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GENERALIZED FRACTIONAL DIFFERINTEGRAL OPERATORS OF THE K-SERIES

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Abstract. In the present paper, we further study the generalized fractional differintegral (integral and differential) operators involving Appell's function F_3 introduced by Saigo-Maeda [9], and are applied to the K-Series defined by Gehlot and Ram [3]. On account of the general nature of our main results, a large number of results obtained earlier by several authors such as Ram et al. [7], Saxena et al. [14], Saxena and Saigo [15] and many more follow as special cases.

1. Introduction and Preliminaries

The K-Series is defined and represented by Gehlot and Ram [3] as follows:

$$\begin{aligned} {}_pK_q^{(\beta,\eta)_m}[z] &= {}_pK_q^{(\beta,\eta)_m}\left(a_1,\cdots,a_p;\, b_1,\cdots,b_q;\, (\beta,\eta)_m;\, z\right) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\ z^n}{\prod_{r=1}^{q}(b_r)_n \prod_{i=1}^{m}\Gamma(\eta_i n+\beta_i)}, \end{aligned} \tag{1.1}$$

where $a_j, b_r, \beta_i \in \mathbb{C};\ \eta_i \in \mathbb{R},\ (j=1,\cdots,p; r=1,\cdots,q; i=1,\cdots,m)$.

The series (1.1) is valid for none of the parameter $b_r\,(r=1,\cdots,q)$ being negative integer or zero. If any parameter $a_j\,(j=1,\cdots,p)$ in (1.1) is zero or negative, then the series terminates into a polynomial in z; and

(i) if $p < q+\sum_{i=1}^{m}\eta_i$, then the power series on the right side of (1.1) is absolutely convergent for all $z \in \mathbb{C}$,

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(ii) if $p = q+\sum_{i=1}^{m} \eta_i$ and $|z| = 1$, then the series is absolutely convergent for all $|z| < \prod_{i=1}^{m} (|\eta_i|)^{\eta_i}$, $|z| = \prod_{i=1}^{m} (|\eta_i|)^{\eta_i}$ and
$\Re\left(\sum_{r=1}^{q} b_r + \sum_{i=1}^{m} \beta_i - \sum_{j=1}^{p} a_j\right) > \frac{2+q+m-p}{2}$.

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\Re(\gamma) > 0$ and $x > 0$. Then the generalized (Saigo-Maeda) fractional integral operators involving Appell function F_3 [9, p. 393, Eqs. (4.12) and (4.13)] are defined as follows:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1}\, F_3\left(\alpha, \alpha', \beta, \beta'; \gamma;\, 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \tag{1.2}$$

and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty t^{-\alpha} (t-x)^{\gamma-1}\, F_3\left(\alpha, \alpha', \beta, \beta'; \gamma;\, 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt. \tag{1.3}$$

Also, the corresponding Saigo-Maeda fractional differential operators [9] are given as follows:

$$\begin{aligned}
\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) &= \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (\Re(\gamma) > 0) \\
&= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x) \quad (\Re(\gamma) > 0;\ k = [\Re(\gamma)] + 1) \\
&= \frac{1}{\Gamma(k-\gamma)} \left(\frac{d}{dx}\right)^k (x)^{\alpha'} \int_0^x (x-t)^{k-\gamma-1}\, t^{\alpha} \\
&\quad \times F_3\left(-\alpha', -\alpha, k-\beta', -\beta, k-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt
\end{aligned} \tag{1.4}$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (\Re(\gamma) > 0)$$

$$= \left(-\frac{d}{dx}\right)^{k} \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x) \quad (\Re(\gamma) > 0;\ k = [\Re(\gamma)] + 1)$$

$$= \frac{1}{\Gamma(k-\gamma)} \left(-\frac{d}{dx}\right)^{k} (x)^{\alpha} \int_{x}^{\infty} (t-x)^{k-\gamma-1}\, t^{\alpha'}$$

(1.5)

$$\times\ F_3\left(-\alpha', -\alpha, -\beta', k-\beta, k-\gamma;\ 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt.$$

Here $F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, \xi)$ is the familiar Appell hypergeometric function of two variables defined by

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, \xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{z^m}{m!} \frac{\xi^n}{n!}$$

(1.6) $\quad (|z| < 1 \quad \text{and} \quad |\xi| < 1),$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\gamma(\lambda)} \begin{cases} \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \\ 1 & (n = 0), \end{cases}$$

it being understood conventionally that $(0)_0$=1 and assumed tacitly that the Γ-quotient exists (see, for details, [16, p. 21]); definitions and properties of the Appell functions are available in the book [2].

The left-hand sided and right-hand sided generalized fractional integration of the type (1.2) and (1.3) for a power function formulas are given by Saigo-Maeda [9, p. 394, Eqs. (4.18) and (4.19)], as follows:

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}$$

(1.7)

$$= \Gamma\left[\begin{array}{c} \rho,\ \rho+\gamma-\alpha-\alpha'-\beta,\ \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha',\ \rho+\gamma-\alpha'-\beta,\ \rho+\beta' \end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1},$$

where $\Re(\gamma) > 0$, $\Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$, $(x > 0)$; and

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}$$

(1.8)

$$= \Gamma\left[\begin{array}{c} 1+\alpha+\alpha'-\gamma-\rho,\ 1+\alpha+\beta'-\gamma-\rho,\ 1-\beta-\rho \\ 1-\rho,\ 1+\alpha+\alpha'+\beta'-\gamma-\rho,\ 1+\alpha-\beta-\rho \end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1},$$

where $\Re(\gamma) > 0, x > 0$, $\Re(\rho) < 1+\min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$.

The symbol occurring in (1.7) and (1.8) is given by

$$\Gamma\left[\begin{array}{c} a,b,c \\ d,e,f \end{array}\right] = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)}.$$

2. Generalized Fractional Integration formulas of the K-Series

In this section we will establish the left-sided and right-sided Saigo-Maeda fractional integration formulas for the K-series.

Theorem 2.1. *Let* $\alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}$, $a \in \mathbb{R}$, $x > 0$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$, *and the convergent conditions* (i) *and* (ii) *of K-series into the account of* (1.1) *be also satisfied. Then the following formula holds true:*

$$\left(I_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)$$

$$= x^{\beta_1-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(ax^{\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=2}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

(2.1)

$$\times\frac{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'-\delta+\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'+\delta'\right)}{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'+\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'-\delta+\gamma\right)\Gamma\left(\eta_1 n+\beta_1+\delta'\right)}.$$

Proof. By using (1.1), we have

$$\left(I_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)$$

$$=\left(I_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(at^{\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\right]\right)(x),$$

whose right-side, on interchanging the order of the integration and summation, becomes

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,(a)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\left(I_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\,t^{(\eta_1 n+\beta_1)-1}\right)(x).$$

Using (1.7) and rearranging the terms, we get

$$= x^{\beta_1-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(ax^{\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=2}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

$$\times\frac{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'-\delta+\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'+\delta'\right)}{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'+\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'-\delta+\gamma\right)\Gamma\left(\eta_1 n+\beta_1+\delta'\right)}.$$

This competes the proof. □

If we take $\alpha = \alpha + \delta$, $\alpha' = \delta' = 0$, $\delta = -\mu$ and $\gamma = \alpha$ in (2.1), we get a known result obtained by Ram et al. [7, p. 408, Eq. (3.1)], as in the following corollary.

Corollary 2.2. *Let* $\alpha, \delta, \mu \in \mathbb{C}$, $\Re(\alpha) > 0$, $a \in \mathbb{R}$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$, $x > 0$, *and the convergent conditions* (i) *and* (ii) *of* K*-series into the account of* (1.1) *be also satisfied. Then we obtain following result:*

$$\begin{aligned}&\left(I_{0+}^{\alpha,\delta,\mu}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)\\&= x^{\beta_1-\delta-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(ax^{\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=2}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\\&\frac{\Gamma\left(\eta_1 n+\beta_1-\delta+\mu\right)}{\Gamma\left(\eta_1 n+\beta_1-\delta\right)\,\Gamma\left(\eta_1 n+\beta_1+\alpha+\mu\right)}.\end{aligned}\tag{2.2}$$

Remark 2.3. *If we take* $p = q = 1$, $a_1 = \rho$, $b_1 = 1$ *and* $\delta = -\alpha$ *in the above equation* (2.2), *we get the result for the Mittag-Leffler function* $E_\rho\left[(\beta, \eta)_m;\, z\right]$ *given by Saxena et al.* [14, Eq. (2.1)]. *Further, if we set* $m = 1$ *then* (2.2) *reduces to the result for the function* $E_{\eta,\beta}^{\rho}\left[z\right]$ *given by Saxena and Saigo* [15, Eq. (14)].

Theorem 2.4. *Let* $\alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}$, $a \in \mathbb{R}$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$, $x > 0$, *and the convergent conditions* (i) *and* (ii) *of* K*-series into the account of* (1.1) *be also satisfied. Then the following formula holds true:*

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{-\gamma-\beta_1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)\\&= x^{-\beta_1-\alpha-\alpha'}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(ax^{-\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\\&\times\frac{\Gamma\left(\eta_1 n+\beta_1+\alpha+\alpha'\right)\Gamma\left(\eta_1 n+\beta_1+\alpha+\delta'\right)\Gamma\left(\eta_1 n+\beta_1-\delta+\gamma\right)}{\Gamma\left(\eta_1 n+\beta_1+\gamma\right)\Gamma\left(\eta_1 n+\beta_1+\alpha+\alpha'+\delta'\right)\Gamma\left(\eta_1 n+\beta_1+\alpha-\delta+\gamma\right)}.\end{aligned}\tag{2.3}$$

Proof. By using (1.1), we arrive at

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{-\gamma-\beta_1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)\\&= \left(I_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{-\gamma-\beta_1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(at^{-\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\right]\right)(x),\end{aligned}$$

next, interchanging the order of the integration and summation, we have

$$= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,(a)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\left(I_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\,t^{(1-(\eta_1 n+\beta_1)-\gamma)-1}\right)(x).$$

Using (1.8) and rearranging the terms, we get

$$= x^{-\beta_1-\alpha-\alpha'} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n \ (ax^{-\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n \prod_{i=1}^{m}\Gamma(\eta_i n+\beta_i)}$$

$$\times \frac{\Gamma(\eta_1 n+\beta_1+\alpha+\alpha')\ \Gamma(\eta_1 n+\beta_1+\alpha+\delta')\ \Gamma(\eta_1 n+\beta_1-\delta+\gamma)}{\Gamma(\eta_1 n+\beta_1+\gamma)\ \Gamma(\eta_1 n+\beta_1+\alpha+\alpha'+\delta')\ \Gamma(\eta_1 n+\beta_1+\alpha-\delta+\gamma)}.$$

This competes the proof. □

If we take $\alpha = \alpha+\delta$, $\alpha' = \delta' = 0$, $\delta = -\mu$ and $\gamma = \alpha$ in (2.3), we obtain a known result given by Ram et al. [7, p. 409, Eq. (4.1)] as follows:

Corollary 2.5. *Let* $\alpha, \delta, \mu \in \mathbb{C}$, $\Re(\alpha) > 0$, $a \in \mathbb{R}$, *the convergent conditions* (i) *and* (ii) *of* K*-series into the account of* (1.1) *be also satisfied, and* $x > 0$. *Then we obtain*

$$\left(I_{-}^{\alpha,\delta,\mu}\left[t^{-\alpha-\beta_1}\ {}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)$$

$$= x^{-\beta_1-\alpha-\delta} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n \ (ax^{-\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n \prod_{i=1}^{m}\Gamma(\eta_i n+\beta_i)}$$

$$\frac{\Gamma(\eta_1 n+\beta_1+\alpha+\delta)\ \Gamma(\eta_1 n+\beta_1+\alpha+\mu)}{\Gamma(\eta_1 n+\beta_1+\alpha)\ \Gamma(\eta_1 n+\beta_1+2\alpha+\delta+\mu)}. \tag{2.4}$$

Remark 2.6. *If we take* $p = q = 1$, $a_1 = \rho$, $b_1 = 1$ *and* $\delta = -\alpha$ *in* (2.4), *then we get the result for the Mittag-Leffler function* $E_\rho[(\beta,\eta)_m;\ z]$ *given by Saxena et al.* [14, Eqn. (2.4)]. *Further, if we set* $m = 1$ *then* (2.4) *reduces to the result for the function* $E_{\eta,\beta}^{\rho}[z]$ *given by Saxena and Saigo* [15, Eq. (23)].

Remark 2.7. *If we set* $\delta = -\alpha$ *in* Corollary 1.1 *and* 2.1 *then we can easily obtain results concerning Riemann-Liouville fractional integral operators.*

3. Generalized Fractional Derivative formulas of the K-Series

In this section we will establish the left- and right-sided Saigo-Maeda fractional differentiation formulas for the K-series.

Theorem 3.1. *Let* $\alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}$, $a \in \mathbb{R}$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$, $x > 0$, *and the convergent conditions* (i) *and* (ii) *of* K*-series into the account* *of* (1.1) *are also satisfied. Then the following formula holds true:*

$$\left(D_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)$$

$$= x^{\beta_1+\alpha+\alpha'-\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (ax^{\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=2}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

(3.1)

$$\times\ \frac{\Gamma\left(\eta_1 n+\beta_1+\alpha+\alpha'+\delta'-\gamma\right)\,\Gamma\left(\eta_1 n+\beta_1+\alpha-\delta\right)}{\Gamma\left(\eta_1 n+\beta_1+\alpha+\alpha'-\gamma\right)\,\Gamma\left(\eta_1 n+\beta_1+\alpha+\delta'-\gamma\right)\,\Gamma\left(\eta_1 n+\beta_1-\delta\right)}.$$

Proof. By using (1.1) and (1.4), we have

$$\left(D_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)$$

$$= \left(D_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\beta_1-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (at^{\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\right]\right)(x),$$

now, interchanging the order of the differentiation and summation, we have

$$= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (a)^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\left(D_{0+}^{\alpha,\alpha',\delta,\delta',\gamma}\,t^{(\eta_1 n+\beta_1)-1}\right)(x).$$

Using the relation (1.4) and taking (1.7) into account, then after rearranging the terms and little simplification, we get the expression as in the right-hand side of (3.1). This competes the proof. □

If we take $\alpha=\alpha+\delta$, $\alpha'=\delta'=0$, $\delta=-\mu$ and $\gamma=\alpha$ in (3.1), we get known result obtained by Ram et al. [7, p. 410, eqn. (5.1)], as given by

Corollary 3.2. *Let* $\alpha,\delta,\mu\in\mathbb{C}$, $\Re(\alpha)>0$, $a\in\mathbb{R}$, $\beta_1\in\mathbb{C}$, $\eta_1\in\mathbb{R}$, $x>0$, *and the convergent conditions* (i) *and* (ii) *of K-series into the account of* (1.1) *be also satisfied. Then we obtain the following formula:*

$$\left(D_{0+}^{\alpha,\delta,\mu}\left[t^{\beta_1-1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{\eta_1}\right)\right]\right)(x)$$

$$= x^{\beta_1+\delta-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (ax^{\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=2}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

$$\frac{\Gamma\left(\eta_1 n+\beta_1+\alpha+\delta+\mu\right)}{\Gamma\left(\eta_1 n+\beta_1+\mu\right)\,\Gamma\left(\eta_1 n+\beta_1+\delta\right)}. \tag{3.2}$$

Remark 3.3. *If we take* $p=q=1$, $a_1=\rho$, $b_1=1$ *and* $\delta=-\alpha$ *in the above corollary, then we get the result for the Mittag-Leffler function* $E_\rho\left[(\beta,\eta)_m;\,z\right]$ *given by Saxena et al.* [14, Eq. (2.6)]. *Further, if we set*

$m = 1$ in (3.2), *then it reduces to the result for the function* $E^{\rho}_{\eta,\beta}[z]$ *given by Saxena and Saigo* [15, Eq. (29)].

Theorem 3.4. *Let $\alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}$, $a \in \mathbb{R}$, $x > 0$, $\beta_1 \in \mathbb{C}$ $\eta_1 \in \mathbb{R}$, and the convergent conditions* (i) *and* (ii) *of K-series into the account of* (1.1) *be also satisfied. Then the following result holds true:*

$$\left(D_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\gamma-\beta_1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)$$

$$= x^{-\beta_1+\alpha+\alpha'}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ \left(ax^{-\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

(3.3)

$$\times\ \frac{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'-\delta\right)\Gamma\left(\eta_1 n+\beta_1+\delta'-\gamma\right)}{\Gamma\left(\eta_1 n+\beta_1-\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'-\delta\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'+\delta'-\gamma\right)}.$$

Proof. By using (1.1) and (1.5), we have

$$\left(D_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\gamma-\beta_1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)$$

$$= \left(D_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\left[t^{\gamma-\beta_1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (at^{-\eta_1})^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\right]\right)(x),$$

whose right-side, interchanging the order of the differentiation and summation, becomes

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ (a)^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}\left(D_{-}^{\alpha,\alpha',\delta,\delta',\gamma}\,t^{\gamma-(\eta_1 n+\beta_1)}\right)(x),$$

by using the relation (1.5), and taking into (1.8), we arrive at

$$= x^{-\beta_1+\alpha+\alpha'}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\ \left(ax^{-\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$

$$\times\ \frac{\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'-\delta\right)\Gamma\left(\eta_1 n+\beta_1+\delta'-\gamma\right)}{\Gamma\left(\eta_1 n+\beta_1-\gamma\right)\Gamma\left(\eta_1 n+\beta_1-\alpha-\alpha'-\delta\right)\Gamma\left(\eta_1 n+\beta_1-\alpha'+\delta'-\gamma\right)}.$$

This competes the proof. □

If we take $\alpha = \alpha+\delta$, $\alpha' = \delta' = 0$, $\delta = -\mu$ and $\gamma = \alpha$ in (3.3), we obtain known result given by Ram et al. [7, p. 412, Eq. (6.1)], as given by

Corollary 3.5. *Let $\alpha, \delta, \mu \in \mathbb{C}$, $\Re(\alpha) > 0$, $a \in \mathbb{R}$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$ and the convergent conditions* (i) *and* (ii) *of K-series into the account of*

(1.1) *be also satisfied, and* $x > 0$. *Then we obtain the following formula:*

$$\left(D_{-}^{\alpha,\delta,\mu}\left[t^{\alpha-\beta_1}\,{}_pK_q^{(\beta,\eta)_m}\left(at^{-\eta_1}\right)\right]\right)(x)$$
$$= x^{-\beta_1+\alpha+\delta}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,\left(ax^{-\eta_1}\right)^n}{\prod_{r=1}^{q}(b_r)_n\,\prod_{i=1}^{m}\Gamma\left(\eta_i n+\beta_i\right)}$$
$$\frac{\Gamma\left(\eta_1 n+\beta_1-\alpha-\delta\right)\Gamma\left(\eta_1 n+\beta_1+\mu\right)}{\Gamma\left(\eta_1 n+\beta_1-\alpha-\delta+\mu\right)\Gamma\left(\eta_1 n+\beta_1-\alpha\right)}. \tag{3.4}$$

Remark 3.6. *If we take* $p = q = 1$, $a_1 = \rho$, $b_1 = 1$ *and* $\delta = -\alpha$ *in* (3.4), *then we get the result given by Saxena et al.* [14, Eq. (2.8)]. *Further, if we set* $m = 1$ *then* (3.4) *reduces to the known result given by Saxena and Saigo* [15, Eq. (35)].

Remark 3.7. *If we set* $\delta = -\alpha$ *in* Corollary 3.1 *and* 4.1 *then we can easily obtain results concerning Riemann-Liouville fractional derivative operators.*

4. Concluding Remarks

In the present paper, we have studied and given new unified fractional calculus (differintegral) formulas associated with the K-Series. The theorems have been developed in terms of series form with the help of Saigo-Maeda power function formulas. Certain special cases of our main results are also pointed out to be related to some earlier works of many authors.

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