{n,q_n}$ is linear and positive, we have

(3.10)

$$\begin{aligned}&|\mathbb{G}_{n,q_n}(f;x)-f(x)|\\ &\leq 2(1+x^2)\Omega(f;\delta)\left\{1+\mathbb{G}_{n,q_n}\left((t-x)^2;x\right)+\mathbb{G}_{n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right\}.\end{aligned}$$

Using Lemma 2.2, we have

$$\mathbb{G}_{n,q_n}\left((t-x)^2;x\right)\leq C_4\frac{1+x^2}{[n+2]_{q_n}} \tag{3.11}$$

for some positive constant C_4. To estimate the second term of (3.10), applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned}(3.12)\quad &\mathbb{G}_{n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\\ &\leq 2\left(\mathbb{G}_{n,q_n}\left(1+(t-x)^4;x\right)\right)^{\frac{1}{2}}\left(\mathbb{G}_{n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}}.\end{aligned}$$

By Lemma 2.2 and (3.11), there exist two positive constants C_5, C_6 such that

$$(3.13)\quad \left(\mathbb{G}_{n,q_n}\left(1+(t-x)^4;x\right)\right)^{\frac{1}{2}} \leq C_5(1+x^2)$$

and

$$(3.14)\quad \left(\mathbb{G}_{n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}} \leq \frac{C_6}{\delta}\sqrt{\frac{1+x^2}{[n+2]_{q_n}}}.$$

Now we take $C_3 = 2 + 2C_4 + 4C_5C_6$ and $\delta = \frac{1}{\sqrt{[n+2]_{q_n}}}$. Combining the above estimates, we obtain the inequality (3.9). The proof is completed. □

3.4. Weighted approximation

Now, we shall discuss the weighted approximation theorem as follows:

Theorem 3.7. *Let the sequence $q = \{q_n\}$ satisfy $0 < q_n < 1$, $q_n \to 1$ and $[n]_{q_n} \to \infty$ as $n \to \infty$. Then for $f \in C^*_{x^2}(0,\infty)$, we have*

$$(3.15)\quad \lim_{n\to\infty} \|\mathbb{G}_{n,q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using Korovkin's theorem (see [12]), it is sufficient to verify the following three conditions:

$$(3.16)\quad \lim_{n\to\infty} \|\mathbb{G}_{n,q_n}(t^v) - x^v\|_{x^2} = 0,\ v = 0,1,2.$$

Since $\mathbb{G}_{n,q_n}(1;x) = 1$, (3.16) holds for $v = 0$.

By Lemma 2.1. we have,

$$\begin{aligned}\|\mathbb{G}_{n,q_n}(t;x) - x\|_{x^2} &= \sup_{x\in(0,\infty)} \frac{1}{1+x^2}\left|\mathbb{G}_{n,q_n}(t;x) - x\right|\\ &= \sup_{x\in(0,\infty)} \frac{1}{1+x^2}\left|\frac{\alpha-\beta x}{[n]_{q_n}} + \beta\right|\\ &\leq \frac{\alpha}{[n]_{q_n}+\beta}\sup_{x\in(0,\infty)}\frac{1}{1+x^2} + \frac{\beta}{[n]_{q_n}+\beta}\sup_{x\in(0,\infty)}\frac{x}{1+x^2}\\ &\leq \frac{\alpha+\beta}{[n]_{q_n}+\beta} \to 0,\ n\to\infty.\end{aligned}$$

and the second condition of (3.16) holds for $v = 1$ as $n \to \infty$.

Similarly we can write

$$\begin{aligned}
&\left\|\mathbb{G}_{n,q_n}(t^2;x)-x^2\right\|_{x^2}\\
&=\sup_{x\in(0,\infty)}\frac{1}{1+x^2}\left|\left(\frac{[n]_{q_n}^2[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2}-1\right)x^2+\frac{2\alpha[n]_{q_n}}{([n]_{q_n}+\beta)^2}x+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\right|\\
&\le\left|\frac{[n]_{q_n}^2[n+2]_{q_n}^2}{q_n[n+1]_{q_n}^2([n]_{q_n}+\beta)^2}-1\right|+\frac{2\alpha[n]_{q_n}}{([n]_{q_n}+\beta)^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\\
&=0,\ n\to\infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\|\mathbb{G}_{n,q_n}(t^2;x)-x^2\right\|_{x^2}=0.$$

Thus the proof is completed. □

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