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PERIMETER CENTROIDS AND CIRCUMSCRIBED QUADRANGLES

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Abstract. For a quadrangle P, we consider the centroid G_0 of the vertices of P, the perimeter centroid G_1 of the edges of P and the centroid G_2 of the interior of P, respectively. If G_0 is equal to G_1 or G_2, then the quadrangle P is a parallelogram. We denote by M the intersection point of two diagonals of P.

In this note, first of all, we show that if M is equal to G_0 or G_2, then the quadrangle P is a parallelogram. Next, we investigate various quadrangles whose perimeter centroid coincides with the intersection point M of diagonals. As a result, for an example, we show that among circumscribed quadrangles rhombi are the only ones whose perimeter centroid coincides with the intersection point M of diagonals.

1. Introduction

For a quadrangle P, we consider the centroid G_0 of the vertices of P, the centroid G_1 of the edges of P and the centroid G_2 of the interior of P, respectively. The centroid G_1 of the edges of P is also called the perimeter centroid of P ([3]). See also [2]. Then we have the following ([9]):

Proposition 1.1. Let P denote a quadrangle. Then the following are equivalent.

(1) P satisfies $G_0 = G_1$.

(2) P satisfies $G_0 = G_2$.

(3) P is a parallelogram.

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If we denote by M the intersection point of two diagonals of P, then we have the following. For a proof, see Section 2.

Proposition 1.2. Let P denote a quadrangle. Then the following are equivalent.

(1) P satisfies $G_0 = M$.

(2) P satisfies $G_2 = M$.

(3) P is a parallelogram.

The perimeter centroid of a parallelogram also coincides with the intersection point of two diagonals. Hence, it is quite natural to ask the following:

Question 1.3. Is $G_1 = M$ a characteristic property of parallelograms?

In this regard, recently in [7], the following characterizations were established.

Proposition 1.4. Suppose that P denotes a convex quadrangle whose two diagonals are perpendicular to each other. We denote by M the intersection point of diagonals of P. Then the following are equivalent.

(1) P satisfies $G_0 = M$.

(2) P satisfies $G_1 = M$.

(3) P satisfies $G_2 = M$.

(4) P is a rhombus.

In this note, we investigate various quadrangles whose perimeter centroid coincides with the intersection point M of diagonals. As a result, in Section 3 we prove the following characterization theorems.

Theorem 1.5. For a trapezoid P, the following are equivalent.

(1) P satisfies $G_1 = M$.

(2) P is a parallelogram.

Theorem 1.6. For a quadrangle P with a pair of opposite edges of equal length, the following are equivalent.

(1) P satisfies $G_1 = M$.

(2) P is a parallelogram.

In Section 4, first of all, we prove

Theorem 1.7. For a circumscribed quadrangle P, the following are equivalent.

(1) P satisfies $G_1 = M$.

(2) P is a parallelogram.

(3) P is a rhombus.

Finally, in Section 4, we state some characterization theorems which can be proved in a similar argument as in the proof of Theorem 1.7.

In case P is a triangle, the centroid G_1 coincides with the center of the Spieker circle, which is the incircle of the triangle formed by connecting midpoint of each side of the original triangle P ([2, p. 249]). In this case, the centroid G_0 always coincides with the centroid $G_2(= G)$, where $G = (A+B+C)/3$. Furthermore, the perimeter centroid G_1 of P satisfies $G_1 = G_2$ if and only if the triangle P is equilateral ([11, Theorem 2]).

In case P is a polygon, the geometric method to find the centroid G_2 of P was given in [4]. In [10], mathematical definitions of centroid G_2 of planar bounded domains were given. For higher dimensions, it was shown that the centroid G_0 of the vertices of a simplex in an n-dimensional space always coincides with the centroid G_n of the simplex ([1, 11]).

Archimedes established some area properties of parabolic sections and then formulated the centroid of parabolic sections ([12]). Using these properties, some characterizations of parabolas were given in [5, 6, 8].

2. Preliminaries and Proposition 1.2

In this section, first of all we recall the centroids of a quadrangle. For centroids of a quadrangle $ABCD$, we have the following, where we use the notations given in Section 1.

Lemma 2.1. Let us denote by P the convex quadrangle $ABCD$. Then we have the following.

(1) The centroid G_0 of P is given by

$$G_0 = \frac{A + B + C + D}{4}. \tag{2.1}$$

(2) The centroid G_1 of P is given by

$$G_1 = \frac{(l_4 + l_1)A + (l_1 + l_2)B + (l_2 + l_3)C + (l_3 + l_4)D}{2l}, \tag{2.2}$$

where we put $l_1 = AB, l_2 = BC, l_3 = CD, l_4 = DA$ and $l = l_1 + \cdots + l_4$.

(3) If $m = \delta + \beta$, where $\delta = \triangle ABC$ and $\beta = \triangle ACD$, then the centroid G_2 of P is given by

$$G_2 = \frac{mA + \delta B + mC + \beta D}{3m}, \tag{2.3}$$

Proof. It is straightforward to prove (1), (2) and (3) or see [4, 9].

Now, we prove Proposition 1.2 stated in Section 1.

Proof of Proposition 1.2. Suppose that P denotes a quadrangle. We denote by M the intersection point of diagonals of P. We may introduce a coordinates system so that the point M is the origin and the vertices of P are given by

$$A(a, 0), B(b, c), C(-d, 0), D = -kB, \tag{2.4}$$

where a, c, d and k are positive real numbers. It follows from Lemma 2.1 that the centroids of P are given by

$$\begin{aligned} G_0 &= \frac{1}{4}(a + b - d - kb, c - kc), \\ G_2 &= \frac{1}{3(k+1)}((a + b - d) + k(a - d - kb), (1 - k^2)c), \end{aligned} \tag{2.5}$$

First, suppose that $G_0 = M$. Then it follows from (2.5) that $k = 1$ and hence $a = d$. This shows that (1) $\Rightarrow$ (3).

Now, suppose that $G_2 = M$. Then (2.5) implies that $k = 1$ and hence $a = d$. This shows that (2) $\Rightarrow$ (3).

Note that every parallelogram satisfies $G_0 = M$ and $G_2 = M$. This completes the proof of Proposition 1.2.

3. Theorems 1.5 and 1.6

In this section, we prove Theorems 1.5 and 1.6 stated in Section 1. First, we prove Theorem 1.5 as follows.

Proof of Theorem 1.5. Since every parallelogram satisfies $G_1 = M$, it suffices to show that (1) $\Rightarrow$ (2).

Suppose that P denotes a trapezoid. We denote by M the intersection point of diagonals of P. We may introduce a coordinates system so that the vertices of P are given by

$$A(a,0), B(b,k), C(c,0), D(-a,0), \tag{3.1}$$

where a and k are positive real numbers and $b > c$. The intersection point M of diagonals of P is as follows.

$$M = \frac{a}{2a+b-c}(b+c, 2k). \tag{3.2}$$

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by

$$G_1 = \frac{1}{2l}((a+b)l_1 + (c-a)l_3 + (b+c)l_2, k(l_1 + 2l_2 + l_3)), \tag{3.3}$$

where we put by l the perimeter of P with

$$l_1 = \sqrt{(a-b)^2 + k^2}, l_2 = b - c, l_3 = \sqrt{(a+c)^2 + k^2}, l_4 = 2a. \tag{3.4}$$

Suppose that $G_1 = M$. Then it follows from (3.2) and (3.3) that

$$4a(l_1 + l_2 + l_3 + l_4) = (l_2 + l_4)(l_1 + 2l_2 + l_3). \tag{3.5}$$

Since $l_4 = 2a$, (3.5) becomes

$$2l_4(l_1 + l_2 + l_3 + l_4) = (l_2 + l_4)(l_1 + 2l_2 + l_3). \tag{3.6}$$

Note that (3.6) can be rewritten as

$$(l_4 - l_2)\{l_1 + l_3 + 2(l_2 + l_4)\} = 0, \tag{3.7}$$

which shows that $l_2 = l_4$. Hence we see that (1) $\Rightarrow$ (2). This completes the proof of Theorem 1.5.

We now prove Theorem 1.6 as follows.

Proof of Theorem 1.6. Since every parallelogram satisfies $G_1 = M$, it suffices to show that (1) $\Rightarrow$ (2).

Suppose that P denotes a quadrangle with a pair of opposite edges of equal length. We denote by M the intersection point of diagonals of P. Using similarity transformation if necessary, we may introduce a coordinates system so that the vertices of P are given by

$$A(x,y), B(0,a), C(-1,0), D(0,-b), \tag{3.8}$$

where a, b and x are positive real numbers. The intersection point M of diagonals of P is as follows.

$$M = \frac{1}{x+1}(0,y). \tag{3.9}$$

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by
(3.10)

$$G_1 = \frac{1}{2l}(x(l_1+l_4)-(l_2+l_3), y(l_1+l_4)+a(l_1+l_2)-b(l_3+l_4)),$$

where we put by l the perimeter of P with
(3.11)

$$l_1 = \sqrt{x^2+(y-a)^2}, l_2 = \sqrt{a^2+1}, l_3 = \sqrt{b^2+1}, l_4 = \sqrt{x^2+(y+b)^2}.$$

Since P has a pair of opposite edges of equal length, we may assume that

$$l_4 = l_2. \tag{3.12}$$

Now, suppose that $G_1 = M$. Then it follows from (3.9) and (3.10) that

$$l_1 + l_4 = \frac{1}{x}(l_2+l_3) \tag{3.13}$$

and

$$al_1 - bl_4 = \frac{l_2}{x}(y-ax) + \frac{l_3}{x}(y+bx). \tag{3.14}$$

Combining (3.12) with (3.13), we get

$$l_1 = \frac{1-x}{x}l_2 + \frac{1}{x}l_3. \tag{3.15}$$

Substituting l_4 and l_1 in (3.12) and (3.15) respectively into (3.14), we obtain

$$(y+bx-a)(l_2+l_3) = 0, \tag{3.16}$$

which implies that $y = -bx + a$, and hence AB is parallel to CD. This shows that the quadrangle P is a trapezoid. Thus, Theorem 1.5 completes the proof of Theorem 1.6.

4. Circumscribed quadrangles and others

In this section, first of all, we prove Theorem 1.7 stated in Section 1 as follows.

Proof of Theorem 1.7. Note that for a circumscribed quadrangle we have (2) $\Leftrightarrow$ (3). Hence, as in the proof of Theorem 1.6 it suffices to show that (1) $\Rightarrow$ (3).

Suppose that P denotes a circumscribed quadrangle. We denote by M the intersection point of diagonals of P. By a suitable similarity transformation if necessary, we may introduce a coordinates system so that the vertices of P are given by

$$A(x,y), B(0,a), C(-1,0), D(0,-b), \tag{4.1}$$

where a, b and x are positive real numbers. Without loss of generality, we may assume that $a \geq b$.

The intersection point M of diagonals of P is as follows.

$$M = \frac{1}{x+1}(0,y). \tag{4.2}$$

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by

$$G_1 = \frac{1}{2l}(x(l_1+l_4)-(l_2+l_3), y(l_1+l_4)+a(l_1+l_2)-b(l_3+l_4)), \tag{4.3}$$

where we put by l the perimeter of P with

$$l_1 = \sqrt{x^2+(y-a)^2}, l_2 = \sqrt{a^2+1}, l_3 = \sqrt{b^2+1}, l_4 = \sqrt{x^2+(y+b)^2}. \tag{4.4}$$

Since P is a circumscribed quadrangle, we get

$$l_1 - l_4 = l_2 - l_3. \tag{4.5}$$

Now, suppose that $G_1 = M$. Then it follows from (4.2) and (4.3) that

$$l_1 + l_4 = \frac{1}{x}(l_2+l_3) \tag{4.6}$$

and

$$al_1 - bl_4 = \frac{l_2}{x}(y-ax) + \frac{l_3}{x}(y+bx). \tag{4.7}$$

It follows from (4.5) and (4.6) that

$$2l_1 = \frac{x+1}{x}l_2 + \frac{1-x}{x}l_3 \tag{4.8}$$

and

$$2l_4 = \frac{1-x}{x}l_2 + \frac{x+1}{x}l_3. \tag{4.9}$$

By substituting l_1 and l_4 in (4.8) and (4.9) respectively into (4.7), we get

$$2y(l_2 + l_3) = x\{(3a + b)l_2 - (a + 3b)l_3\} + (a - b)(l_2 + l_3). \tag{4.10}$$

Squaring the both sides of (4.8) and (4.9) respectively, we obtain

$$2y = (a - b)\frac{x-1}{x}. \tag{4.11}$$

Finally, we substitute y in (4.11) into (4.10). Then we have

$$x^2\{(3a + b)l_2 - (a + 3b)l_3\} = (b - a)(l_2 + l_3). \tag{4.12}$$

Since $a \geq b$, we have two cases as follows.

Case 1. $a = b$. Then we have $l_2 = l_3$ and hence it follows from (4.5) that $l_1 = l_4$. Furthermore, we get from (4.7) that $y = 0$. Together with (4.6), this shows that $x = 1$. Hence, the quadrangle P is a rhombus.

Case 2. $a > b$. Then we have $l_2 > l_3$ and $3a+b > a+3b$. Hence the left side of (4.12) is positive. However the right side of (4.12) is negative, which is a contradiction.

Summarizing the above cases, we see that the quadrangle P is a rhombus, which shows that (1) $\Rightarrow$ (3). This completes the proof of Theorem 1.7.

Finally, we state some characterization theorems which can be proved in a similar argument as in the proof of Theorem 1.7. We will omit the proofs of them.

Theorem 4.1. For a quadrangle P with a pair of adjacent edges of equal length, the following are equivalent.

(1) P satisfies $G_1 = M$.

(2) P is a parallelogram.

(3) P is a rhombus.

Theorem 4.2. Suppose that a quadrangle P has two pairs of adjacent edges such that the total length of a pair equals to that of the other pair. Then the following are equivalent.

(1) P satisfies $G_1 = M$.

(2) P is a parallelogram.

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