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G-CW COMPLEX STRUCTURES OF PROPER SEMIALGEBRAIC G-SETS

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Abstract. Let G be a semialgebraic group which is not necessarily compact. Let X be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. In this paper we prove that X has a finite open straight G-CW complex structure.

1. Introduction

A semialgebraic set is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities. Throughout this paper we consider the semialgebraic sets in $\mathbb{R}^n$ equipped with the subspace topology induced by the usual topology of $\mathbb{R}^n$. A continuous map $f\colon X \to Y$ between semialgebraic sets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ is called *semialgebraic* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. Note that all semialgebraic maps are assumed to be continuous.

In this paper we discuss topological properties of semialgebraic sets with semialgebraic actions of semialgebraic groups. Let X be a semialgebraic set and let G be a semialgebraic group which is not necessarily compact. We say X is a semialgebraic G-set if the action $\theta\colon G \times X \to X$ is semialgebraic. A semialgebraic G-set X is called *proper* if the associated action

$$\vartheta_*\colon G \times X \to X \times X, \quad (g, x) \mapsto (\theta(g, x), x)$$

is proper. We remark that when G is compact, every semialgebraic G-set is proper.

A fundamental question in transformation group theory is whether a given G-space has a G-CW complex structure. Illman showed in [10]

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that a subanalytic space with proper action of a Lie group with the compact orbit space has a G-CW complex structure. On the other hand Park and Suh showed in [17] and [18] that a semialgebraic space with a semialgebraic action of a compact Lie group has a finite open G-CW complex structure. We extend this result to a semialgebraic set with a proper semialgebraic action of a semialgebraic group which is not necessarily compact. Namely, we have Theorem 3.6.

We remark that the stability under finite union is the wide difference between semialgebraic category and the other (topological, subanalytic, semi-analytic, or smooth) categories. For example, the infinite union of locally finite semialgebraic sets is not a semialgebraic set. Thus the attaching map of infinite semialgebraic maps, which are well-defined in the intersections, is not semialgebraic in general.

2. Semialgebraic actions

In this section we gather some results about semialgebraic actions. For more details, see [4, 5, 11, 14, 15, 16, 18].

The class of semialgebraic sets in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real valued polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite union, finite intersection and complement.

It is easy to see that finite unions and finite intersections of semialgebraic sets are semialgebraic and that the complement of a semialgebraic set is semialgebraic. Furthermore, the closure, and thus the interior, of a semialgebraic set are semialgebraic. In addition, every connected component of a semialgebraic set is semialgebraic and the family of the connected components of a semialgebraic set is finite. For the general theory of semialgebraic sets and semialgebraic maps, we refer the reader to [1, 7].

The definition of a semialgebraic group is given obviously, i.e., a semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. A semialgebraic homomorphism between two semialgebraic groups is a semialgebraic map that is a group homomorphism. If H is a subgroup and semialgebraic subset of a semialgebraic group G, then H is called a *semialgebraic subgroup* of G.

Proposition 2.1. (1) *Every semialgebraic group has a Lie group structure, and hence locally compact.*

(2) *Every semialgebraic subgroup of a semialgebraic group is closed.*
(3) *The normalizer $N(H)$ of a semialgebraic subgroup H of a semialgebraic group G is also a semialgebraic subgroup of G.*
(4) *Let G be a semialgebraic group and H a semialgebraic subgroup of G. If $gHg^{-1} \subset H$ for some $g \in G$, then $gHg^{-1} = H$.*

For (topological) proper actions the following proposition appears in [8, Section 1.3] whose proofs are straightforward.

Proposition 2.2. *Let X be a proper semialgebraic G-set and let $x \in X$, then*

(1) *the isotropy subgroup $G_x = \{g \in G \mid g(x) = x\}$ is compact and semialgebraic,*
(2) *the orbit $G(x) = \{gx \in X \mid g \in G\}$ is a closed semialgebraic subset of X,*
(3) *the evaluation map $\theta_x \colon G \to X$, $g \mapsto gx$, is proper,*
(4) *the fixed point set $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$ is a closed semialgebraic subset of X.*

Let X be a proper semialgebraic G-set. If H is a semialgebraic subgroup H of G, then the restriction $\vartheta_*| \colon H \times X \to X \times X$ of ϑ_* is proper; hence X is a proper semialgebraic H-set. Moreover, if A is a G-invariant semialgebraic subset of X, then the restriction $\vartheta_*| \colon G \times A \to A \times A$ is proper; hence A is also a proper semialgebraic G-set.

Working in semialgebraic category requires a lot of nontrivial efforts to establish some of the properties which are easy or well-known in topological or smooth category. One of such properties is the existence of semialgebraic structure on the orbit space of a semialgebraic G-set X. Here is a natural question. Does there exist a semialgebraic structure of the orbit space X/G in the natural sense? A *semialgebraic structure* (N, f) of X/G is a semialgebraic set $N \subset \mathbb{R}^k$ together with a semialgebraic map $f \colon X \to N$ which is topologically quotient map of X by G. In this case we can substitute X/G and the orbit map $\pi \colon X \to X/G$ with N and f respectively. Scheiderer [19] gave us a partially positive answer of this question as follows.

Theorem 2.3 ([19])**.** *Let X be a proper semialgebraic G-set which is locally compact. Then the orbit space X/G has a semialgebraic structure such that the orbit map $\pi \colon X \to X/G$ is semialgebraic.*

When G is compact, this was proved by Brumfiel [3] without the assumption that X is locally compact.

As a specific example of a proper semialgebraic action, we consider the following situation; let G be a semialgebraic group and H a semialgebraic subgroup of G. Then G can be seen as a proper semialgebraic H-set where H acts by the right multiplication on G. Note that every semialgebraic group is locally compact, hence that the quotient space G/H has a semialgebraic structure.

We say two homogeneous semialgebraic G-sets are equivalent if they are semialgebraically G-homeomorphic. Let (G/H) denote the equivalence class of G/H. Moreover, the set of all equivalence classes of homogeneous semialgebraic G-sets has the natural partial ordering defined as $(G/K) \le (G/H)$ if there exists a semialgebraic G-map $G/H \to G/K$. Then $(G/K) \le (G/H)$ if and only if H is conjugate to a subgroup of K.

When a homogeneous space X of G is equivalent to G/H, the conjugacy class (H) of H in G is called the *isotropy type* of X. For semialgebraic subgroups H, K of G, a partial ordering $\le$ is given by $(H) \le (K)$ if and only if $(G/H) \ge (G/K)$. For more details, see [2, 8].

Let X be a semialgebraic G-set. As the theory of Lie group actions, the natural map

$$\alpha_x \colon G/G_x \to G(x), \quad gG_x \mapsto gx$$

is semialgebraic G-homeomorphism([18]). We define *orbit type map* from X to the set of equivalence classes of homogeneous semialgebraic G-sets by $\mathrm{type}(x) = (G/G_x)$. We call $\mathrm{type}(x)$ the *orbit type* of x. The following theorem is one of the main results of [15].

Theorem 2.4 ([15])**.** *Every proper semialgebraic G-set has only finitely many orbit types.*

For a semialgebraic G-set X and a semialgebraic subgroup H of G, we set

$$\begin{aligned} X_{(H)} &= \{x \in X \mid (G_x) = (H)\} \\ &= \{x \in X \mid G_x = gHg^{-1} \ \text{ for some } g \in G\}. \end{aligned}$$

Then $X_{(H)}$ is the set of points on orbits of type (G/H).

Corollary 2.5. *Let G be a semialgebraic group and H a semialgebraic subgroup of G, and X a proper semialgebraic G-set. Then $X_{(H)}$ is a proper semialgebraic G-subset of X.*

Proof. In fact, it is proved in [18, Proposition 2.8] when X has finitely many orbit types. So it follows from Theorem 2.4. □

Let X be a proper semialgebraic G-set and H a semialgebraic subgroup of G. Then X^H is a proper semialgebraic $N(H)$-set by Proposition 2.2 where $N(H)$ is the normalizer of H in G.

Remark 2.6 ([11]). *For a semialgebraic subgroup H of G and a proper semialgebraic G-set X whose orbit space has a semialgebraic structure, the set*

$$\begin{aligned} X_{(\geq H)} &= \{x \in X \mid (G_x) \geq (H)\} \\ &= \{x \in X \mid G_x \supset gHg^{-1} \ \text{ for some } g \in G\} \end{aligned}$$

is a closed semialgebraic G-subset of X because $X_{(\geq H)} = GX^H$. Since (G/H) is the largest orbit type occurring in GX^H, $X^H/N(H)$ has a semialgebraic structure such that the inclusion $j\colon X^H \hookrightarrow GX^H$ induces a semialgebraic homeomorphism $\tilde{j}\colon X^H/N(H) \to GX^H/G$.

Let X be a G-space and H a closed subgroup of G. A subset S of X is called *H-kernel* if there exists a continuous G-map $f\colon GS \to G/H$ such that $f^{-1}(eH) = S$, where e is the identity of G.

Proposition 2.7. *Let X be a G-space and H a compact subgroup of G. Let $S \subset X$ be an H-kernel. Then the action map $\theta\colon G \times S \to GS$ is proper.*

Let G be a semialgebraic group and H a semialgebraic subgroup of G. Let X be a semialgebraic G-set. A semialgebraic subset S of X will be called a *semialgebraic H-slice* if GS is an open semialgebraic subset of X and there exists a semialgebraic G-map $f\colon GS \to G/H$ such that $f^{-1}(eH) = S$. For $x \in X$ a *semialgebraic slice* at x means a semialgebraic G_x-slice S in X such that $x \in S$. We call GS a *semialgebraic G-tube* about $G(x)$.

Theorem 2.8 ([11]). *Let G be a semialgebraic group, and let X be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. Then*

(1) *for each $x \in X$, there exists a semialgebraic G_x-slice S at x, and*
(2) *X can be covered by a finite number of semialgebraic G-tubes.*

Moreover, we have the following properties.

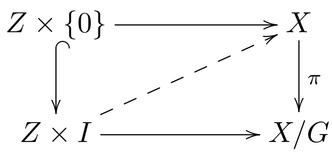
- The map $\varphi\colon G \times_{G_x} S \to GS$ defined by $[g, s] \mapsto gs$ is a semialgebraic G-homeomorphism.
- The map $\kappa\colon S/G_x \to GS/G$ defined by $[s] \mapsto [s]$ is a semialgebraic homeomorphism.

Note that the definition of a semialgebraic fiber bundle is analogous to that of the bundles in other categories, except that we require finiteness of the covering of locally trivial open sets in the semialgebraic category.

Theorem 2.9. *Let X be a proper semialgebraic G-set with only one orbit type. Suppose that X/G has a semialgebraic structure. Then the orbit map $\pi\colon X \to X/G$ is a semialgebraic fiber bundle.*

Proof. Since X has one orbit type (G/H), each tube has the form $G \times_H S$ and H acts trivially on S, see the proof of Theorem 5.8 of Chapter II of [2]. Therefore each tube defines a trivial bundle over S with G/H as the fiber. By Theorem 2.8 X can be covered by finitely many G-tubes. Hence $\pi\colon X \to X/G$ has a semialgebraic bundle structure. □

Corollary 2.10. *Let X be a proper semialgebraic G-set with only one orbit type. Suppose that X/G has a semialgebraic structure. Then the orbit map $\pi\colon X \to X/G$ is a semialgebraic fibration, that is, the following commutative diagram can always be completed in the semialgebraic category.*



3. Semialgebraic G-CW complex structures

In this section we show that every proper semialgebraic G-set whose orbit space is semialgebraic has a finite open G-CW complex structure.

Firstly, we deal with the semialgebraic triangulation of semialgebraic sets. Let $a_0, \dots, a_n$ be generically independent points of $\mathbb{R}^m$. The *n-simplex* $\langle a_0, \dots, a_n\rangle$ spanned by $a_0, \dots, a_n$ is defined by

$$\langle a_0, \dots, a_n\rangle = \Big\{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\Big\}.$$

The *open n-simplex* $(a_0, \dots, a_n)$ spanned by $a_0, \dots, a_n$ is defined by

$$(a_0, \dots, a_n) = \Big\{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i > 0\Big\}.$$

Note that the open 0-simplex (a) is equal to $\langle a\rangle$ from the definition. Clearly both $\langle a_0, \dots, a_n\rangle$ and $(a_0, \dots, a_n)$ are semialgebraic sets in $\mathbb{R}^m$.

A *finite open simplicial complex* $\big(K, \{\sigma_i \mid i \in I\}\big)$ is defined as a subset of some $\mathbb{R}^m$ equipped with a partition $\{\sigma_i \mid i \in I\}$ composed of a finite number of open simplices σ_i in $\mathbb{R}^m$, such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two open simplices σ_i and σ_j is either empty or a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$. Thus a finite open simplicial complex K is obtained by deleting some open simplices from a "usual" finite simplicial complex. The following theorem can be seen in many places, for instance, [6, Section 2], [1, Chapter 9].

Theorem 3.1. *Let X be a semialgebraic set, and let $X_1, \dots, X_k$ be semialgebraic subsets of X. Then there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\tau\colon |K| \to X$ such that each X_i is a finite union of some of $\tau(\sigma)$, where σ is an open simplex of K.*

We call such (K, τ) a *finite open semialgebraic triangulation* of X compatible with $X_1, \dots, X_k$.

Definition 3.2. *Let G be a topological group. An* open G-CW complex *is a pair $(X, \{c_i \mid i \in I\})$ of a Hausdorff G-space X and a family of open G-cells c_i such that*

(1) *the orbit space X/G is a Hausdorff space,*
(2) *for each open G-n-cell c_i, there exist a subgroup H_{c_i} of G and the characteristic G-map $f_{c_i}\colon G/H_{c_i} \times \delta \to \bar{c}_i \subset X$ such that the restriction $f_{c_i}|\colon G/H_{c_i} \times \overset{\circ}{\delta} \to c_i$ is a G-homeomorphism and the boundary ∂c_i is equal to $f_{c_i}(G/H_{c_i} \times \partial\delta)$ where δ is a subset of a compact standard n-simplex Δ^n obtained by removing some finite open lower dimensional faces of Δ^n and $\partial\delta = \delta - \overset{\circ}{\delta}$. Note that $\overset{\circ}{\delta} = (\Delta^n)^\circ$. Moreover*
(3) *the closure $\bar{c}_i$ of each open G-cell c_i in X contains only finitely many open G-cells, and*
(4) *X has the weak topology with respect to the closed covering $\{\bar{c}_i\}$ of X.*

An open G-CW complex X is said to be *finite* if X has only a finite number of open G-cells. To distinguish an open G-CW complex from classical G-CW complex, we call the latter a *complete* G-CW complex here. An open G-CW complex is *straight* if the restriction $f_c|_{\{eH_c\}\times\delta}\colon \{eH_c\} \times \delta \to f_c(\{eH_c\} \times \delta)$ of the characteristic map $f_c\colon G/H_c \times \delta \to \bar{c}$ for each open G-n-cell c of X is a homeomorphism.

Usually in other categories (such as topological, subanalytic, semianalytic, or smooth) we can find 'complete' (possibly infinite) simplicial or CW complex structures. But in the semialgebraic category we can only obtain 'finite open' simplicial or CW complex structures. The reason is that the semialgebraic category is not stable under infinite union. For example, even if $f_i\colon A_i \to X$ are semialgebraic for $i \in I$, $|I| = \infty$ such that $f_i = f_j$ on $A_i \cap A_j$ for all $i, j \in I$, the attaching map $\cup f_i\colon \cup A_i \to X$ need not be semialgebraic.

We now construct a finite open straight G-CW complex structure of a proper semialgebraic G-set. For this we need the following lemma.

Lemma 3.3 ([5, Lemma 3.2]). *Let X be a semialgebraic set and A a closed semialgebraic subset of X. Suppose that A is a semialgebraic strong deformation retract of X. Then for a given semialgebraic neighborhood U of A there is a closed semialgebraic neighborhood N of A contained in U with a semialgebraic map $\rho\colon X \to U$ such that $\rho(x) = x$ for $x \in N$ and $\rho(X - N) \subset U - N$.*

Let δ be an n-dimensional simplex which is not necessarily closed. Namely δ is obtained from a closed n-simplex by deleting some lower dimensional open faces. A *straight filtration* of δ is a filtration

$$\varnothing = \delta^{-1} \subset \delta^0 \subset \delta^1 \subset \cdots \subset \delta^{n-1} \subset \delta^n = \delta$$

where δ^k is a face of δ which is closed in δ such that if $\delta^0 = \delta^1 = \cdots = \delta^{k_0-1} = \varnothing$ but $\delta^{k_0} \neq \varnothing$, then $\dim \delta^k = k$ for all $k \geq k_0$.

Lemma 3.4. *Let G be a compact semialgebraic group. Let X be a semialgebraic G-set with the orbit space X/G equals to an n-dimensional simplex δ (which is not compact) with a straight filtration*

$$\delta^0 \subset \delta^1 \subset \cdots \subset \delta^n = \delta$$

such that $\pi^{-1}(\delta^k - \delta^{k-1})$ has a constant orbit type for each $0 \leq k \leq n$. Then there is a semialgebraic section $s\colon \delta \to X$ of the orbit map $\pi\colon X \to X/G = \delta$ such that $s(\delta^k - \delta^{k-1})$ has a constant isotropy subgroup for each $0 \leq k \leq n$.

Proof. We prove the lemma by the triple induction on n, on the dimension of G, and on the number of components of G. By the induction hypothesis we assume that there exists a semialgebraic section $s'\colon \delta^{n-1} \to \pi^{-1}(\delta^{n-1})$ such that $s'(\delta^k - \delta^{k-1})$ has a constant isotropy subgroup for each $0 \leq k \leq n-1$.

We first claim that X has a global slice, i.e., there exists a semialgebraic G-map $p\colon X \to G/H$ where H is an isotropy subgroup of a point in $\pi^{-1}(\delta^{k_0})$, and δ^{k_0} is the first nonempty stratum in the straight filtration of δ. We will construct p by constructing G-maps $p_k\colon \pi^{-1}(\delta^k) \to \pi^{-1}(\delta^{k-1})$ for $k_0 + 1 \leq k \leq n$ and $p_{k_0}\colon \pi^{-1}(\delta^{k_0}) \to G/H$. First of all $\pi^{-1}(\delta^{k_0})$ is a semialgebraic G-set of one orbit type. Therefore $\pi|_{\pi^{-1}(\delta^{k_0})}\colon \pi^{-1}(\delta^{k_0}) \to \delta^{k_0}$ has a semialgebraic fiber bundle structure with G/H as the fiber, which is trivial because δ^{k_0} is contractible. Hence we can find a semialgebraic G-map

$$p_{k_0}\colon \pi^{-1}(\delta^{k_0}) \to G/H.$$

To construct semialgebraic G-map $p_k\colon \pi^{-1}(\delta^k) \to \pi^{-1}(\delta^{k-1})$ for $k_0 + 1 \leq k \leq n-1$, note that δ^k retracts into δ^{k-1} semialgebraically. Therefore $s'(\delta^k)$ retracts into $s'(\delta^{k-1})$ semialgebraically, and this retraction induces a semialgebraic G-retraction $p_k\colon \pi^{-1}(\delta^k) \to \pi^{-1}(\delta^{k-1})$ for $k_0+1 \leq k \leq n-1$. It remains to construct a semialgebraic G-map $p_n\colon \pi^{-1}(\delta^n) \to \pi^{-1}(\delta^{n-1})$. Let H_n be the isotropy subgroup of a point in $\pi^{-1}(\delta^n - \delta^{n-1})$, then $\pi^{-1}(\delta^n - \delta^{n-1})$ is a semialgebraic G-set with one orbit type (G/H_n). By Theorem 2.9 $\pi\colon \pi^{-1}(\delta^n - \delta^{n-1}) \to \delta^n - \delta^{n-1}$ has a semialgebraic bundle structure, and since $\delta^n - \delta^{n-1}$ is contractible the bundle structure is trivial. Therefore there exists a semialgebraic section $s''\colon \delta^n - \delta^{n-1} \to \pi^{-1}(\delta^n - \delta^{n-1})$. Moreover we can find s'' so that its image $s''(\delta^n - \delta^{n-1})$ lies in $\pi^{-1}(\delta^n - \delta^{n-1})^{H_n}$. Now let $\tilde{\delta}$ be the closure of $s''(\delta^n - \delta^{n-1})$ and $\tilde{\delta}^{n-1} = \tilde{\delta} - s''(\delta^n - \delta^{n-1})$.

We now claim that there exists a semialgebraic retraction $\tilde{r}\colon \tilde{\delta} \to \tilde{\delta}^{n-1}$. First triangulate $\tilde{\delta}$ compatible with $\tilde{\delta}^{n-1}$ using Theorem 3.1 and take its barycentric subdivision. Let $\widetilde{U}$ be the open regular neighborhood of $\tilde{\delta}^{n-1}$ in $\tilde{\delta}$. Since $\widetilde{U}$ is open and π is an open map, $U = \pi(\widetilde{U})$ is an open semialgebraic neighborhood of δ^{n-1} in δ^n. Obviously δ^{n-1} is a semialgebraic deformation retract of δ^n, we can apply Lemma 3.3 to find a closed semialgebraic neighborhood N of δ^{n-1} in U and a semialgebraic map $\rho\colon \delta^n \to U$ such that $\rho(x) = x$ for $x \in N$ and $\rho(\delta^n - N) \subset U - N$. Now define $r'\colon \tilde{\delta} \to \pi^{-1}(U) \cap \tilde{\delta} = \widetilde{U}$ by

$$r'(x) = \begin{cases} s'' \circ \rho \circ \pi(x), & x \in \tilde{\delta} - \pi^{-1}(\delta^{n-1}) \\ x, & x \in \tilde{\delta}^{n-1}. \end{cases}$$

Since $\rho|_N = \mathrm{id}$, $s'' \circ \rho \circ \pi(x) = s'' \circ \pi(x) = x$ for $x \in \pi^{-1}(N - \delta^{n-1}) \cap \tilde{\delta}$. Therefore the map r' is continuous. Since the regular neighborhood $\widetilde{U}$ has a semialgebraic retraction to $\tilde{\delta}^{n-1}$ the composition of r' followed by this retraction gives a semialgebraic retraction $\tilde{r}\colon \tilde{\delta} \to \tilde{\delta}^{n-1}$.

Since any element in $\pi^{-1}(\delta)$ is of the form gx for some $g \in G$ and $x \in \tilde{\delta}$, the retraction $\tilde{r}$ induces a semialgebraic G-map $p_n\colon \pi^{-1}(\delta^n) \to \pi^{-1}(\delta^{n-1})$, $p_n(gx) = g\tilde{r}(x)$ where $x \in \tilde{\delta}$, because $s''(\delta^n - \delta^{n-1}) \subset \pi^{-1}(\delta^n - \delta^{n-1})^{H_n}$. Now let

$$p = p_{k_0} \circ p_{k_0+1} \circ \cdots \circ p_n\colon X \to G/H.$$

Then p is a semialgebraic G-map and hence we have shown that X has a global slice.

Now a desired section $s\colon \delta \to X$ is defined as follows: If $H \neq G$, consider the slice $S = p^{-1}(eH)$. Then S is a semialgebraic H-set with the orbit space $S/H \cong X/G$. By the induction hypothesis, we can find a semialgebraic section $s\colon \delta \to S \subset \pi^{-1}(\delta) = X$ of the orbit map $S \to S/H$, and this section is a desired section.

On the other hand if $H = G$, then $X^G \neq \varnothing$. Let $X' = X - X^G$. Then $\delta' = \delta - \pi(X^G)$ is again a (non closed) simplex, and by the previous argument of the case when $H \neq G$, we can find a semialgebraic section $s'\colon \delta' \to X'$ of the orbit map $X' \to X'/G$. We now define a semialgebraic section $s\colon \delta \to X$ by

$$s(x) = \begin{cases} s'(x), & x \in \delta' \\ \pi^{-1}(x), & x \in \pi(X^G). \end{cases}$$

Obviously such defined s is continuous on δ'. In order to see that s is continuous at $y = \pi(x)$ for $x \in X^G$, it is enough to show that for a given open neighborhood U of x there exists a G-invariant neighborhood V of x such that $V \subset U$. If we choose $V = X - G(X - U)$ then, since G is compact, $G(X - U)$ is a closed subset of X and hence V is open. Thus V is a desired open neighborhood of x. □

Note that compactness of G is necessary to show the map s is continuous in the last part of the above proof. However we can extend the previous lemma to proper actions of noncompact groups in the following sense.

Lemma 3.5. *Let G be a semialgebraic group, and let X be a proper semialgebraic G-set whose orbit space X/G has a semialgebraic structure. Let $\{X_i\}$ be a finite collection of G-invariant semialgebraic subsets of X. Then there exists a finite open semialgebraic triangulation (K, τ) of X/G compatible with $\{X_i/G\}$ such that*

(1) *any simplex δ of K has a semialgebraic cross section $s_\delta\colon \delta \to X$ of the restriction of the orbit map $\pi\colon X \to X/G$ to $\pi^{-1}(\delta)$.*

(2) *The image $s_\delta(\delta)$ and its orbit $G(s_\delta(\delta))$ are closed semialgebraic subsets of X, and*
(3) *for each n-simplex δ of K, there exists a straight filtration $\delta^0 \subset \delta^1 \subset \cdots \subset \delta^n = \delta$ of δ such that $s_\delta(\delta^k - \delta^{k-1})$ has a constant isotropy subgroup for each $0 \le k \le n$.*

Proof. By Theorem 2.4 X has finitely many orbit types, say $\{(G/H_j)\}$. Moreover by Theorem 2.8 X can be covered by a finite number of semialgebraic G-tubes $\{U_l\}$. Therefore by Theorem 3.1 we can find a semialgebraic triangulation (K, τ) of X/G compatible with $\{X_i/G\} \cup \{X_{(\ge H_j)}/G\} \cup \{U_l/G\}$ such that $\pi^{-1}(\delta)$ is contained in a semialgebraic G-tube U_l for each simplex δ of K. We replace K by its barycentric subdivision. Then for each simplex δ of K we can find a straight filtration $\delta^0 \subset \delta^1 \subset \cdots \subset \delta^n = \delta$ such that $\pi^{-1}(\delta^k - \delta^{k-1})$ has only one orbit type for each $0 \le k \le n$.

We now show that such K satisfies the conditions (1)-(3). Let δ be a given simplex of K. Since $\pi^{-1}(\delta)$ is contained in a semialgebraic G-tube U of X, there exists a semialgebraic G-map $f\colon U \to G/H$ where $H = G_x$ for some $x \in U$. Since the G-action on X is proper, H is compact. Let $S' = f^{-1}(eH)$ be the slice of the G-tube U, and let $S = S' \cap \pi^{-1}(\delta)$. Then GS and S are closed in X since $GS = \pi^{-1}(\delta)$ and $S = (f|_{GS})^{-1}(eH)$. Furthermore S is H-invariant and $S/H = \delta$. Since H is compact, we can apply Lemma 3.4 to the H-space S to get a semialgebraic section $s_\delta\colon \delta \to S$ of the orbit map $\pi_S\colon S \to S/H = \delta$. The conditions (1) and (3) directly follow from Lemma 3.4. Since $s_\delta(\delta)$ is closed in S, so is in X. Moreover, since the action $G \times S \to GS$ is proper by Proposition 2.7, $G(s_\delta(\delta))$ is closed in X, so the condition (2) is satisfied. □

Theorem 3.6. *Let G be a semialgebraic group, and let X be a proper semialgebraic G-set whose orbit space X/G has a semialgebraic structure. Let A be a closed semialgebraic G-subset of X. Then there exists a pair (Y, B) of finite open straight G-CW complexes such that*

(1) *the underlying spaces Y and B are equal to X and A, respectively,*
(2) *Y/G is a finite open simplicial complex compatible with the orbit types and A/G such that the orbit map $\pi_Y\colon Y \to Y/G$ is a semialgebraic cellular map,*
(3) *each open G-cell c of Y is a semialgebraic G-set, and hence its closure $\bar{c}$ is also a semialgebraic G-set.*
(4) *for each open G-cell c of Y, the characteristic G-map $f_c\colon G/H_c \times \delta \to \bar{c}$ is semialgebraic whose restriction $f_{c_i}|\colon G/H_{c_i} \times \mathring{\delta} \to c_i$ is a semialgebraic G-homeomorphism,*

(5) *for each open G-cell c of Y, the restriction $\pi_Y|_{\bar{c}}\colon \bar{c} \to \pi_Y(\bar{c})$ has semialgebraic section $s\colon \pi_Y(\bar{c}) \to \bar{c}$, and*
(6) *each n-simplex δ of Y/G has a straight filtration $\delta^0 \subset \delta^1 \subset \cdots \subset \delta^n = \delta$ such that $s(\delta^k - \delta^{k-1})$ has a constant isotropy subgroup for each $0 \le k \le n$.*

In particular, if X is compact, we can take Y to be a complete finite G-CW complex.

Proof. Let (K, τ) be a finite triangulation of X/G compatible with orbit types and A/G as Lemma 3.5. Set $\sigma = s_\delta(\overset{\circ}{\delta})$ for each simplex δ of K, and also set $c = G\overset{\circ}{\sigma}$. From Lemma 3.5(2) and continuity of s_δ we can see that $\bar{\sigma} = s_\delta(\delta)$ and $\bar{c} = Gs_\delta(\delta)$. We define the characteristic G-map $f_c\colon G/H_c \times \delta \to \bar{c}$ by $(gH_c, x) \mapsto gs_\delta(x)$, where H_c is the isotropy subgroup of $s_\delta(\delta^n - \delta^{n-1})$. Then f_c is semialgebraic. Let Y be the pair $(X, \{c\})$. To show that it defines an open G-CW complex structure on X, it is enough to check its topology, namely, we have to show that X has the weak topology with respect to the closed covering $\{\bar{c}\}$ of X. This follows from the fact that $\bar{\sigma} = s_\delta(\delta)$. The conditions (1)-(6) follow easily from the construction. □

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