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STRONG CONVERGENCE THEOREMS FOR A QUASI CONTRACTIVE TYPE MAPPING EMPLOYING A NEW ITERATIVE SCHEME WITH AN APPLICATION

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Abstract. In this paper, we introduce a new scheme namely: CUIA-iterative scheme and utilize the same to prove a strong convergence theorem for quasi contractive mappings in Banach spaces. We also establish the equivalence of our new iterative scheme with various iterative schemes namely: Picard, Mann, Ishikawa, Agarwal et al., Noor, SP, CR *etc* for quasi contractive mappings besides carrying out a comparative study of rate of convergences of involve iterative schemes. The present new iterative scheme converges faster than above mentioned iterative schemes whose detailed comparison carried out with the help of different tables and graphs prepared with the help of MATLAB.

1. Introduction and Preliminaries

There exist different techniques to solve the problems involving nonlinear equations employing the approximation fixed points of corresponding contractive type operators. Let (X, d) be a complete metric space and $T : X \to X$ a self map. Suppose that $F(T) = \{p \in X \mid Tp = p\}$. Over the years various researchers have been approximating the fixed point of operators using several iterative processes.

In a complete metric space, Picard iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, \cdots \tag{1.1}$$

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has been employed to approximate the fixed points of the mapping satisfying the Banach contraction condition

$$d(Tx,Ty) \leq kd(x,y), \text{ for all } x,y \in X \text{ and } k \in [0,1). \tag{1.2}$$

After this natural iterative procedure due to Picard, there came a host of iterative procedures namely: Mann, Ishikawa, Agarwal et al., Noor, SP, CR *etc.* For the sake of completeness, we describe these iterative schemes in the following lines:

- Mann iterative scheme (see [10]) is defined as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \tag{1.3}$$

  where $\{\alpha_n\}$ is a sequence of positive numbers in [0, 1].
- Ishikawa iterative scheme (see [8]) is defined as

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n; \\ \quad y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{1.4}$$

  where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1].
- Agarwal et al. iterative scheme (see [1]) is defined as

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n; \\ \quad y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{1.5}$$

  where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1].
- The new two step iteration (see [19]) of Thianwan is defined as

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n; \\ \quad y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{1.6}$$

  where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1].
- Noor iterative scheme (see [11]) is as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n; \\ \quad y_n = (1-\beta_n)x_n + \beta_n Tz_n; \\ \quad z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{1.7}$$

  where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0,1].
- SP iterative scheme (see [13]) of Phuengrattana and Suantai is

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n; \\ \quad y_n = (1-\beta_n)z_n + \beta_n Tz_n; \\ \quad z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{1.8}$$

  where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0,1].

- CR iterative scheme (see [6]) of Chugh et al. is defined as

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n; \\ y_n = (1-\beta_n)Tx_n + \beta_n T z_n; \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0,1] with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 1.1: Observe that if:

- $\gamma_n = 0$, then (1.8) reduces to the new two step iteration of Thianwan (1.6),
- $\gamma_n = \beta_n = 0$, then (1.8) reduces to Mann iteration (1.3),
- $\gamma_n = 0$, then (1.7) reduces to Ishikawa iteration (1.4),
- $\gamma_n = \beta_n = 0$, then (1.7) reduces to Mann iteration (1.3),
- $\beta_n = 0$, then (1.6) reduces to Mann iteration (1.3) and
- $\beta_n = 0$, then (1.4) reduces to Mann iteration (1.3).

In order to prove our results, we need the following lemma, definitions and theorems:

Lemma 1.1 [4]**.** Let k be a real number with $0 \le k < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers with $\lim_{n\to\infty} \epsilon_n = 0$. If any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfies

$$u_{n+1} \le k u_n + \epsilon_n, \ n = 0, 1, 2, \cdots,$$

then $\lim_{n\to\infty} u_n = 0$.

Definition 1.1. [3] Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real convergent sequences with limits α and β respectively. Then $\{\alpha_n\}$ converges faster than $\{\beta_n\}$ if

$$\lim_{n\to\infty} \left| \frac{\alpha_n - \alpha}{\beta_n - \beta} \right| = 0.$$

Definition 1.2 [3]**.** Suppose that $\{u_n\}$ and $\{v_n\}$ are two fixed point iterative procedures both converging to the same fixed point p with the error estimates

$$\|u_n - p\| \le \alpha_n, \ n = 0, 1, 2, \cdots$$

$$\|v_n - p\| \le \beta_n, \ n = 0, 1, 2, \cdots$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real convergent sequences converging to 0. If $\{\alpha_n\}$ converges faster than $\{\beta_n\}$, then we say $\{u_n\}$ converges faster than $\{v_n\}$ to p.

In 1972, Zamfirescu [22] proved a attractive fixed point theorem which as follows:

Theorem 1.1 [22]**.** Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b and c satisfying $a \in (0, 1)$ and $b, c \in (0, \frac{1}{2})$ such that for each pair $x, y \in X$, at least one of the following conditions hold

$$\left.\begin{aligned} (i)\ & d(Tx, Ty) \leq ad(x, y), \\ (ii)\ & d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \\ (iii)\ & d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]. \end{aligned}\right\} \tag{1.10}$$

Then T has a fixed point (say, p) and for any arbitrary $x_0 \in X$, Picard iteration $\{T^n x_0\}_{n=0}^{\infty}$ converges to p.

Recall that, the operator T satisfying condition (1.10) is called Zamfirescu operator.

Lemma 1.2 [2]**.** Let $T : X \to X$ be a mapping satisfying Zamfirescu condition (1.10). Then T satisfies the following, $\forall x, y \in X$ and $k \in [0, 1)$

$$d(Tx, Ty) \leq 2kd(x, Tx) + kd(x, y). \tag{1.11}$$

In 2004, Berinde [2] proved a fixed point theorem of Zamfirescu operators defined on an arbitrary Banach space which is as follows:

Theorem 1.2 [2]**.** Let K be a non-empty closed convex subset of an arbitrary Banach space X and $T : K \to K$ a mapping satisfying (1.10). Let $\{x_n\}$ be a sequence defined by Ishikawa iteration (1.4) and $x_0 \in X$ where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.

In the last four decades, numerous researchers (see [14, 15, 16, 18, 17, 23]) have shown keen interest to introduce improved and generalised iteration procedures and to employ them to approximate fixed points of various classes of mappings such as non-expansive, generalise non-expansive, quasi-contractive mappings etc. Researchers like Rhoades and Soltuz [16], Berinde [2], Soltuz [17], Olaleru [12] made valuable contribution to establish equivalence amongst various type of iterative schemes. Chugh and Kumar [5] have shown that for quasi-contractive operators satisfying (1.11), the iterative procedures due to Picard, Mann, Ishikawa, Noor and SP are equivalent. Recently, Chugh et al. [6] proved that CR iterative scheme is equivalent to Picard, Mann, Ishikawa, Noor and SP iterative scheme for quasi-contractive operators. Phuengratana and Suantai [13] demonstrate that SP iterative scheme converges faster than Mann, Ishikawa and Noor iterative schemes for increasing functions. These iterative schemes have wide range of application in sciences especially in Physics. Wazwaz [21] used iterations to solve linear and non-linear Schrodinger equations. Montri [20] used iterations to get exact solution to linear and non-linear Fokker- Plank equations which have various applications in plasma physics, surface physics, biophysics, laser physics *etc.* Yucheng Liu [9] used iteration method to solve free vibration problems for an Euler-Bernoulli beam under various supporting conditions. Iterative schemes have wide applications to Statistical Physics particularly coding theory.

The purpose of this paper is to introduce a new iterative scheme under the name "CUIA-iteratives scheme" and utilize the same to show that convergence of our newly introduced scheme for quasi-contractive operators satisfying (1.10) and to compare CUIA-iteration with the other existing iterative schemes such as: Picard, Mann, Ishikawa, Noor, Agarwal et al., SP and CR. Also, we provide examples to authenticate our results by using C-programme and MATLAB.

2. Main Results

Now, we are introducing a new iterative scheme with the name CUIA-iterative process which runs as follows:

Let X be a Banach space, $T : X \to X$ a self map on X. Take an arbitrary element x_0 in X. Define a sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n; \\ y_n = (1-\beta_n)Tw_n + \beta_n T z_n; \\ z_n = (1-\gamma_n)Tx_n + \gamma_n T w_n; \\ w_n = (1-\delta_n)x_n + \delta_n T x_n; \end{cases} \tag{1.12}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in [0,1] with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We prove our main result as follows:

Theorem 2.1. Let K be a non-empty closed convex subset of an arbitrary Banach space X and $T : K \to K$ a mapping satisfying (1.10). For any $x_0 \in$ X, we define a sequence $\{x_n\}_{n=0}^{\infty}$ with the help of CUIA-iteration (1.12), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences of real numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. Theorem 1.1. ensures that T has a unique fixed point in K (say, p). From (1.12), we have

$$\|x_{n+1} - p\| \leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|Ty_n - p\|. \tag{2.1}$$

Using Lemma 1.2 and (2.1), we have

$$\|x_{n+1} - p\| \leq [1-\alpha_n(1-k)]\|y_n - p\|. \tag{2.2}$$

Using (1.12) and (1.11), we get

$$\begin{aligned}
\|x_{n+1} - p\| &\leq [1-\alpha_n(1-k)](1-\beta_n)\|Tw_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n\|Tz_n - p\| \\
&\leq [1-\alpha_n(1-k)](1-\beta_n)k\|w_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n k\|z_n - p\| \\
&\leq [1-\alpha_n(1-k)](1-\beta_n)k(1-\delta_n)\|x_n - p\| \\
&\quad +[1-\alpha_n(1-k)](1-\beta_n)k\delta_n\|Tx_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n k(1-\gamma_n)\|Tx_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n k\gamma_n\|Tw_n - p\| \\
&\leq [1-\alpha_n(1-k)](1-\beta_n)k(1-\delta_n)\|x_n - p\| \\
&\quad +[1-\alpha_n(1-k)](1-\beta_n)k^2\delta_n\|x_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n k^2(1-\gamma_n)\|x_n - p\| \\
&\quad +[1-\alpha_n(1-k)]\beta_n k^2\gamma_n\|w_n - p\| \\
&\leq [1-\alpha_n(1-k)]k\big\{(1-\beta_n)(1-\delta_n)\|x_n - p\| \\
&\quad +(1-\beta_n)k\delta_n\|x_n - p\| + k\beta_n(1-\gamma_n)\|x_n - p\| \\
&\quad +k\beta_n\gamma_n\|w_n - p\|\big\} \\
&\leq [1-\alpha_n(1-k)]k\big\{(1-\beta_n)(1-\delta_n)\|x_n - p\| \\
&\quad +k(1-\beta_n)\delta_n\|x_n - p\| + k\beta_n(1-\gamma_n)\|x_n - p\| \\
&\quad +k\beta_n\gamma_n(1-\delta_n)\|x_n - p\| + k\beta_n\gamma_n\delta_n\|Tx_n - p\|\big\}
\end{aligned}$$

$$\begin{aligned} &\leq [1-\alpha_n(1-k)]k\big\{(1-\beta_n)(1-\delta_n)+k(1-\beta_n)\delta_n \\ &\quad +k\beta_n(1-\gamma_n)+k\beta_n\gamma_n(1-\delta_n)+k^2\beta_n\gamma_n\delta_n\big\}\|x_n-p\| \\ &= [1-\alpha_n(1-k)]k\big\{1-\beta_n(1-k)-\delta_n(1-k) \\ &\quad +\beta_n\delta_n(1-k)-k\beta_n\gamma_n\delta_n(1-k)\big\}\|x_n-p\|. \end{aligned}$$

$$\therefore \|x_{n+1}-p\| \leq k^{n+1}\prod_{i=0}^{n}[1-\alpha_i(1-k)]\|x_0-p\|,\ n=0,1,2,\cdots. \quad (2.3)$$

Since, $0 \leq k < 1$, $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, we get

$$\lim_{n\to\infty} k^n \prod_{i=0}^{n}[1-\alpha_i(1-k)] = 0.$$

This implies $\lim_{n\to\infty}\|x_{n+1}-p\| = 0$. Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 2.2. Let K be a non-empty closed convex subset of a Banach space X and $T : K \to K$ a mapping satisfying (1.10). If the initial point is same, $\alpha_n \geq A > 0$, $\forall n \in N$, then Mann iteration (1.3) converges to p implies CUIA-iteration (1.12) converges to p and vice-versa.

Proof: First we assume that Mann iteration $\{u_n\}_{n\in\mathbb{N}}$ (1.3) converges to p. To show that CUIA-iteration $\{x_n\}_{n\in\mathbb{N}}$ (1.12) also converges to p, *i.e.*, to prove $u_n \to p$ implies $x_n \to p$.

On using Mann iteration (1.3) and CUIA-iteration (1.12), we have

$$\|x_{n+1}-u_{n+1}\| \leq (1-\alpha_n)\|y_n-u_n\| + \alpha_n\|Ty_n-Tu_n\|. \quad (2.4)$$

Using Lemma 1.2, we get

$$\begin{aligned} \|x_{n+1}-u_{n+1}\| &\leq (1-\alpha_n)\|y_n-u_n\| + k\alpha_n\|y_n-u_n\| \\ &\quad +2k\alpha_n\ \|Tu_n-u_n\| \\ &= [1-(1-k)\alpha_n]\|y_n-u_n\| + 2k\alpha_n\|Tu_n-u_n\|. \quad (2.5) \end{aligned}$$

Again on using (1.11), we have

$$\begin{aligned}\|y_n-u_n\| &\le (1-\beta_n)\,\|Tw_n-u_n\|+\beta_n\|Tz_n-u_n\|\\ &= (1+2k)\,\|Tu_n-u_n\|+k\,(1-\beta_n)\,\|w_n-u_n\|\\ &\quad +k\beta_n\|z_n-u_n\|. \qquad (2.6)\end{aligned}$$

$$\begin{aligned}\text{Now, }\|z_n-u_n\| &\le (1-\gamma_n)\,\|Tx_n-u_n\|+\gamma_n\|Tw_n-u_n\|\\ &= (1+2k)\,\|Tu_n-u_n\|+k\,(1-\gamma_n)\,\|x_n-u_n\|\\ &\quad +k\gamma_n\|w_n-u_n\| \qquad (2.7)\\ \text{and }\|w_n-u_n\| &\le (1-\delta_n)\,\|x_n-u_n\|+\delta_n\|Tx_n-u_n\|\\ &\le \big(1-(1-k)\delta_n\big)\|x_n-u_n\|\\ &\quad +(1+2k)\delta_n\|Tu_n-u_n\|. \qquad (2.8)\end{aligned}$$

Substituting (2.7) in (2.6), we have

$$\begin{aligned}\|y_n-u_n\| &\le (1+2k)\,(1+k\beta_n)\|Tu_n-u_n\|+[k\,(1-\beta_n)\\ &\quad +k^2\beta_n\gamma_n]\|w_n-u_n\|+(1-\gamma_n)k^2\beta_n\|x_n-u_n\|. \qquad (2.9)\end{aligned}$$

Substituting (2.8) in (2.9), we have

$$\begin{aligned}\|y_n-u_n\| &\le (1+2k)\,\big(1+(1-k\beta_n)\gamma_n+k\beta_n(1+k\gamma_n\delta_n)\big)\|Tu_n-u_n\|\\ &\quad +k\big(1-(1-k)\beta_n-(1-k)\gamma_n+(1-k)\beta_n\gamma_n\\ &\quad -k(1-k)\beta_n\gamma_n\delta_n\big)\|x_n-u_n\| \qquad (2.10)\end{aligned}$$

Now, on substituting (2.10) in (2.5), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1+2k)\big(1-(1-k)\alpha_n\big)\big(1+(1-k\beta_n)\delta_n\\ &\quad +k(1+k\gamma_n\delta_n)\beta_n\big)\|Tu_n-u_n\|+k\big(1-(1-k)\alpha_n\big)\\ &\quad \big(1-(1-k)\beta_n-(1-k)\delta_n+(1-k)\beta_n\delta_n-k(1-k)\\ &\quad \times\beta_n\gamma_n\delta_n\big)\|x_n-u_n\|+2k\alpha_n\|Tu_n-u_n\|. \qquad (2.11)\end{aligned}$$

Also,

$$\begin{aligned}\|Tu_n-u_n\| &\le \|Tu_n-p\|+\|u_n-p\|\\ &\le (1+k)\|u_n-p\|.\end{aligned}$$

Substitute this in (2.11), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq (1+2k)\big(1-(1-k)\alpha_n\big)\big(1+(1-k\beta_n)\delta_n\\ &\quad +k(1+k\gamma_n\delta_n)\beta_n\big)(1+k)\|u_n-p\|\\ &\quad +k\big(1-(1-k)\alpha_n\big)\big(1-(1-k)\beta_n-(1-k)\delta_n\\ &\quad +(1-k)\beta_n\delta_n-k(1-k)\beta_n\gamma_n\delta_n\big)\|x_n-u_n\|\\ &\quad +2k(1+k)\alpha_n\|u_n-p\|\\ &= H\|x_n-u_n\|+L\|u_n-p\| \qquad (2.12)\end{aligned}$$

where
$H=k\big(1-(1-k)\alpha_n\big)\big(1-(1-k)\beta_n-(1-k)\delta_n+(1-k)\beta_n\delta_n-k(1-k)$
$\beta_n\gamma_n\delta_n\big)<1$ (as $\alpha_n\geq A>0, \forall n\in\mathbb{N}$) and
$L=(1+2k)\big(1-(1-k)\alpha_n\big)\big(1+(1-k\beta_n)\delta_n+k(1+k\gamma_n\delta_n)\beta_n\big)(1+k)+$
$2k(1+k)\alpha_n$

Due to the fact $u_n\to p$ as $n\to\infty$, using Lemma 1.1 and (2.12) yields

$$\|x_n-u_n\|\to 0 \text{ as } n\to\infty.$$

By triangular inequality, $\|x_n-p\|\leq\|x_n-u_n\|+\|u_n-p\|$, we have

$$x_n\to p \text{ as } n\to\infty.$$

Conversely, suppose CUIA-iteration (1.12) converges to p. Then we have to show that Mann iteration (1.3) also converges to p, *i.e.*, to prove

$$x_n\to p \text{ implies } u_n\to p.$$

Using (1.3) and (1.11), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq (1-\alpha_n)\|y_n-u_n\|+\alpha_n\|Ty_n-Tu_n\|\\ &\leq \big(1-(1-k)\alpha_n\big)\|y_n-u_n\|+2k\alpha_n\|Ty_n-y_n\|. \qquad (2.13)\end{aligned}$$

On using (2.12), we have

$$\begin{aligned}\|y_n-u_n\| &\leq (1-\beta_n)\|Tw_n-u_n\|+\beta_n\|Tz_n-u_n\|\\ &\leq (1-\beta_n)\|Tw_n-x_n\|+(1-\beta_n)\|x_n-u_n\|+\beta_n\|Tz_n-x_n\|\\ &\quad +\beta_n\|x_n-u_n\|\\ &\leq (1-\beta_n)\|Tw_n-p\|+\|x_n-p\|+\|x_n-u_n\|+\beta_n\|Tz_n-p\|\\ &\leq (1-\beta_n)k\|w_n-p\|+\|x_n-p\|+\|x_n-u_n\|\\ &\quad +k\beta_n\|z_n-p\|. \qquad (2.14)\end{aligned}$$

From (1.12) and on using (1.11), we have

$$\begin{aligned}\|w_n-p\| &\leq (1-\delta_n)\|x_n-p\|+\delta_n\|Tx_n-p\|\\ &\leq \big(1-(1-k)\delta_n\big)\|x_n-p\| \qquad (2.15)\end{aligned}$$

From (1.12), on using (1.11) and (2.15), we have

$$\begin{aligned}\|z_n - p\| &\leq (1-\gamma_n)\|Tx_n - p\| + \gamma_n\|Tw_n - p\| \\ &\leq k(1-\gamma_n)\|x_n - p\| + k\gamma_n\|w_n - p\| \\ &\leq k(1-\gamma_n)\|x_n - p\| + k\gamma_n\big(1-(1-k)\delta_n\big)\|x_n - p\| \\ &= k\big(1-(1-k)\gamma_n\delta_n\big)\|x_n - p\|. \qquad (2.16)\end{aligned}$$

Substituting (2.15) and (2.16) in (2.14), we have

$$\begin{aligned}\|y_n - u_n\| &\leq (1-\beta_n)k\big(1-(1-k)\delta_n\big)\|x_n - p\| + \|x_n - p\| \\ &\quad +\|x_n - u_n\| + k^2\big(1-(1-k)\beta_n\gamma_n\delta_n\big)\|x_n - p\| \\ &\leq \|x_n - u_n\| + \big(1+k-k(1-k)\beta_n - k(1-k)\delta_n \\ &\quad +k(1-k)\beta_n\delta_n - k(1-k)\beta_n\gamma_n\delta_n\big)\|x_n - p\|. \qquad (2.17)\end{aligned}$$

Also,

$$\begin{aligned}\|Ty_n - y_n\| &\leq \|Ty_n - p\| + \|y_n - p\| \\ &\leq (1+k)\|y_n - p\| \quad \text{(using (1.11))} \\ &\leq (1+k)(1-\beta_n)\|Tw_n - p\| + (1+k)\beta_n\|Tz_n - p\| \\ &\leq k(1+k)(1-\beta_n)\|w_n - p\| + k(1+k)\beta_n\|z_n - p\| \\ &\leq k(1+k)(1-\beta_n)\big(1-(1-k)\delta_n\big)\|x_n - p\| \\ &\quad +k(1+k)\beta_n k\big(1-(1-k)\gamma_n\delta_n\big)\|x_n - p\| \\ &= k(1+k)\Big(1-(1-k)\beta_n - (1-k)\delta_n + (1-k)\beta_n\delta \\ &\quad -k(1-k)\beta_n\gamma_n\delta_n\Big)\|x_n - p\|. \qquad (2.18)\end{aligned}$$

On substituting (2.17) and (2.18) in (2.13), we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq \big(1-(1-k)\alpha_n\big)\|y_n - u_n\| + 2k\alpha_n\|Ty_n - y_n\| \\ &\leq \big(1-(1-k)\alpha_n\big)\|x_n - u_n\| + \big(1-(1-k)\alpha_n\big) \times \\ &\quad \big(1+k-k(1-k)\beta_n - k(1-k)\delta_n + k(1-k)\beta_n\delta_n \\ &\quad -k(1-k)\beta_n\gamma_n\delta_n\big)\|x_n - p\| + 2k\alpha_n k(1+k) \times \\ &\quad \big(1-(1-k)\beta_n - (1-k)\delta_n + (1-k)\beta_n\delta \\ &\quad -k(1-k)\beta_n\gamma_n\delta_n\big)\|x_n - p\|. \qquad (2.19)\end{aligned}$$

Since $\alpha_n \geq A > 0, \forall\, n \in \mathbb{N}$, so $0 \leq 1-(1-k)\alpha_n < 1, \forall n \in \mathbb{N}$. Also

$$x_n \to p \text{ as } n \to \infty.$$

Hence on using Lemma 1.1, (2.19) gives

$$\|x_n - u_n\| \to 0 \text{ as } n \to \infty$$

Since, $\|u_n - p\| \leq \|x_n - u_n\| + \|x_n - p\|$ and $x_n \to p$ as $n \to \infty$, we get

$$u_n \to p \text{ as } n \to \infty.$$

This completes the proof.

Results of Soltuz [17, 18], Chugh and Kumar [5] and Chugh et al. [6] lead to the following corollary:

Corollary 2.1. Let K be a non-empty closed convex subset of a Banach space X and $T : K \to K$ a mapping satisfying (1.10). If the initial point is x_0 and there exists a real number A such that $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$, then the following are equivalent:

- Picard iteration (1.1) converges to p.
- Mann iteration (1.3) converges to p.
- Ishikawa iteration (1.4) converges to p.
- Noor iteration (1.7) converges to p.
- Agarwal et al. iteration (1.5) converges to p.
- SP iteration (1.8) converges to p.
- CR iteration (1.9) converges to p.
- CUIA- iteration (1.12) converges to p.

A critical examination of the recent literature reveals the fact that in the course of last four decades researchers have worked extensively to improve the rate of convergence of different iterative procedures. Most recently, Chugh et al. [6] proved that their iteration (*i.e.*, CR iterative scheme) converges faster than other iterative schemes. In the following lines, we also show that our iterative scheme (*i.e.*, CUIA-iterative scheme) is converges faster than CR-iterative scheme and henceforth than all other schemes such as: Mann, Ishikawa, Picard and Noor iteration processes for quasi-contractive operators satisfying (1.10).

Example 3.1. Let $T : [0,1] \to [0,1]$ be a mapping defined by $T(x) = \frac{x}{2}, \alpha_n = \beta_n = \gamma_n = \delta_n = 0$ for $n = 1, 2, \cdots, 15$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{4}{\sqrt{n}}$, for $n \geq 16$. Then T is a quasi-contractive operator satisfying (1.10) with a unique fixed point 0. Also $T, \alpha_n, \beta_n, \gamma_n,$ and δ_n satisfy all the conditions of strong convergence of Theorem 2.1.

Now, we show that CUIA-iterative scheme is faster than CR and Picard iterative scheme. To accomplish this, let $n \geq 16$ and $p_0 = x_0$.

Then from (1.12), we have CUIA-iteration as

$$\begin{aligned} w_n &= \Big(1-\frac{4}{\sqrt{n}}\Big)x_n + \frac{2}{\sqrt{n}}x_n = \Big(1-\frac{2}{\sqrt{n}}\Big)x_n; \\ z_n &= \Big(1-\frac{4}{\sqrt{n}}\Big)x_n + \frac{4}{\sqrt{n}}\Big(1-\frac{2}{\sqrt{n}}\Big)\frac{x_n}{2} = \Big(1-\frac{8}{\sqrt{n}}\Big)\frac{x_n}{2}; \\ y_n &= \Big(1-\frac{4}{\sqrt{n}}+\frac{8}{n}-\frac{16}{n\sqrt{n}}\Big)\frac{x_n}{2}; \\ x_{n+1} &= (1-\frac{4}{\sqrt{n}}\Big)\Big(1-\frac{4}{\sqrt{n}}+\frac{8}{n}-\frac{16}{n\sqrt{n}}\Big)\frac{x_n}{2} \\ &\quad +\frac{4}{\sqrt{n}}\Big(1-\frac{4}{\sqrt{n}}+\frac{8}{n}-\frac{16}{n\sqrt{n}}\Big)\frac{x_n}{4} \\ &= \Big(\frac{1}{2}-\frac{3}{\sqrt{n}}+\frac{8}{n}-\frac{16}{n\sqrt{n}}+\frac{16}{n^2}\Big)x_n \\ &= \prod_{i=16}^{n}\Big(\frac{1}{2}-\frac{3}{\sqrt{i}}+\frac{8}{i}-\frac{16}{i\sqrt{i}}+\frac{16}{i^2}\Big)x_0. \end{aligned}$$

In [6], for CR iteration (1.9) Chugh shows that

$$x_{n+1} = \prod_{i=16}^{n}\Big(\frac{1}{2}-\frac{1}{\sqrt{i}}-\frac{4}{i}+\frac{8}{i\sqrt{i}}\Big)x_0.$$

So,

$$\begin{aligned} \frac{x_{n+1}(CUIA)}{x_{n+1}(CR)} &= \frac{\prod_{i=16}^{n}\Big(\frac{1}{2}-\frac{3}{\sqrt{i}}+\frac{8}{i}-\frac{16}{i\sqrt{i}}+\frac{16}{i^2}\Big)x_0}{\prod_{i=16}^{n}\Big(\frac{1}{2}-\frac{1}{\sqrt{i}}-\frac{4}{i}+\frac{8}{i\sqrt{i}}\Big)x_0} \\ &\leq \prod_{i=16}^{n}\Big(1-\frac{(2i\sqrt{i}-12i+24\sqrt{i}-\ 16)}{(i^2-i\sqrt{i}-2i+4\sqrt{i})}\Big). \end{aligned}$$

Thus

$$\begin{aligned} 0 \leq &\prod_{i=16}^{n}\Big(1-\frac{(2i\sqrt{i}-12i+24\sqrt{i}-\ 16)}{(i^2-i\sqrt{i}-2i+4\sqrt{i})}\Big) \\ &\leq \lim_{n\to\infty}\prod_{i=16}^{n}\Big(1-\frac{1}{i}\Big) = \lim_{n\to\infty}\frac{15}{n} = 0. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}\Big|\frac{x_{n+1}(CUIA)}{x_{n+1}(CR)}\Big| = 0.$$

Therefore, CUIA-iterative scheme converges faster than CR iterative scheme to the fixed point 0 of T.

Now, we have to show that CUIA-iterative scheme is faster than Picard iterative scheme.
Let $n \geq 16$. Then

$$\begin{aligned}
\Big|\frac{x_{n+1}(CUIA)}{x_{n+1}(Picard)}\Big| &= \Big|\frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{3}{\sqrt{i}}+\frac{8}{i}-\frac{16}{i\sqrt{i}}+\frac{16}{i^2}\right)x_0}{\left(\frac{1}{2}\right)^{n+1}x_0}\Big|.\\
&\leq \frac{\big|(\frac{1}{2})^{n-15}\prod_{i=16}^{n}\left(1-\frac{6}{\sqrt{i}}+\frac{16}{i}-\frac{32}{i\sqrt{i}}+\frac{32}{i^2}\right)\big|}{\left(\frac{1}{2}\right)^{n+1}}.\\
&= \big|(\frac{1}{2})^{-16}\prod_{i=16}^{n}\Big(1-\frac{6}{\sqrt{i}}+\frac{16}{i}-\frac{32}{i\sqrt{i}}+\frac{32}{i^2}\Big)\big|.
\end{aligned}$$

Since,

$$\Big(1-\frac{6}{\sqrt{i}}+\frac{16}{i}-\frac{32}{i\sqrt{i}}+\frac{32}{i^2}\Big) < 1,\ \forall\ i \geq 16,$$

therefore,

$$\lim_{n\to\infty}\Big|\frac{x_{n+1}(CUIA)}{x_{n+1}(Picard)}\Big| = 0.$$

Thus, CUIA-iterative scheme converges faster than Picard iterative scheme.

Considering the aforementioned results of Example 2.1 as well as results of Círíc et al. [7], Chugh et al. [6], we conclude that CUIA-iterative scheme is faster than other iterative schemes for a certain class of quasi-contractive operators.

3. Applications

In this section, we compare the rate of convergence of various iteration namely: Picard, Mann, Ishikawa, Noor, Agarwal et al., SP, CR and CUIA-iteration via different examples employing C-programming and MATLAB. We are enlisting the outcomes in the form of Tables 1-3 and the graphs are plotted for above mentioned iterations.

3.1. Decreasing function

Let $f : [0,1] \to [0,1]$ be a function defined by $f(x) = (1-x)^m$, $m = 7, 8, \cdots$. Then f is a decreasing function. By taking $m = 11$, initial approximation $x_0 = 0.4$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{2/3}}$, the comparison of convergence of the above mentioned iterative processes to the exact fixed point $p = 0.155602$ is listed in Table 1.

3.2. Increasing function

Let $f : [0, 8] \to [0, 8]$ be a function defined by $f(x) = \frac{x^2+9}{10}$. Then f is a increasing function. By taking $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{2/3}}$, the aforementioned iterations converge to the fixed point $p = 1$ listed in Table 2.

3.3. Super linear functions with multiple roots

The function defined by $f(x) = 2x^3 - 7x^2 + 8x - 2$ is a super linear function with multiple real roots. By taking $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{2/3}}$, the comparison shows the convergence of the different iterative schemes to the exact fixed point $p = 1$ is listed in Table 3.

Table 1. Decreasing function.

	CUIA-iteration		Picard iteration		Mann iteration		Ishikawa iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.003628	1.000000	0.003628	0.003628	0.003628	0.003628	0.003628	0.960809
1	0.000000	0.224159	0.960809	0.960809	0.960809	0.606614	0.000000	0.360554
2	0.061306	0.145146	0.000000	0.000000	0.000035	0.315001	0.007309	0.234093
3	0.178160	0.158682	1	1	0.015581	0.196176	0.053203	0.197760
4	0.149474	0.155534	0.000000	0.000000	0.090526	0.160044	0.088583	0.180095
5	0.155741	0.155598	1	1	0.146832	0.156043	0.112564	0.170255
6	0.155611	0.155602	0.00000	0.00000	0.154712	0.155679	0.128348	0.164511
7	0.155603	0.155602	1	1	0.155447	0.155621	0.138467	0.161086
8	0.155602	0.155602	0.00000	0.00000	0.155565	0.155608	0.144841	0.159017
9	0.155602	0.155602	1	1	0.155591	0.155604	0.148819	0.157753
10	0.155602	0.155602	0.000000	0.000000	0.155599	0.155603	0.151297	0.156974
11	0.155602	0.155602	1	1	0.155601	0.155603	0.152845	0.156487
12	0.155602	0.155602	0.000000	0.000000	0.155602	0.155603	0.153819	0.156180
13	0.155602	0.155602	1	1	0.155602	0.155603	0.154436	0.155984
14	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.154831	0.155857
15	0.155602	0.155602	1	1	0.155602	0.155602	0.155086	0.155775
16	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155254	0.155720
17	0.155602	0.155602	1	1	0.155602	0.155602	0.155364	0.155683
18	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155438	0.155659
19	0.155602	0.155602	1	1	0.155602	0.155602	0.155488	0.155642
20	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155522	0.155630
21	0.155602	0.155602	1	1	0.155602	0.155602	0.155546	0.155622
22	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155562	0.155617
23	0.155602	0.155602	1	1	0.155602	0.155602	0.155573	0.155613
24	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155581	0.155610
25	0.155602	0.155602	1	1	0.155602	0.155602	0.155587	0.155608
26	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155591	0.155607
27	0.155602	0.155602	1	1	0.155602	0.155602	0.155594	0.155606
28	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155596	0.155605
29	0.155602	0.155602	1	1	0.155602	0.155602	0.155598	0.155604
30	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155599	0.155604

	CUIA-iteration		Picard iteration		Mann iteration		Ishikawa iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
31	0.155602	0.155602	1	1	0.155602	0.155602	0.155600	0.155603
32	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155600	0.155603
33	0.155602	0.155602	1	1	0.155602	0.155602	0.155601	0.155603
34	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155601	0.155603
35	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155603
36	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155603
37	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155603
38	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155603
39	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155603
40	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155603
41	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155603
42	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155603
43	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155603
44	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155603
45	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155602
46	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155602
47	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155602
48	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155602
49	0.155602	0.155602	1	1	0.155602	0.155602	0.155602	0.155602
50	0.155602	0.155602	0.000000	0.000000	0.155602	0.155602	0.155602	0.155602

Table. 1A

	Agarwal et al. Iteration		Noor iteration		SP iteration		CR iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.003628	0.960809	0.003628	0.000000	0.003628	0.960809	0.003628	0.000000
1	0.000000	0.005018	1	0.619883	0.000000	0.053701	1	0.140839
2	0.946171	0.491875	0.000018	0.332635	0.544895	0.142181	0.188288	0.151784
3	0.000583	0.008589	0.011695	0.222284	0.185076	0.126595	0.163521	0.154855
4	0.909478	0.603626	0.062956	0.179212	0.225618	0.180183	0.157123	0.155572
5	0.000038	0.000771	0.113904	0.164063	0.112432	0.140948	0.155665	0.155606
6	0.991547	0.728959	0.139287	0.158946	0.188024	0.163496	0.155595	0.155601
7	0.000001	0.000042	0.148958	0.157069	0.140330	0.152324	0.155604	0.155603
8	0.999539	0.781370	0.152656	0.156302	0.162378	0.156576	0.155602	0.155602
9	0.000000	0.000006	0.154191	0.155959	0.153640	0.155431	0.155603	0.155693
10	0.999931	0.814620	0.154882	0.155793	0.155950	0.155597	0.155602	0.155602
11	0.000000	0.000001	0.155216	0.155709	0.155614	0.155615	0.155603	0.155603
12	0.999985	0.839264	0.155386	0.155664	0.155578	0.155600	0.155602	0.155602
13	0.000000	0.000000	0.155477	0.155639	0.155608	0.155602	0.155602	0.155602
14	0.999996	0.858377	0.155527	0.155625	0.155603	0.155603	0.155602	0.155602
15	0.000000	0.000000	0.155556	0.155617	0.155602	0.155602	0.155602	0.155602
16	0.999999	0.873651	0.155574	0.155612	0.155602	0.155602	0.155602	0.155602
17	0.000000	0.000000	0.155584	0.155608	0.155602	0.155602	0.155602	0.155602
18	.9999999	0.886135	0.155590	0.155606	0.155602	0.155602	0.155602	0.155602
19	0	0.000000	0.155595	0.155605	0.155602	0.155602	0.155602	0.155602
20	1	0.896526	0.155597	0.155604	0.155602	0.155602	0.155602	0.155602
21	0	0.000000	0.155599	0.155604	0.155602	0.155602	0.155602	0.155602
22	1	0.905304	0.155600	0.155603	0.155602	0.155602	0.155602	0.155602
23	0	0.000000	0.155601	0.155603	0.155602	0.155602	0.155602	0.155602
24	1	0.912812	0.155601	0.155603	0.155602	0.155602	0.155602	0.155602
25	0	0.000000	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
26	1	0.919303	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
27	0	0.000000	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
28	1	0.924967	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
29	0	0.000000	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
30	1	0.929950	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
31	0	0.000000	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602
32	1	0.934364	0.155602	0.155603	0.155602	0.155602	0.155602	0.155602

	Agarwal et al. Iteration		Noor iteration		SP iteration		CR iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
33	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
34	1	0.938301	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
35	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
36	1	0.941831	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
37	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
38	1	0.945013	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
39	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
40	1	0.947895	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
41	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
42	1	0.950516	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
43	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
44	1	0.952910	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
45	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
46	1	0.955103	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
47	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
48	1	0.957119	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
49	0	0.000000	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602
50	1	0.958979	0.155602	0.155602	0.155602	0.155602	0.155602	0.155602

Table. 1B

Table 2. Increasing function

	CUIA-iteration		Picard iteration		Mann iteration		Ishikawa iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.916000	0.999362	0.916000	0.916000	0.916000	0.916000	0.916000	0.983906
1	0.999873	0.999993	0.983906	0.983906	0.983906	0.969897	0.996807	0.995753
2	0.999999	1	0.996807	0.996807	0.994070	0.986658	0.999152	0.998436
3	1	1	0.999362	0.999362	0.997349	0.993393	0.999687	0.999323
4	1	1	0.999873	0.999873	0.998683	0.996487	0.999865	0.999677
5	1	1	0.999975	0.999975	0.999299	0.998034	0.999935	0.999835
6	1	1	0.999995	0.999995	0.999607	0.998856	0.999967	0.999911
7	1	1	0.999999	0.999999	0.999771	0.999314	0.999982	0.999950
8	1	1	1	1	0.999863	0.999578	0.999990	0.999971
9	1	1	1	1	0.999916	0.999735	0.999994	0.999983
10	1	1	1	1	0.999947	0.999830	0.999997	0.999990
11	1	1	1	1	0.999966	0.999889	0.999998	0.999994
12	1	1	1	1	0.999978	0.999927	0.999999	0.999996
13	1	1	1	1	0.999985	0.999951	0.999999	0.999997
14	1	1	1	1	0.999990	0.999967	0.999999	0.999998
15	1	1	1	1	0.999993	0.999978	1	0.999999
16	1	1	1	1	0.999996	0.999985	1	0.999999
17	1	1	1	1	0.999997	0.999989	1	1
18	1	1	1	1	0.999998	0.999992	1	1
19	1	1	1	1	0.999998	0.999995	1	1
20	1	1	1	1	0.999999	0.999996	1	1
21	1	1	1	1	0.999999	0.999997	1	1
22	1	1	1	1	0.999999	0.999998	1	1
23	1	1	1	1	1	0.999999	1	1
24	1	1	1	1	1	0.999999	1	1
25	1	1	1	1	1	0.999999	1	1
26	1	1	1	1	1	0.999999	1	1
27	1	1	1	1	1	1	1	1
28	1	1	1	1	1	1	1	1
29	1	1	1	1	1	1	1	1

Table. 2A

	Agarwal et al. Iteration		Noor iteration		SP iteration		CR iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.916000	0.983906	0.916000	0.996807	0.916000	0.996807	0.916000	0.996807
1	0.996807	0.998415	0.999362	0.999207	0.999362	0.999845	0.999362	0.999885
2	0.999683	0.999805	0.999842	0.999717	0.999969	0.999986	0.999977	0.999994
3	0.999961	0.999973	0.999943	0.999880	0.999997	0.999998	0.999999	1
4	0.999995	0.999996	0.999976	0.999943	1	1	1	1
5	0.999999	0.999999	0.999989	0.999971	1	1	1	1
6	1	1	0.999994	0.999985	1	1	1	1
7	1	1	0.999997	0.999992	1	1	1	1
8	1	1	0.999998	0.999995	1	1	1	1
9	1	1	0.999999	0.999997	1	1	1	1
10	1	1	0.999999	0.999998	1	1	1	1
11	1	1	1	0.999999	1	1	1	1
12	1	1	1	0.999999	1	1	1	1
13	1	1	1	1	1	1	1	1
14	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1
16	1	1	1	1	1	1	1	1
17	1	1	1	1	1	1	1	1
18	1	1	1	1	1	1	1	1
19	1	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1	1
21	1	1	1	1	1	1	1	1
22	1	1	1	1	1	1	1	1
23	1	1	1	1	1	1	1	1
24	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1
26	1	1	1	1	1	1	1	1
27	1	1	1	1	1	1	1	1
28	1	1	1	1	1	1	1	1
29	1	1	1	1	1	1	1	1

Table. 2B

Table 3. Super linear function with multiple roots

	CUIA-iteration		Picard iteration		Mann iteration		Ishikawa iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.712000	0.999516	0.712000	0.712000	0.712000	0.712000	0.712000	0.869280
1	1	1	0.869280	0.869280	0.869280	0.836833	0.978445	0.971354
2	1	1	0.978445	0.978445	0.964689	0.925483	0.999132	0.991154
3	1	1	0.999516	0.999516	0.993620	0.968407	0.999920	0.996720
4	1	1	1	1	0.998939	0.986262	0.999990	0.998637
5	1	1	1	1	0.999806	0.993715	0.999998	0.999387
6	1	1	1	1	0.999960	0.996980	1	0.999707
7	1	1	1	1	0.999990	0.998485	1	0.999854
8	1	1	1	1	0.999998	0.999212	1	0.999924
9	1	1	1	1	1	0.999578	1	0.999959
10	1	1	1	1	1	0.999768	1	0.999977
11	1	1	1	1	1	0.999869	1	0.999987
12	1	1	1	1	1	0.999925	1	0.999992
13	1	1	1	1	1	0.999956	1	0.999996
14	1	1	1	1	1	0.999974	1	0.999997
15	1	1	1	1	1	0.999984	1	0.999998
16	1	1	1	1	1	0.999990	1	0.999999
17	1	1	1	1	1	0.999994	1	1
18	1	1	1	1	1	0.999996	1	1
19	1	1	1	1	1	0.999998	1	1
20	1	1	1	1	1	0.999998	1	1

n	CUIA-iteration fx_n	CUIA-iteration x_{n+1}	Picard iteration fx_n	Picard iteration x_{n+1}	Mann iteration fx_n	Mann iteration x_{n+1}	Ishikawa iteration fx_n	Ishikawa iteration x_{n+1}
21	1	1	1	1	1	0.999999	1	1
22	1	1	1	1	1	0.999999	1	1
23	1	1	1	1	1	1	1	1
24	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1
26	1	1	1	1	1	1	1	1
27	1	1	1	1	1	1	1	1
28	1	1	1	1	1	1	1	1

Table. 3A

n	Agarwal et al. Iteration fx_n	Agarwal et al. Iteration x_{n+1}	Noor iteration fx_n	Noor iteration x_{n+1}	SP iteration fx_n	SP iteration x_{n+1}	CR iteration fx_n	CR iteration x_{n+1}
0	0.712000	0.869280	0.712000	0.978445	0.712000	0.978445	0.712000	0.978445
1	0.978445	0.993875	0.999516	0.995537	0.999516	0.999790	0.999516	0.999976
2	0.999962	0.999986	0.999980	0.998630	1	0.999994	1	1
3	1	1	0.999998	0.999493	1	1	1	1
4	1	1	1	0.999789	1	1	1	1
5	1	1	1	0.999905	1	1	1	1
6	1	1	1	0.999955	1	1	1	1
7	1	1	1	0.999977	1	1	1	1
8	1	1	1	0.999988	1	1	1	1
9	1	1	1	0.999994	1	1	1	1
10	1	1	1	0.999997	1	1	1	1
11	1	1	1	0.999998	1	1	1	1
12	1	1	1	0.999999	1	1	1	1
13	1	1	1	0.999999	1	1	1	1
14	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1
16	1	1	1	1	1	1	1	1
17	1	1	1	1	1	1	1	1
18	1	1	1	1	1	1	1	1
19	1	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1	1
21	1	1	1	1	1	1	1	1
22	1	1	1	1	1	1	1	1
23	1	1	1	1	1	1	1	1
24	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1
26	1	1	1	1	1	1	1	1
27	1	1	1	1	1	1	1	1
28	1	1	1	1	1	1	1	1

Table. 3B

Plots of different iterative schemes for decreasing, increasing and super linear function with multiple roots: In all plots (figures) blue line indicates for decreasing function, green line for increasing function and red line shows the result for super linear function with multiple roots.

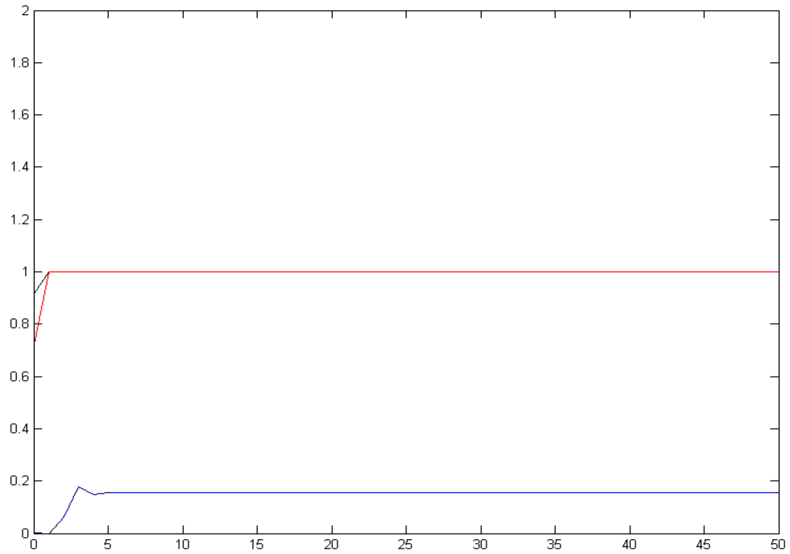

Figure 1. CUIA-iteration

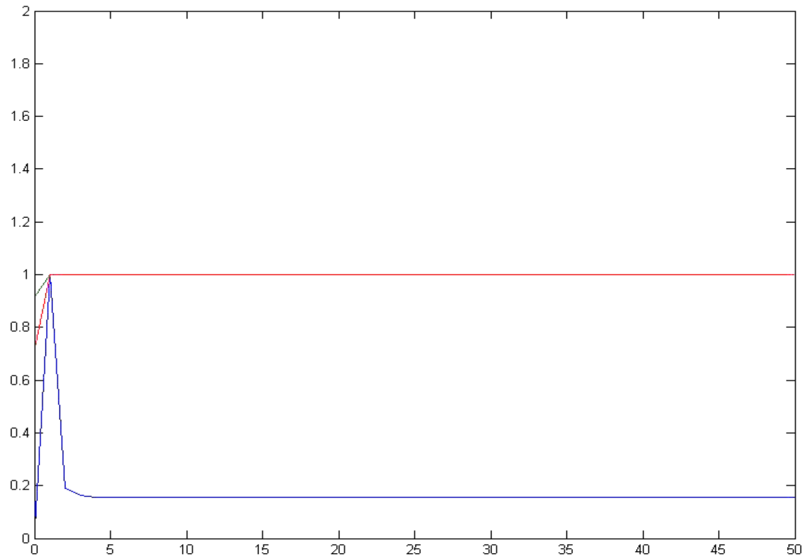

Figure 2. CR-iteration

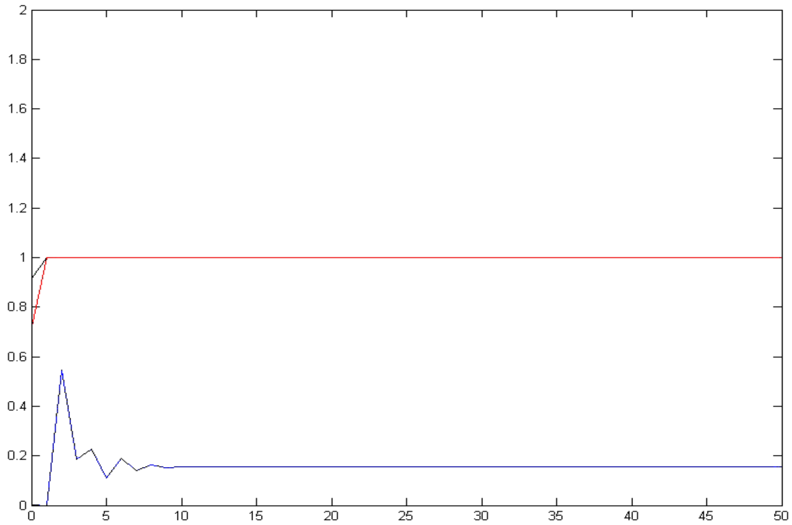

Figure 3. SP-iteration

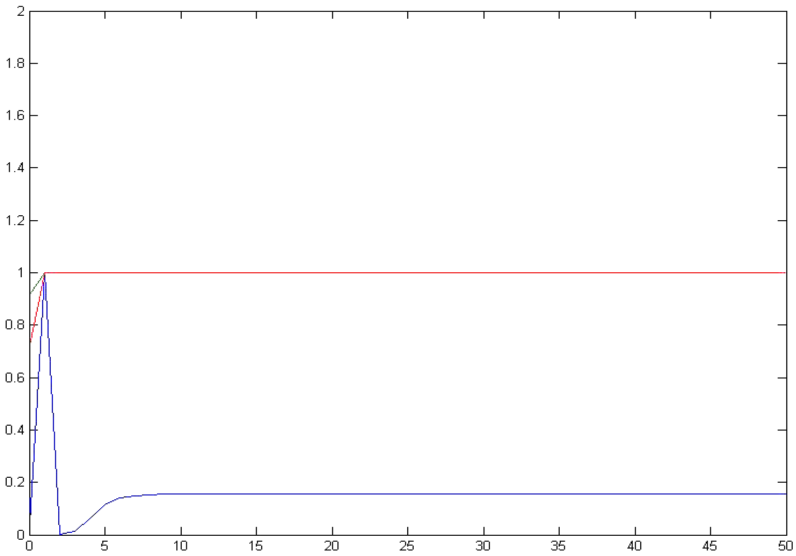

Figure 4. Noor-iteration

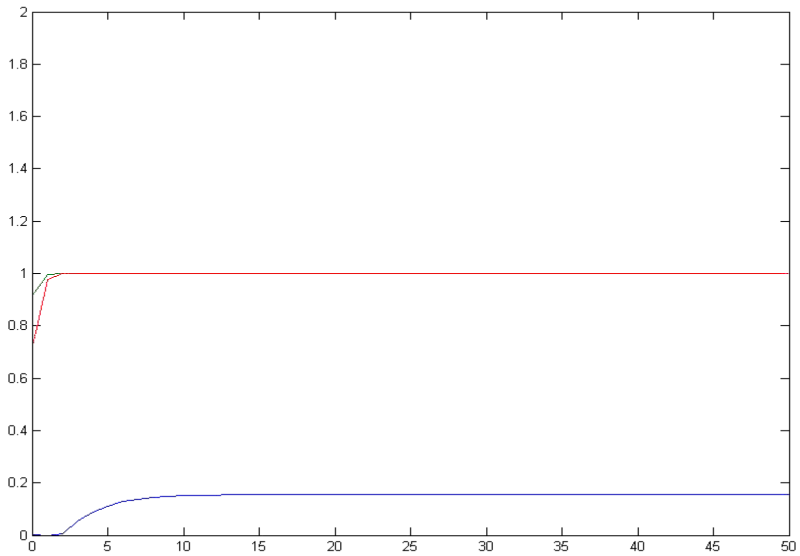

Figure 5. Ishikawa-iteration

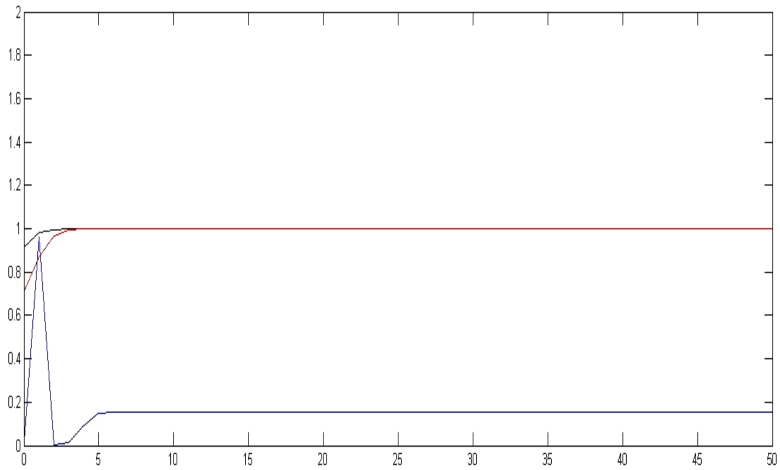

Figure 6. Mann-iteration

4. Observations

4.1. Decreasing function $(1-x)^m$.

(i) For $m = 11$ and on taking the initial approximation as $x_0 = 0.4$, Picard iteration oscillate between 0 and 1, Mann iterative process converges in 14^{th} iterations, Ishikawa iterative process in 45^{th} iteration, Agarwal et al. scheme does not converge, Noor iterative scheme in 33^{rd} iteration, SP iterative scheme in 15^{th} iteration, CR iterative scheme in 12^{th} iteration and CUIA-iterative scheme in 8^{th} iteration for $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{\frac{2}{3}}}$.

(ii) If $m = 11$ and the starting point is $x_0 = 0.6$, Picard iteration oscillate between 0 and 1, Mann iterative process converges in 166^{th} iterations, the number of iterations for Ishikawa iterative process are 2358, Agarwal et al. scheme requires 84 iterations, Noor iterative scheme 424 iterations, SP iterative scheme in 118^{th} iteration, CR iterative scheme needs 76 iterations and CUIA-iterative scheme converges in 55^{th} iteration for $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{0.1}}$.

4.2. Increasing function $\frac{(x^2+9)}{10}$

(i) By taking the initial value $x_0 = 0.4$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{1/3}}$, Picard iteration converges in 8^{th} iteration, Mann scheme in 27^{th} iteration, Ishikawa iterative process in 17^{th} iteration, Agarwal et al. iterative scheme in 6^{th} iteration, Noor iterative scheme in 13^{th} iteration, SP scheme in 4^{th} iteration, CR schemes converges in 4^{th} iteration and CUIA-iterative process in 3^{rd} iteration.

(ii) If we take the initial approximation $x_0 = 0.8$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{0.1}}$, Picard iteration converges in 7^{th} iteration, Mann scheme requires 10^{th} iterations, Ishikawa iterative process needs 6^{th} iterations, Agarwal iterative scheme converges in 4^{th} iteration, Noor iterative scheme will converge in 5^{th} iteration, SP scheme needs 3^{rd} iterations, CR schemes also converges in 3^{rd} iteration and CUIA-iterative process requires the minimum that is 2^{nd} iterations only.

4.3. Super linear equation with multiple roots $2x^3 - 7x^2 + 8x - 2$.

(i) If the initial value is $x_0 = 0.6$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{1/3}}$, Picard iteration converges in 4^{th} iteration, Mann scheme in 23^{rd} iteration, Ishikawa iterative process in 17^{th} iteration, Agarwal et al. iterative scheme in 3^{rd} iteration, Noor iterative scheme in 14^{th} iteration, SP scheme in 4^{th} iteration, CR schemes converges in 2^{nd} iteration and CUIA-iterative process in 1^{st} iteration.

(ii) If we take the initial approximation as $x_0 = 0.7$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^{3/5}}$, we need 3^{rd} iterations for Picard iterative scheme to converge, 78^{th} iterations for Mann iterative scheme, 62^{th} iterations for Ishikawa process, Agarwal et al. needs 2^{nd} iteration to converge, Noor iterative scheme also converges in 22^{nd} iteration , SP iterative scheme in 46^{th} iterations, CR iterative scheme in 1^{st} iteration and CUIA-iteration also in the first iteration.

5. Conclusions

1. We may conclude that CUIA-iterative scheme is equivalent to Mann, Picard, Ishikawa, Agarwal et al., Noor, SP, CR iterative schemes.
2. The rate of convergence of CUIA-iterative scheme is faster than the other entire two steps and three steps iterative procedures.
3. For decreasing function, we may conclude that:
   (i) Picard scheme oscillate between 0 and 1 as the value of initial approximation increases from 0.1 to 1.
   (ii) Agarwal et al. scheme does not converge as the value of q varies from 0.3 to 0.6 in $\alpha_n = \beta_n = \frac{1}{(1+n)^q}$ for the initial approximations from 0.1 to 1.
   (iii) Rate of convergence of Mann, Ishikawa, SP, CR, and CUIA-iterative scheme decreases as the value of q varies from 0.1 to 0.6 in $x_0 = 0.6$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^q}$ whatever the initial approximation may be from 0.1 to 1.

   Notice that, in this case all results show that CUIA-iterative scheme is faster than other iterative schemes.
4. For the increasing function, we may conclude the following.
   (i) The rate of convergence is increasing from Mann, Ishikawa, Noor, Picard, Agarwal et al., SP, CR and CUIA-iteration in which CUIA-iterations leads.
   (ii) The number of iterations required to get fixed point decreases as the value of initial approximation becomes closer to the fixed point but it almost remain same for each approximation from 0.1 to the fixed point in CUIA-iterative scheme.
5. For the cubic equation with multiple roots:
   (i) The number of iterations decreases as the value of initial approximation becomes closer to the fixed point.

(ii) The number of iterations increases on varying the values of q from 0.1 to 0.6 in $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{1}{(1+n)^q}$ but almost remains same for CUIA-iterative.

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