Honam Mathematical J. **39** (2017), No. 1, pp. 41–47
https://doi.org/10.5831/HMJ.2017.39.1.41

THE ORIENTABILITY OF REAL TORIC MANIFOLDS

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Abstract. The aim of this short paper is to give a new necessary and sufficient condition, topological in nature, for certain real toric manifolds to be orientable in terms of the connectedness of their particular submanifolds of codimension zero.

1. Introduction and main results

Let P be a connected manifold with corners of dimension n that is also *nice* in that every codimension-k face is a connected component of the intersections of k facets. Here a facet of P is defined to be a face of codimension-1 face of P. A typical example of a nice manifold with corners is a simple convex polytope, and there are examples of a manifold with corners that is not nice. Note also that not every nice manifold with corners has connected intersections of faces.

Given a nice manifold P with corners, let $\mathcal{F}$ denote the collection of all facets $F_1, F_2, \ldots, F_m$ of P, and let

$$\lambda : \mathcal{F} \to \mathbb{Z}^n$$

be a characteristic function on $\mathcal{F}$ such that

(1) $\lambda(F_i)$ is a primitive vector for each $i \in [m] := \{1, 2, \ldots, m\}$, and
(2) for a non-empty $P_I := \cap_{i\in I} F_i$ for $I \subset [m]$, $\lambda(F_i)$'s are linearly independent over $\mathbb{Q}$.

Let S^1 be the unit circle of complex numbers in $\mathbb{C}$, and let $T^n = (S^1)^n$. For a non-empty P_I, we can form an abelian subgroup T_I^n of T^n generated by $\lambda(F_i)$'s for $i \in I$. Then one can construct a manifold $X(P, \lambda)$

Received July 05, 2016. Accepted December 30, 2016.
2010 Mathematics Subject Classification. 55N10, 17R15, 14M25.
Key words and phrases. real toric manifolds, nice manifolds with corners, simple polytopes, orientability.
This study was supported by research fund from Chosun University, 2016.
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by using the quotient space

$$X(P,\lambda) = (P \times T^n)/\sim .$$

Here, the equivalence relation $\sim$ on the product space $P \times T^n$ is given by

$$(x,t) \sim (y,s) \text{ if and only if } x = y \text{ and } t^{-1}s \in T^n_I,$$

where I is a subset of $[m]$ such that P_I is the minimal face of P containing $x = y$. The manifold $X(P,\lambda)$ is usually called a *toric manifold*, and, in general, X is just an orbifold. Further, it admits a T^n-action induced from the natural T^n-action on the second factor of $P \times T^n$ whose orbit space is P itself. Hence there is a quotient map

$$\pi : X(P,\lambda) \to P = X(P,\lambda)/T^n.$$

For the sake of simplicity, we shall also use the notation X for $X(P,\lambda)$ if there is no confusion. A typical example of a toric manifold can be provided by the natural action of T^n on the complex projective space $\mathbb{CP}^n$ associated to the n-simplex Δ^n. See [1], [2], and [3] for more details.

Instead of S^1 and T^n, we may repeat the above construction with $\mathbb{Z}_2 = \{0,1\}$ and

$$\mathbb{Z}_2^n = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$$

to obtain a *real toric manifold* $X(P,\lambda_{\mathbb{R}})$ of dimension n for a characteristic function $\lambda_{\mathbb{R}} : \mathcal{F} \to \mathbb{Z}_2^n$. However, note that the image $\lambda(F_i)$ of a characteristic function $\lambda_{\mathbb{R}}$ is always primitive and that every linearly independent vectors in $\mathbb{Z}_2^n$ is a part of a basis of $\mathbb{Z}_2^n$. So the quotient space $X(P,\lambda_{\mathbb{R}})$ with the quotient map $\pi : X(P,\lambda_{\mathbb{R}}) \to P$ is always smooth. As in the case of $X(P,\lambda)$, $X(P,\lambda_{\mathbb{R}})$ has a $\mathbb{Z}_2^n$-fixed point if and only if P has a vertex. As in the case of $X(P,\lambda)$, we shall also use the notation $X_{\mathbb{R}}$ for $X(P,\lambda_{\mathbb{R}})$ if there is no confusion.

One example of a real toric manifold can be given by the natural action of $\mathbb{Z}_2^n$ on the real projective space $\mathbb{RP}^n$ associated to the n-simplex Δ^n. When P is a simple convex polytope, $X_{\mathbb{R}}$ is very often called a *small cover* in the literature (see [3]).

In order to explain our main result more precisely, we first set

$$P^{(n-2)} = \bigcup_{\substack{I \subset [m], \\ |I| \geq 2}} P_I,$$

where P_I denotes the intersection of all facets F_i's for $i \in I \subset [m]$, i.e., $P_I = \cap_{i \in I \subset [m]} F_i$. Let P' be a small tubular neighborhood of $P^{(n-2)}$ in P, $X'_{\mathbb{R}} = \pi^{-1}(P')$, and $(X'_{\mathbb{R}})^\circ$ the interior of $X'_{\mathbb{R}}$.

The aim of this paper is to characterize the orientability of real toric manifolds in terms of the connectedness of their particular submanifolds, as follows.

Theorem 1.1. *Let P be a nice manifold with corners of dimension $n \geq 2$ with m facets, and let $X_{\mathbb{R}} = X(P, \lambda_{\mathbb{R}})$ be a real toric manifold for a characteristic function $\lambda_{\mathbb{R}}$ with the quotient map $\pi : X_{\mathbb{R}} \to P$. Assume that*

$$P^{(n-2)} = \bigcup_{\substack{I \subset [m], \\ |I| \geq 2}} P_I$$

is non-empty and that $X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}$ is orientable. Then we have

$$H^n(X_{\mathbb{R}}; \mathbb{Z}) \cong H_0(X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}; \mathbb{Z}).$$

As a consequence, in this case $X_{\mathbb{R}}$ is orientable if and only $X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}$ is connected.

As is remarked as above, simple convex polytopes are typical examples of nice manifolds P with corners in Theorem 1.1. There are some earlier works [6, Theorem 1.7], [7, Theorem 3.2], and [5, Section 4] which give a necessary and sufficient condition for the orientability of small covers or the so-called 2-torus manifolds. Here a 2-torus manifold of dimension n is a closed smooth manifold of dimension n with an effective action of $\mathbb{Z}_2^n$. Theorem 1.1 gives a new necessary and sufficient condition for real toric manifolds to be orientable, which is topological in nature.

We organize this paper, as follows. In Section 2, we give a proof of Theorem 1.1. In the same section, we also give some immediate but also relevant consequences of Theorem 1.1 for simple convex polytopes (refer to Corollaries 2.2 and 2.3).

Finally, we remark that this paper has been partially motivated by the paper [4] and the technique of Yeroshkin in [8] that deletes a small neighborhood of the singular set in $X(P, \lambda)$ in order to obtain a smooth part and investigates the relation of the cohomology groups between $X(P, \lambda)$ and the smooth part. To be a little more precise, the paper [4] studies the torsion in the integral cohomology of a certain family of $2n$-dimensional orbifolds $X(P, \lambda)$ with actions of the n-dimensional compact torus. However, it should be also remarked that at the moment there seems to be some problem in [4, Lemma 6.3] and so their main Theorem in [4, p. 2] seems to be affected accordingly.

2. Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1.

To do so, as before let P be a nice manifold with corners of dimension $n \geq 2$ with m facets, and let $X_{\mathbb{R}} = (P, \lambda_{\mathbb{R}})$ be a real toric manifold for a characteristic function $\lambda_{\mathbb{R}}$ with the quotient map $\pi : X_{\mathbb{R}} \to P$. For each $2 \leq s \leq n$, let

$$P^{(n-s)} = \bigcup_{\substack{I \subset [m], \\ |I| \geq s}} P_I,$$

where P_I denotes the intersection of all F_i's for $i \in I \subset [m]$, i.e., $P_I = \cap_{i \in I \subset [m]} F_i$. Let P' be a small tubular neighborhood of $P^{(n-s)}$ in P, and let

$$X'_{\mathbb{R}} = \pi^{-1}(P').$$

From now on, all cohomology groups will be taken with integer coefficients, unless stated otherwise.

For the case of $s = 2$, we first have the following

Theorem 2.1. *Let P and $X_{\mathbb{R}}$ be the same as above. Assume that $P^{(n-2)} = \bigcup_{\substack{I \subset [m], \\ |I| \geq 2}} P_I$ is non-empty and that $X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}$ is orientable. Then we have*

$$H^n(X_{\mathbb{R}}; \mathbb{Z}) \cong H_0(X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}; \mathbb{Z}).$$

As a consequence, in this case $X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^{\circ}$ is connected if and only if $X_{\mathbb{R}}$ is orientable.

Proof. To prove it, note first that $X'_{\mathbb{R}}$ is homotopy equivalent to $\pi^{-1}(P^{(n-2)})$. Since the dimension of $\pi^{-1}(P^{(n-2)})$ is equal to $n - 2$, we should have

$$H^l(X'_{\mathbb{R}}; \mathbb{Z}) = 0 \tag{1}$$

for all $l \geq n - 1$.

Next we consider the following long exact sequence for the pair $(X, X'_{\mathbb{R}})$, as follows.

$$\begin{aligned} &\longrightarrow H^{n-1}(X_{\mathbb{R}}, X'_{\mathbb{R}}) \longrightarrow H^{n-1}(X_{\mathbb{R}}) \longrightarrow H^{n-1}(X'_{\mathbb{R}}) \\ &\longrightarrow H^n(X_{\mathbb{R}}, X'_{\mathbb{R}}) \longrightarrow H^n(X_{\mathbb{R}}) \longrightarrow H^n(X'_{\mathbb{R}}) \\ &\longrightarrow 0. \end{aligned} \tag{2}$$

Then it easily follows from (1) and (2) that we have

$$H^n(X_{\mathbb{R}}, X'_{\mathbb{R}}; \mathbb{Z}) \cong H^n(X_{\mathbb{R}}; \mathbb{Z}). \tag{3}$$

Note also that by excision and Poincaré-Lefschetz duality we have

$$H^n(X_{\mathbb{R}}, X'_{\mathbb{R}}) \cong H^n(X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^\circ, \partial X'_{\mathbb{R}}) \cong H_0(X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^\circ), \tag{4}$$

where as before $(X'_{\mathbb{R}})^\circ$ denotes the interior of $X'_{\mathbb{R}}$, and excision (resp. Poincaré-Leschetz duality) is used in the first (resp. second) equality. Note that we need the orientabilty of $X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^\circ$ in order to apply the Poincaré-Lefschetz duality as in the above equation (4). Hence by (3) and (4) we have

$$H^n(X_{\mathbb{R}}; \mathbb{Z}) \cong H_0(X_{\mathbb{R}} \backslash (X'_{\mathbb{R}})^\circ; \mathbb{Z}),$$

as desired. □

As before, let $\mathcal{F}$ be the collection of all facets F_i of P, and let

$$\lambda^i_{\mathbb{R}} : \mathcal{F}_i \to \mathbb{Z}_2^{n-1}$$

be the characteristic function given by composing $\lambda_{\mathbb{R}} : \mathcal{F} \to \mathbb{Z}_2^n$ and the quotient map

$$\mathbb{Z}_2^n \to \mathbb{Z}_2^n / \langle \lambda_{\mathbb{R}}(F_i) \rangle \cong \mathbb{Z}_2^{n-1}.$$

Note that each facet F_i is a nice manifold with corners, whenever P is so. Thus, for each $1 \leq i \leq m$ we can obtain a real toric manifold

$$X_{\mathbb{R}}(F_i) = X_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}})$$

which is equivariantly diffeomorphic to the preimage $\pi^{-1}(F_i)$ of F_i.

For a real toric manifold (or small cover) of a simplex convex polytope and its facets F, if the submanifolds $\pi^{-1}(F)$ are all orientable, then the integral homology group $H_k(X_{\mathbb{R}}; \mathbb{Z})$ is a free abelian group of rank h_k ([3, Corollary 3.8]). Here h_k denotes the k-th component of the h-vector $(h_0, h_1, \ldots, h_n)$ of P (see [3, p.430]).

As a consequence of Theorem 1.1, we have the following corollary.

Corollary 2.2. *Let P be a simple convex polytope of dimension $n \geq 3$ with m facets F_1, F_2,…, F_m, and let $X_{\mathbb{R}} = X(P, \lambda_{\mathbb{R}})$ be a real toric manifold for a characteristic function $\lambda_{\mathbb{R}}$ with the quotient map $\pi : X_{\mathbb{R}} \to P$. Assume that the following statements hold.*

(1) $F_i^{(n-3)}$ *is non-empty for each $1 \leq i \leq m$.*
(2) $X_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}}) \backslash (X'_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}}))^\circ$ *is connected and orientable for each $1 \leq i \leq m$.*

Then the integral homology group $H_k(X_{\mathbb{R}}; \mathbb{Z})$ is a free abelian group of rank h_k for each $0 \leq k \leq n$.

Proof. For the proof, it suffices to note that if $X_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}}) \backslash (X'_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}}))^\circ$ is connected and orientable for each $1 \leq i \leq m$, then the submanifold $X_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}})$ is orientable by Theorem 2.1. Since $\pi^{-1}(F_i)$ is same as $X_{\mathbb{R}}(F_i, \lambda^i_{\mathbb{R}})$, the corollary follows immediately from [3, Corollary 3.8]. □

It is well-known that the real projective space $\mathbb{RP}^n$ associated to a n-simplex Δ^n is orientable if and only if n is odd. Hence the following corollary also holds.

Corollary 2.3. *Let $\mathbb{RP}^n$ be the real projective space of even dimension $n \geq 2$ with the quotient map $\pi : \mathbb{RP}^n \to \Delta^n$. Let $(\Delta^{n-2})'$ be a small tubular neighborhood of $(\Delta^n)^{(n-2)}$ in Δ^n, and let $(\mathbb{RP}^n)' = \pi^{-1}((\Delta^{n-2})')$. Then the submanifold $\mathbb{RP}^n \backslash ((\mathbb{RP}^n)')^\circ$ of $\mathbb{RP}^n$ is non-orientable.*

Proof. For the proof, note first that $\mathbb{RP}^n \backslash ((\mathbb{RP}^n)')^\circ$ is connected. Indeed, $\mathbb{RP}^n$ is path-connected and $\pi^{-1}((\Delta^n)^{(n-2)})$ is of codimension 2 in $\mathbb{RP}^n$, so that $\mathbb{RP}^n \backslash ((\mathbb{RP}^n)')^\circ$ should be also path-connected.

Now, suppose that $\mathbb{RP}^n \backslash ((\mathbb{RP}^n)')^\circ$ is orientable. Then it follows from Theorem 2.1 that $\mathbb{RP}^n$ would be orientable for even integer $n \geq 2$, which is a contradiction. This completes the proof of Corollary 2.3. □

Acknowledgements: The author is grateful to the referees for their valuable comments on this paper. This study was supported by research fund from Chosun University, 2016.

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