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ON KANTOROVICH FORM OF GENERALIZED SZÁSZ-TYPE OPERATORS USING CHARLIER POLYNOMIALS

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Abstract. The aim of this article is to introduce a new form of Kantorovich Szász-type operators involving Charlier polynomials. In this manuscript, we discuss the rate of convergence, better error estimates. Further, we investigate order of approximation in the sense of local approximation results with the help of Ditzian-Totik modulus of smoothness, second order modulus of continuity, Peetre's K-functional and Lipschitz class.

1. Introduction

Bernstein [2] defined the positive linear operators using binomial distribution and proved pointwise and uniform approximation in the space of continuous functions on $[0, 1]$. These operators provide the powerful tool for numerical analysis, computer added geometric design (CAGD) and solutions of differential equations. But these operators are not suitable for discontinuous functions. Later on, Kantorovich [9] generalized the Bernstein operators for integrable functions. Szász [21] introduced

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linear positive operators in the sense of exponential growth on non-negative semi axes. Several generalizations of these operators have been studied by different researchers ([1, 6, 7, 13, 18–20, 23–26]). Some interesting results can be seen in ([5, 12, 14–17]). Many operators preserve the constant and linear functions but these operators do not preserve x^2. King [10] introduced a method in order to preserve x^2 for the Bernstein operators. Recently, Varma and Taşdelen [27] gave a Kantorovich-Szász type operators by means of Charlier polynomials [8] having the generating function of the form

$$e^t\left(1-\frac{t}{a}\right)^u=\sum_{k=0}^{\infty}C_k^{(a)}(u)\frac{t^k}{k!},\ \ |t|<a, \tag{1}$$

and the explicit representation

$$C_k^{(a)}(u)=\sum_{r=0}^{k}\binom{k}{r}(-u)_r\left(\frac{1}{a}\right)^r,$$

where $(\alpha)_k$ is the Pochhammer's symbol given by

$$(\alpha)_0=1,(\alpha)_k=\alpha(\alpha+1)...(\alpha+k-1),\ \ k\in\mathbb{N}.$$

We note that for $a>0$ and $u\leq 0$, Charlier polynomials are positive. They [27] defined Kantorovich-Szász-type operators as

$$\begin{aligned} L_n^*(f;x,a) &= ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!} \\ &\quad\times\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt, \end{aligned} \tag{2}$$

where $a>1$, $n\in\mathbb{N}$ and $x\geq 0$. Motivated by the above development, we define a new sequence of Kantorovich-Szász-type operators which preserves constant and quadratic test functions i.e. $e_0(x)$ and $e_2(x)$ $(e_i(x)=x^i, i=0,2)$

$$\begin{aligned} K_{n,a}^*(f;r_{n,a}^*(x)) &= ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}^*(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}^*(x))}{k!} \\ &\quad\times\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(s)ds, \end{aligned} \tag{3}$$

where

$$r^*_{n,a}(x) = \frac{-(4+\frac{1}{a-1}) + \sqrt{(4+\frac{1}{a-1})^2 + 4(n^2x^2 - \frac{10}{3})}}{2n} \tag{4}$$

and $r^*_{n,a}(x) \geq 0$ for $x \in \left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$.

We observe that

(i) For a fixed $x \in \left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$, $r^*_{n,a}(x) \to x$ as $n \to \infty$ on $[0,\infty)$ and operators (3) reduce to operators (2)

and

(ii) For a fixed x, $r^*_{n,a}(x) = x$ as $a \to \infty$ and taking $x - \frac{1}{n}$ instead of x, operators (3) reduce to the Classical Kantorovich-Szász operators [22].

In the present paper, we discuss the rate of convergence for continuous functions, first order derivative of the function. Further, we investigate some direct and local approximation results using Ditzian-Totik modulus of smoothness, second order modulus of continuity, Peetre's K-functional and Lipschitz space.

2. Basic Estimates

Lemma 2.1. *From generating function (1) and differentiation, we have*

$$\sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr^*_{n,a}(x)},$$

$$\sum_{k=0}^{\infty} k\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr^*_{n,a}(x)}(1+nr^*_{n,a}(x)),$$

$$\sum_{k=0}^{\infty} k^2\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr^*_{n,a}(x)} \times \left(2+\left(3+\frac{1}{a-1}\right)nr^*_{n,a}(x) + n^2r^{*2}_{n,a}(x)\right).$$

Lemma 2.2. *Let $e_i(t) = t^i, i = 0, 1, 2$ and $r^*_{n,a}(x) \geq 0$. Then for the operators $K^*_{n,a}$ defined by (3), we have*

$$\begin{aligned} K^*_{n,a}(e_0; r^*_{n,a}(x)) &= 1, \\ K^*_{n,a}(e_1; r^*_{n,a}(x)) &= \frac{-(1+\frac{1}{a-1}) + \sqrt{(4+\frac{1}{a-1})^2 + 4(n^2x^2 - \frac{10}{3})}}{2n}, \\ K^*_{n,a}(e_2; r^*_{n,a}(x)) &= x^2. \end{aligned}$$

Proof. Using Lemma 2.1 and equation (4), we prove Lemma 2.2. □

Lemma 2.3. *Let $\psi^i_x(t) = (t-x)^i, i = 0, 1, 2$ and $r^*_{n,a}(x) \geq 0$, $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} K^*_{n,a}(\psi^0_x; r^*_{n,a}(x)) &= 1, \\ K^*_{n,a}(\psi^1_x; r^*_{n,a}(x)) &= -\frac{\left(1+\frac{1}{a-1}\right)}{2n} + \frac{\frac{8}{3} + \frac{8}{a-1} + \frac{1}{(a-1)^2}}{2n\left(\sqrt{(4+\frac{1}{a-1})^2 + 4(n^2x^2 - \frac{10}{3})} + 2nx\right)}, \\ K^*_{n,a}(\psi^2_x; r^*_{n,a}(x)) &= \left(1 + \frac{1}{a-1}\right) \cdot \frac{x}{n} - \frac{\frac{8}{3} + \frac{8}{a-1} + \frac{1}{(a-1)^2}}{\sqrt{(4+\frac{1}{a-1})^2 + 4(n^2x^2 - \frac{10}{3})} + 2nx} \\ &\times \frac{x}{n} + x^2. \end{aligned}$$

Proof. In view of Lemma 2.2 and linearity property, we can easily prove this Lemma. □

3. Rate of convergence

For $f \in C[0, \infty)$, where $C[0, \infty)$ is the set of all continuous functions on $[0, \infty)$, the modulus of continuity for a uniformly continuous f is

$$\omega(f; \delta) = \sup_{|t-y| \leq \delta} |f(t) - f(y)|, \qquad t, y \in [0, \infty).$$

For a uniformly continuous f in $C[0, \infty)$ and $\delta > 0$, one has

$$|f(t) - f(y)| \leq \left(1 + \frac{(t-y)^2}{\delta^2}\right)\omega(f; \delta). \tag{5}$$

THEOREM 3.1. *Let f be uniformly continuous and bounded in $C\left[\frac{\sqrt{\frac{10}{3}}}{n},\infty\right)$, $n\in\mathbb{N}$. Then for the operators $K_{n,a}^{*}$, we have*

$$|K_{n,a}^{*}(f;x)-f(x)|\leq 2\omega\biggl(f;\delta_{n,a}\biggr),$$

where $\delta_{n,a}=\sqrt{K_{n,a}^{*}(\psi_x^2;x)}$, holds uniformly in each compact subset of $\left[\frac{\sqrt{\frac{10}{3}}}{n},\infty\right)$.

Proof. From (5), we have

$$\begin{aligned}
&|K_{n,a}^{*}(f;x)-f(x)|\\
&\leq\ ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}^{*}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}^{*}(x))}{k!}\\
&\times\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)-f(x)|dt\\
&\leq\ \left\{ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}^{*}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}^{*}(x))}{k!}\right.\\
&\times\left.\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(1+\frac{(t-x)^2}{\delta_{n,a}^2}\right)dt\right\}\omega(f;\delta_{n,a})\\
&\leq\ \left\{1+\frac{K_{n,a}^{*}(\psi_x^2;x)}{\delta_{n,a}^2}\right\}\omega(f;\delta_{n,a})\\
&=\ 2\omega(f;\delta_{n,a}),
\end{aligned}$$

where $\delta_{n,a}=\sqrt{K_{n,a}^{*}(\psi_x^2;x)}$, holds uniformly in each compact subset of $[0,\infty)$. □

REMARK 3.2. For the Kantorovich-Szász type operators L_n^* given by (2), we have, for every $f\in C[0,\infty)\cap E$, where

$$E:=\left\{f:[0,\infty)\to\mathbb{R}:|f(x)|\leq Me^{Ax},A\in\mathbb{R},M\in\mathbb{R}^{+}\right\},$$

one has

$$|L_n^*(f;x,a) - f(x)| \leq 2\omega(f;\delta),$$

where $\delta = \sqrt{\frac{x}{n}\left(1+\frac{1}{a-1}\right)+\frac{10}{3n^2}}$ (see [27]). Here, we shall show that our operators $K^*_{n,a}$ has the better approximation than the operators L_n^*. Since

$$\frac{x}{n}\left(1+\frac{1}{a-1}\right) < \frac{x}{n}\left(1+\frac{1}{a-1}\right)+\frac{10}{3n^2},$$

which shows that $\delta_{n,a} < \delta$. This implies that our operators (3) converge uniformly more rapidly than operators defined by (2).

Theorem 3.3. *Let $\omega_1(f;\delta)$ is the modulus of continuity of $f'(x)$. For $f \in C\left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$ and bounded on $\left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$, $n \in \mathbb{N}$ with continuous derivative, we have*

$$|K^*_{n,a}(f;x) - f(x)| \leq \omega_1\big(f'; n^{-\frac{1}{2}}\big)\sqrt{K^*_{n,a}(\psi_x^2(t);x)}\left\{1+\sqrt{n}\sqrt{K^*_{n,a}(\psi_x^2(t);x)}\right\},$$

where $\delta_n = n^{-\frac{1}{2}}$.

Proof. It is known that

$$\begin{aligned} f(x_1) - f(x_2) &= (x_1-x_2)f'(\xi) \\ &= (x_1-x_2)f'(x_1) + (x_1-x_2)[f'(\xi) - f'(x_1)], \end{aligned} \tag{6}$$

for $x_1, x_2 \in \left[\frac{\sqrt{\frac{10}{3}}}{n}, b\right]$, $b > \frac{\sqrt{\frac{10}{3}}}{n}$ and $x_1 < \xi < x_2$. Also, we have (see [11], Theorem 1.6.2, pp. 21)

$$|(x_1-x_2)[f'(\xi) - f'(x_1)]| \leq |x_1 - x_2|(\lambda+1)\omega_1(\delta), \quad \lambda = \lambda(x_1, x_2; \delta). \tag{7}$$

Next, we find

$$
\begin{aligned}
|K^*_{n,a}(f;x)-f(x)| &= \left| ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\right. \\
&\times \left. \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)-f(x)dt\right|. \qquad (8)
\end{aligned}
$$

Using (6) and (7), we get

$$
\begin{aligned}
|K^*_{n,a}(f;x)-f(x)| &\leq \left| ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}\right. \\
&\times \left. \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(x-t)f'(x)dt\right| \\
&\quad +\omega_1(f;\delta_n)ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \\
&\times \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|(\lambda+1)dt \\
&\leq \omega_1(f;\delta_n)\left\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}\right. \\
&\times \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dt \\
&+ e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}
\end{aligned}
$$

$$\begin{aligned}
&\times \quad \sum_{\lambda\geq 1} C_k^{(a)}(-(a-1)nr^*_{n,a}(x)) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\lambda(x,t;\delta)dt\Bigg\} \\
&\leq \quad \omega_1(f';\delta_n)\Bigg\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dt \\
&+ \quad \frac{1}{\delta_n}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^2dt\Bigg\} \\
&\leq \quad \omega_1(f';\delta_n)\left(\sqrt{K^*_{n,a}(\psi_x^2;x)}+\frac{K^*_{n,a}(\psi_x^2;x)}{\delta_n}\right) \\
&= \quad \omega_1(f';\delta_n)\sqrt{K^*_{n,a}(\psi_x^2;x)}\left\{1+\frac{\sqrt{K^*_{n,a}(\psi_x^2;x)}}{\delta_n}\right\}.
\end{aligned}$$

Taking $\delta_n = n^{-\frac{1}{2}}$, we get

$$|K^*_{n,a}(f;x)-f(x)| \leq \omega_1(f';n^{-\frac{1}{2}})\sqrt{K^*_{n,a}(\psi_x^2(t);x)}\left\{1+\sqrt{n}\sqrt{K^*_{n,a}(\psi_x^2(t);x)}\right\}.$$

□

4. Direct Estimate

Ditzian-Totik Modulus of smoothness [4] is defined for continuous and bounded functions as:

$$\begin{aligned}
&\omega^2_{\varphi^\lambda}(f;\delta) \\
&= \sup_{0<h\leq\delta} \parallel \Delta^2_{h\varphi(x)}f(x) \parallel \\
&= \sup_{0<h\leq\delta} \sup_{x\pm h\varphi^\lambda(x)\in[0,b],b<\infty} |f(x-h\varphi^\lambda(x))-2f(x)+f(x+h\varphi^\lambda(x))|,
\end{aligned}$$

where $\varphi^2(x) = x$ and Peetre's K-functional [4] is given by

$$K_{\varphi^\lambda}(f,\delta^2)=\inf_g\left(\|f-g\|_{C[0,b],b<\infty}+\delta^2\|\varphi^2\lambda g''\|_{C[0,b],b<\infty}\right),\ g,g'\in AC_{loc}. \tag{9}$$

The K-functional is equivalent to the modulus of smoothness, i.e.,

$$C^{-1}K_{\varphi^\lambda}(f,\delta^2)\le\omega^2_{\varphi^\lambda}(f,\delta)\le CK_{\varphi^\lambda}(f,\delta^2) \text{ where } C>0. \tag{10}$$

THEOREM 4.1. *For all the continuous functions f defined on* $\left[\frac{\sqrt{\frac{10}{3}}}{n},b\right],\frac{\sqrt{\frac{10}{3}}}{n}<b<\infty$, $n\in\mathbb{N}$ *and* $a>1$, *we have*

$$|K^*_{n,a}(f;x)-f(x)|\le C\omega^2_{\varphi^\lambda}\big(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\big) \text{ for the large value of } n,$$

where $0\le\lambda\le1$ *and* $\varphi^2(x)=x$.

Proof. Using (9), (10), we have

$$\|\ f-g\ \|_{C[0,b],b<\infty}\le A\omega^2_{\varphi^\lambda}\big(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\big), \tag{11}$$

$$n^{-1}\varphi(x)^{2-2\lambda}\|\varphi^{2\lambda}g''\|_{C[0,b],b<\infty}\le B\omega^2_{\varphi^\lambda}\big(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\big). \tag{12}$$

Next, we can choose $g_n\equiv g_{n,x,\lambda}$ for fixed x and λ such that

$$\begin{aligned}&|K^*_{n,a}(f;x)-f(x)|\\ &\quad\le\ |K^*_{n,a}(f-g_n;x)-(f-g_n)(x)|+|K^*_{n,a}(g_n;x)-g_n(x)|,\\ &\quad\le\ 2\,\|\ f-g_n\ \|_{C[0,b],b<\infty}+|K^*_{n,a}(g_n;x)-g_n(x)|.\end{aligned}$$

From (11), we get

$$|K^*_{n,a}(f;x)-f(x)|\ \le2A\omega^2_{\varphi^\lambda}\big(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\big)+|K^*_{n,a}(g_n;x)-g_n(x)|. \tag{13}$$

Now, the last term can be calculated by using Taylor's formula

$$
\begin{aligned}
&|K^*_{n,a}(g_n(t)-g_n(x);x)| \\
&\leq |g'_n(x)K^*_{n,a}((t-x);x)| + \left|K^*_{n,a}\left(\int_t^x (x-u)g''_n(u)du;x\right)\right| \\
&\leq K^*_{n,a}\left(\frac{|x-\frac{k}{n}|}{\varphi^{2\lambda}(x)}\int_{\frac{k}{n}}^{x}\varphi^{2\lambda}(u)|g''_n(u)du|;x\right) \\
&\leq \parallel \varphi^{2\lambda}g''_n \parallel_{C[0,b],b<\infty} \frac{1}{\varphi^{2\lambda}(x)}K^*_{n,a}((t-x)^2;x) \\
&\leq \parallel \varphi^{2\lambda}g''_n \parallel_{C[0,b],b<\infty} \frac{1}{\varphi^{2\lambda}(x)}\frac{x}{n}\frac{nK^*_{n,a}((t-x)^2;x)}{x} \\
&\leq \parallel \varphi^{2\lambda}g''_n \parallel_{C[0,b],b<\infty} \frac{xn^{-1}}{\varphi^{2\lambda}(x)}\frac{nK^*_{n,a}((t-x)^2;x)}{x}.
\end{aligned}
$$

For sufficiently large value of n, we get

$$\frac{nK^*_{n,a}((t-x)^2;x)}{x} \leq \left(1+\frac{1}{a-1}\right).$$

Therefore

$$(14)\quad |K^*_{n,a}(g_n(t)-g_n(x);x)| \leq \left(1+\frac{1}{a-1}\right)B\omega^2_{\varphi^\lambda}\left(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right).$$

Using (13) and (14), we get

$$|K^*_{n,a}(f(t)-f(x);x)| \leq M\omega^2_\lambda\left(f,n^{\frac{-1}{2}}\varphi(x)^{1-\lambda}\right),$$

where $M = max\left(2A,\left(1+\frac{1}{a-1}\right)B\right)$. □

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{0 \leq x < \infty} |f(x)|.$$

Then, for any $\delta > 0$, Peeter's K-functional is defined as

$$K_2(f, \delta) = inf\{\|f - g\| + \delta\|g''\| : g \in C_B^2[0, \infty)\},$$

where $C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By Devore and Lorentz [[3], p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f; \delta) \leq C\omega_2(f; \sqrt{\delta}),$$

where $\omega_2(f; \delta)$ is the second order modulus of continuity is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Theorem 4.2. *Let* $f \in C_B^2\left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$, $n \in \mathbb{N}$. *Then for all* $x \in \left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$, $n \in \mathbb{N}$ *there exist a constant* $C > 0$ *such that*

$$| K_{n,a}^*(f; x) - f(x) | \leq C\omega_2(f; \sqrt{\Pi_{n,a}(x)}) + \omega(f; K_{n,a}^*(\psi_x; x))$$

where $\Pi_{n,a}(x) = K_{n,a}^*(\psi_x^2; x) + \left(K_{n,a}^*(\psi_x; x)\right)^2$.

Proof. First, we define the auxiliary operators as follows

$$\widehat{K}_{n,a}^*(f; x) = K_{n,a}^*(f; x) + f(x) - f\big(\Lambda_{n,a}(x)\big), \tag{15}$$

where $\Lambda_{n,a}(x) = K_{n,a}^*(\psi_x; x) + x$. We find that

$$\widehat{K}_{n,a}^*(1; x) = 1,$$

$$\widehat{K}_{n,a}^*(\psi_x(t); x) = 0,$$

$$|\widehat{K}^*_{n,a}(f;x)| \leq 3\|f\|. \tag{16}$$

Let $g \in C_B^2\left[\frac{\sqrt{\frac{10}{3}}}{n}, \infty\right)$, using Taylor's theorem, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Now

$$\begin{aligned}
\widehat{K}^*_{n,a}(g;x) - g(x) &= g'(x)\widehat{K}^*_{n,a}(t-x;x) + \widehat{K}^*_{n,a}\Big(\int_x^t (t-v)g''(v)dv;x\Big) \\
&= \widehat{K}^*_{n,a}\Big(\int_x^t (t-v)g''(v)dv;x\Big) \\
&= K^*_{n,a}\Big(\int_x^t (t-v)g''(v)dv;x\Big) - \int_x^{\Lambda_{n,a}} (\Lambda_{n,a}-v)g''(v)dv.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\widehat{K}^*_{n,a}(g;x) - g(x)| \\
&\leq \left|K^*_{n,a}\Big(\int_x^t (t-v)g''(v)dv;x\Big)\right| + \left|\int_x^{\Lambda_{n,a}} (\Lambda_{n,a}-v)g''(v)dv\right|.
\end{aligned} \tag{17}$$

Since

$$\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2 \parallel g'' \parallel, \tag{18}$$

and

$$\left| \int_x^{\Lambda_{n,a}} (\Lambda_{n,a} - v) g''(v) dv \right| \leq (\Lambda_{n,a} - x)^2 \parallel g'' \parallel . \tag{19}$$

Then from (17), (18) and (19), we have

$$\begin{aligned} |\widehat{K}^*_{n,a}(g;x) - g(x)| &\leq \left\{ K^*_{n,a}((t-x)^2;x) + (\Lambda_{n,a} - x)^2 \right\} \|g''\| \\ &= \Pi_{n,a}(x)\|g''\|. \end{aligned} \tag{20}$$

Next, we have

$$\begin{aligned} |K^*_{n,a}(f;x) - f(x)| &\leq |\widehat{K}^*_{n,a}(f-g;x)| + |(f-g)(x)| + |\widehat{K}^*_{n,a}(g;x) - g(x)| \\ &+ \left| f(\Lambda_{n,a}) - f(x) \right|, \end{aligned}$$

using (20), we have

$$\begin{aligned} |K^*_{n,a}(f;x) - f(x)| &\leq 4\|f-g\| + |\widehat{K}^*_{n,a}(g;x) - g(x)| + \left| f(\Lambda_{n,a}) - f(x) \right| \\ &\leq 4\|f-g\| + \Pi_{n,a}(x)\|g''\| + \omega\Big(f; K^*_{n,a}(\psi_x;x)\Big). \end{aligned}$$

By the definition of Peetre's K-functional, we find

$$|K^*_{n,a}(f;x) - f(x)| \leq C\omega_2\big(f; \sqrt{\Pi_{n,a}(x)}\big) + \omega(f; K^*_{n,a}(\psi_x;x)).$$

This completes the proof of Theorem 4.2. □

Now, we discuss a local result in Lipschitz class

$$Lip^*_M(\alpha) = \{f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\alpha}{(t+x)^{\frac{\alpha}{2}}} : x, t \in (0,\infty)\},$$

where M is a constant and $0 < \alpha \leq 1$.

THEOREM 4.3. *Let $f \in Lip^*_M(\alpha)$ and $x \in \left(\frac{\sqrt{\frac{10}{3}}}{n}, \infty \right)$, $n \in \mathbb{N}$. Then, we have*

$$|K^*_{n,a}(f;x)-f(x)| \le M\left[\frac{\Theta_{n,a}(x)}{x}\right]^{\frac{\alpha}{2}},$$

where $\Theta_{n,a}(x)=K^*_{n,a}((t-x)^2;x)$.

Proof. Let $\alpha=1$ and $x\in(0,\infty)$. Then, for $f\in Lip^*_M(1)$, we have

$$\begin{aligned}
&|K^*_{n,a}(f;x)-f(x)|\\
\le\ & ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\\
\times\ & \int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)-f(x)|dt\\
\le\ & Mne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\\
\times\ & \int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{|t-x|}{\sqrt{t+x}}dt.\\
\le\ & \frac{M}{\sqrt{x}}ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\\
\times\ & \int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dt\\
\le\ & \frac{M}{\sqrt{x}}K^*_{n,a}(|t-x|;x)\\
\le\ & M\frac{\sqrt{K^*_{n,a}((t-x)^2;x)}}{\sqrt{x}}\\
=\ & M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, the assertion holds for $\alpha=1$. Now, we will prove for $\alpha\in(0,1)$. From the Hölder Inequality with $p=\frac{1}{\alpha}$,
$q=\frac{1}{1-\alpha}$, we have

$$
\begin{aligned}
&|K^*_{n,a}(f;x)-f(x)| \\
= & \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\right. \\
\times & \left.\left(n\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t)-f(x)|dt\right)^{\frac{1}{\alpha}}\right)^{\alpha} \\
& \times\big(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\big)^{1-\alpha} \\
\leq & \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\right. \\
\times & \left.\left(n\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t)-f(x)|dt\right)^{\frac{1}{\alpha}}\right)^{\alpha}.
\end{aligned}
$$

Since $f \in Lip^*_M$, we obtain

$$
\begin{aligned}
&|K^*_{n,a}(f;x)-f(x)| \\
\leq & M\left(ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\right. \\
\times & \left.\int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}}dt\right)^{\alpha} \\
\leq & \frac{M}{x^{\frac{\alpha}{2}}}\left(ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr^*_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr^*_{n,a}(x))}{k!}\right. \\
\times & \left.\int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|dt\right)^{\alpha} \\
= & \frac{M}{x^{\frac{\alpha}{2}}}\big(K^*_{n,a}(|t-x|;x)\big)^{\alpha} \\
\leq & M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{\alpha}{2}}.
\end{aligned}
$$

This completes the proof of Theorem 4.3. □

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