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# ESTIMATION OF NON-INTEGRAL AND INTEGRAL QUADRATIC FUNCTIONS IN LINEAR STOCHASTIC DIFFERENTIAL SYSTEMS

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Abstract. This paper focuses on estimation of an non-integral quadratic function (NIQF) and integral quadratic function (IQF) of a random signal in dynamic system described by a linear stochastic differential equation. The quadratic form of an unobservable signal indicates useful information of a signal for control. The optimal (in mean square sense) and suboptimal estimates of NIQF and IQF represent a function of the Kalman estimate and its error covariance. The proposed estimation algorithms have a closed-form estimation procedure. The obtained estimates are studied in detail, including derivation of the exact formulas and differential equations for mean square errors. The results we demonstrate on practical example of a power of signal , and comparison analysis between optimal and suboptimal estimators is presented.

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## 1. Introduction

The Kalman filtering and its variations are well-known signal estimation techniques in wide use in a variety of applications such as navigation, target tracking, vehicle state estimation, communications engineering, air traffic control, biomedical and chemical processing and many other areas [1-8].

However, some applications require the estimation of not only a signal but also an nonlinear functions of the signal, which express practical and worthwhile information for control systems. For instance, in a mechanical application, such functions include displacement, energy or work which can be interpreted as a quadratic form of a random signal. Aside from the aforementioned papers, most authors have not focused on estimation of nonlinear functions of a signal but have considered signal estimation or filtering only. To the best of our knowledge, there are no methods for estimation of an nonlinear functions in a linear stochastic differential systems in the literature.

Therefore, the aim of this paper is to develop estimators for an arbitrary non-integral quadratic function (NIQF) and integral quadratic function (IQF) in linear systems described by stochastic differential equations. We propose an optimal (in the mean square error sense) and suboptimal estimates for NIQF and IQF, and demonstrate their theoretical and practical effectiveness.

This paper is organized as follows. Section 2 presents a statement of the estimation problem for NIQF and IQF within the continuous-time Kalman filtering framework. In Section 3, the optimal estimates for NIQF and IQF are derived. The simple suboptimal estimates for the functions are also considered. In Section 4, we study an unbiased property of the obtained estimates. In Section 5, we derive the exact formulas and differential equations for the mean square errors. In Section 6, the numerical efficiency of the proposed estimators is studied. Finally, we conclude the paper in Section 7.

## 2. Problem Statement

Let consider a linear dynamic system described by the Ito stochastic differential equation

$$dx_t = F_t x_t dt + G_t dv_t, \quad t \geq 0 \tag{1}$$

where $x_t \in \Re^n$ is an unobservable random process (signal), and $v_t \in \Re^r$ is a Wiener process with the intensity $Q_t$ , i.e., $\mathbf{E}\left(dv_t dv_t^T\right) = Q_t dt$, and $F_t \in \Re^{n\times n}$, $G_t \in \Re^{n\times r}$, and $Q_t \in \Re^{r\times r}$.

Suppose that an observable process $y_t \in \Re^m$ is determined by the Ito stochastic differential equation

$$dy_t \ = \ H_t x_t dt \ + \ dw_t, \tag{2}$$

where $w_t \in \Re^m$ represents a Wiener process (observation error) with intensity $R_t$ i.e., $\mathbf{E}\left(dw_t dw_t^T\right) = R_t dt$, and $H_t \in \Re^{m\times n}$.

We assume that the initial condition $x_0 \sim \mathcal{N}\left(\bar{x}_0, P_0\right)$, and Wiener processes are independent.

Let consider the non-integral quadratic function (NIQF) and integral quadratic function (IQF) of the unobservable random process,

$$\text{NIQF}: \ z_t \ = \ x_t^T \Omega_t x_t \ + \ d_t^T x_t, \tag{3}$$

and

$$\text{IQF}: \ u_t \ = \ \int_0^t \left(x_s^T \Omega_s x_s \ + \ d_s^T x_s\right) ds, \tag{4}$$

respectively.

Here $\Omega_t = \Omega_t^T \geq 0$, and $d_t$ are an arbitrary matrix and vector, respectively, and $A^T$ denotes the transposition of a matrix $A$.

A problem associated with the partially observable process $(x_t, y_t)$ is that of estimation of an NIQF and IQF from the overall noisy observations $y_0^t = \{y_s : \ 0 \leq s \leq t\}$.

Simple examples of such a quadratic functions may be the Euclidean square distance (norm) $z_t = \|x_t - \tilde{x}_t\|^2$ between two vector processes $x_t$ and $\tilde{x}_t$, or the integral $u_t \ = \ \int_0^t x_s^T x_s ds$ representing an accumulated energy-like function of an object.

We propose an optimal and suboptimal estimation algorithms for an NIQF and IQF, and investigate their statistical properties in the subsequent Sections 3 and 4.

## 3. Optimal and Suboptimal Estimates for NIQF and IQF

In this section, the best optimal (in the mean square sense) estimation algorithms for an NIQF and IQF are derived. A simple suboptimal estimates for the functions are also proposed.

The optimal estimation algorithms include two stages: the optimal Kalman estimate of the unobservable random process $\hat{x}_t$ computed at the first stage is used at the second stage for the best estimation of an NIQF or IQF.

### 3.1. First stage – Kalman estimate for the unobservable random process

The optimal mean square estimate $\hat{x}_t = \mathbf{E}\left(x_t|y_0^t\right)$ of the process $x_t$ based on the overall observations $y_0^t$, and its error covariance $P_t = \mathbf{E}\left(e_t e_t^T\right)$, $e_t = x_t - \hat{x}_t$ are given by the continuous Kalman filter (KF) equations [5,8]:

$$\begin{aligned} d\hat{x}_t &= F_t\hat{x}_t dt + K_t\left(dy_t - H_t\hat{x}_t dt\right), \quad t \geq 0, \quad \hat{x}_{t=0} = \bar{x}_0, \\ dP_t &= \left(F_tP_t + P_tF_t^T - P_tH_t^TR_t^{-1}H_tP_t + \tilde{G}_t\right)dt, \quad P_{t=0} = P_0, \\ K_t &= P_tH_t^TR_t^{-1}, \ \tilde{G}_t = G_tQ_tG_t^T. \end{aligned} \tag{5}$$

### 3.2. Second stage for NIQF – Formula for the optimal estimate

The optimal mean square estimate of the NIQF (3) based on the overall observations $y_0^t$ also represents a conditional mean,

$$\hat{z}_t^{opt} = \mathbf{E}\left(z_t|y_0^t\right). \tag{6}$$

The conditional mean (6) can be explicitly calculated in terms of the Kalman estimate $\hat{x}_t$ and its error covariance $P_t$. We have

Theorem 3.1. *The optimal mean square estimate $\hat{z}_t^{opt}$ is given by*

$$\hat{z}_t^{opt} = tr\left[\Omega_t\left(P_t + \hat{x}_t\hat{x}_t^T\right)\right] + d_t^T\hat{x}_t, \tag{7}$$

*where $tr(A)$ is the trace of a matrix $A$, and the Kalman estimate $\hat{x}_t$ and error covariance $P_t$ satisfy (5).*

*Proof.* Using the formula for a second-order vector moment $\mathbf{E}\left(x^Tx\right) = \mu^T\mu + \mathrm{tr}(C)$, where $\mu = \mathbf{E}(x)$, $C = \mathrm{Cov}(x,x) = \mathbf{E}\left[(x-\mu)(x-\mu)^T\right]$, it is easy to derive that

$$\mathbf{E}\left(x^T\Omega x\right) = \mathrm{tr}\left[\Omega\left(C + \mu\mu^T\right)\right]. \tag{8}$$

Using the fact (8) we obtain the optimal mean square estimate (6) for the NIQF,

$$\begin{aligned} \hat{z}_t^{opt} &= \mathbf{E}\left(x_t^T\Omega_t x_t + d_t^T x_t|y_0^t\right) = \mathbf{E}\left(x_t^T\Omega_t x_t|y_0^t\right) + d_t^T\mathbf{E}\left(x_t|y_0^t\right) \\ &= \mathrm{tr}\left\{\Omega_t\left[P_t + \mathbf{E}\left(x_t|y_0^t\right)\mathbf{E}\left(x_t^T|y_0^t\right)\right]\right\} + d_t^T\hat{x}_t^T \\ &= \mathrm{tr}\left[\Omega_t\left(P_t + \hat{x}_t\hat{x}_t^T\right)\right] + d_t^T\hat{x}_t. \end{aligned} \tag{9}$$

This completes the derivation of (7). □

### 3.3. Second stage for IQF – Differential equation for the optimal estimate

The integral function (4) is described by the differential equation

$$\dot{u}_t = x_t^T \Omega_t x_t + d_t^T x_t, \quad t \geq 0, \quad u_0 = 0. \tag{10}$$

THEOREM 3.2. *The optimal mean square estimate $\hat{u}_t^{opt}$ is given by*

$$\dot{\hat{u}}_t^{opt} = \hat{x}_t^T \Omega_t \hat{x}_t + tr\left(\Omega_t P_t\right) + d_t^T \hat{x}_t, \quad \hat{u}_0 = 0, \tag{11}$$

*where the Kalman estimate $\hat{x}_t$ and its error covariance $P_t$ satisfy (5).*

*Proof.* Taking the conditional expectation of both parts of the equation (10) and using the formula (8) we obtain,

$$\begin{aligned}
\dot{\hat{u}}_t^{opt} &= \mathbf{E}\left(\dot{u}_t | y_0^t\right) = \mathbf{E}\left(x_t^T \Omega_t x_t + d_t^T x_t | y_0^t\right) \\
&= \mathbf{E}\left(x_t^T \Omega_t x_t | y_0^t\right) + d_t^T \mathbf{E}\left(x_t | y_0^t\right) \\
&= \mathrm{tr}\left\{\Omega_t \left[P_t + \mathbf{E}\left(x_t | y_0^t\right) \mathbf{E}\left(x_t^T | y_0^t\right)\right]\right\} + d_t^T \hat{x}_t^T \\
&= \mathrm{tr}\left[\Omega_t \left(P_t + \hat{x}_t \hat{x}_t^T\right)\right] + d_t^T \hat{x}_t \\
&= \mathrm{tr}\left(\Omega_t P_t\right) + \mathrm{tr}\left(\Omega_t \hat{x}_t \hat{x}_t^T\right) + d_t^T \hat{x}_t \\
&= \hat{x}_t^T \Omega_t \hat{x}_t + \mathrm{tr}\left(\Omega_t P_t\right) + d_t^T \hat{x}_t.
\end{aligned}$$

This completes the derivation of (11). □

In parallel to the optimal estimates (7) and (11) we propose a simple suboptimal estimates for the NIQF and IQF,

$$\hat{z}_t^{sub} = \hat{x}_t^T \Omega_t \hat{x}_t + d_t^T \hat{x}_t, \quad \dot{\hat{u}}_t^{sub} = \hat{x}_t^T \Omega_t \hat{x}_t + d_t^T \hat{x}_t, \tag{12}$$

respectively.

## 4. Unbiased and Biased Estimates

Here we study the unbiased property of the optimal and suboptimal estimates for the NIQF and IQF.

THEOREM 4.1. *The optimal mean square estimate $\hat{z}_t^{opt}$ is unbiased.*

*Proof.* Using the unbiased and orthogonality properties of the Kalman estimate [5,8],

$$\begin{gathered}
\mathbf{E}\left(\hat{x}_t\right) = \mathbf{E}\left(x_t\right), \ \mathbf{E}\left[\left(x_t - \hat{x}_t\right) \hat{x}_t^T\right] = 0, \\
P_t = \mathbf{E}\left(x_t x_t^T\right) - \mathbf{E}\left(\hat{x}_t \hat{x}_t^T\right),
\end{gathered} \tag{13}$$

and the formula $x_t^T \Omega_t x_t = \mathrm{tr}\left(\Omega_t x_t x_t^T\right)$, we obtain

$$\begin{aligned}
&\mathbf{E}\left(\hat{z}_t^{opt}\right) = \mathrm{tr}\left(\Omega_t P_t\right) + \mathrm{tr}\left[\Omega_t \mathbf{E}\left(\hat{x}_t \hat{x}_t^T\right)\right] + d_t^T \mathbf{E}\left(\hat{x}_t\right) = \mathrm{tr}\left(\Omega_t P_t\right) \\
&+\mathrm{tr}\left\{\Omega_t\left[\mathbf{E}\left(x_t x_t^T\right) - P_t\right]\right\} + d_t^T \mathbf{E}\left(x_t\right) = \mathrm{tr}\left[\Omega_t \mathbf{E}\left(x_t x_t^T\right)\right] + d_t^T \mathbf{E}\left(x_t\right), \\
&\text{and} \\
&\mathbf{E}\left(z_t\right) = \mathbf{E}\left(x_t^T \Omega_t x_t\right) + d_t^T \mathbf{E}\left(x_t\right) = \mathrm{tr}\left[\Omega_t \mathbf{E}\left(x_t x_t^T\right)\right] + d_t^T \mathbf{E}\left(x_t\right).
\end{aligned}$$

So, $\mathbf{E}\left(\hat{z}_t^{opt}\right) = \mathbf{E}\left(z_t\right)$ This completes the proof. □

Theorem 4.2. *The optimal mean square estimate $\hat{u}_t^{opt}$ is unbiased.*

*Proof.* Note that $u_t = \int_0^t z_s ds$ and $\hat{u}_t^{opt} = \int_0^t \hat{z}_s^{opt} ds$. Then using the unbiased property of the estimate $\hat{z}_t^{opt}$ we obtain,

$$\mathbf{E}\left(\hat{u}_t^{opt}\right) = \int_0^t \mathbf{E}\left(\hat{z}_s^{opt}\right) ds = \int_0^t \mathbf{E}\left(z_s\right) ds = \mathbf{E}\left(\int_0^t z_s\right) ds = \mathbf{E}\left(u_t\right).$$

□

Corollary 4.1. *The suboptimal estimates $\hat{z}_t^{sub}$ and $\hat{u}_t^{sub}$ are biased.*

## 5. Calculation of Mean Square Errors

Here we study the estimation accuracy of the optimal and suboptimal estimates of the NIQF and IQF.

The following result completely define the actual mean square errors (MSEs)

$$P_{z,t}^{opt} = \mathbf{E}\left(\varepsilon_t^2\right), \quad P_{z,t}^{sub} = \mathbf{E}\left(\tilde{\varepsilon}_t^2\right), \quad \varepsilon_t = z_t - \hat{z}_t^{opt}, \quad \tilde{\varepsilon}_t = z_t - \hat{z}_t^{sub} \tag{14}$$

for the non-integral optimal and suboptimal estimates $\hat{z}_t^{opt}$ and $\hat{z}_t^{sub}$, respectively.

Theorem 5.1. *The actual mean square errors $P_{z,t}^{opt}$ and $P_{z,t}^{sub}$ for the NIQF are given by*

$$\begin{aligned}
P_{z,t}^{opt} &= 4tr\left(\Omega_t P_t \Omega_t C_t\right) - 2tr\left(\Omega_t P_t \Omega_t P_t\right) + 4\mu_t \Omega_t P_t \Omega_t \mu_t \\
&\quad + d_t^T P_t d_t + 4\mu_t^T \Omega_t P_t d_t,
\end{aligned} \tag{15}$$

*and*

$$\begin{aligned}
P_{z,t}^{sub} &= 4tr\left(\Omega_t P_t \Omega_t C_t\right) - 2tr\left(\Omega_t P_t \Omega_t P_t\right) + tr^2\left(\Omega_t P_t\right) \\
&\quad + 4\mu_t \Omega_t P_t \Omega_t \mu_t + d_t^T P_t d_t + 4\mu_t^T \Omega_t P_t d_t,
\end{aligned} \tag{16}$$

*respectively. Here the unconditional mean* $\mu_t = \mathbf{E}(x_t)$ *and covariance* $C_t = Cov(x_t, x_t)$ *of the unobservable process* $x_t$ *are determined by the Lyapunov equations [5-7],*

$$\dot{\mu}_t = F_t \mu_t, \quad \mu_0 = \bar{x}_0, \quad \dot{C}_t = F_t C_t + C_t F_t^T + G_t Q_t G_t^T, \quad C_0 = P_0. \tag{17}$$

The derivation of the MSEs (15) and (16) is based on the following Lemma.

LEMMA 5.1. *Let* $X \in \Re^{3n}$ *be a composite multivariate normal vector,*

$$X \sim \mathcal{N}(\mu_x, S_x), \quad X^T = \begin{bmatrix} U^T & V^T & W^T \end{bmatrix}, \quad U, V, W \in \Re^n,$$
$$\mu_x = \mathbf{E}(X) = \begin{bmatrix} \mu_u \\ \mu_v \\ \mu_w \end{bmatrix}, \quad S_x = Cov(X, X) = \begin{bmatrix} S_{uu} & S_{uv} & S_{uw} \\ S_{vu} & S_{vv} & S_{vw} \\ S_{wu} & S_{wv} & S_{ww} \end{bmatrix}.$$

*Then the third- and fourth-order vector moments of the composite random vector* $X$ *are given by*

$$\begin{aligned}
&\text{(i)}\ \ \mathbf{E}\left(U^T V W^T\right) = \mu_u^T \mu_v \mu_w^T + tr(S_{uv})\, \mu_w^T + \mu_v^T S_{uw} + \mu_u^T S_{vw},\\
&\text{(ii)}\ \ \mathbf{E}\left(U^T U V^T V\right) = \mu_u^T \mu_u \mu_v^T \mu_v + 2tr(S_{uv} S_{vu})\\
&\qquad + tr(S_{uu})\, tr(S_{vv}) + tr(S_{uu})\, \mu_v^T \mu_v\\
&\qquad + tr(S_{vv})\, \mu_u^T \mu_u + 4\mu_u^T S_{uv} \mu_v,\\
&\text{(iii)}\ \ \mathbf{E}\left(U^T V V^T U\right) = \mu_u^T \mu_v \mu_v^T \mu_u + tr(S_{uu} S_{vv})\\
&\qquad + tr(S_{uv})\, tr(S_{vu}) + tr(S_{uv}^2) + \mu_v^T S_{uu} \mu_v\\
&\qquad + \mu_u^T S_{vv} \mu_u + 2tr(S_{uv})\, \mu_u^T \mu_v + \mu_v^T S_{uv} \mu_u\\
&\qquad + \mu_u^T S_{vu} \mu_v,\\
&\text{(iv)}\ \ \mathbf{E}\left(U^T V W^T U\right) = \mu_u^T \mu_v \mu_w^T \mu_u + tr(S_{uv})\, tr(S_{uw})\\
&\qquad + tr(S_{uu} S_{wv}) + tr(S_{uw} S_{uv}) + tr(S_{uv})\, \mu_u^T \mu_w\\
&\qquad + tr(C_{uw})\, \mu_u^T \mu_v + \mu_v^T S_{uu} \mu_w + \mu_v^T S_{uw} \mu_u\\
&\qquad + \mu_u^T S_{vu} \mu_w + \mu_u^T S_{vw} \mu_u.
\end{aligned} \tag{18}$$

The derivation of the vector formulas (18) for calculating the high-order moments is based on their scalar versions [9, 10],

$$\begin{aligned}
&\mathbf{E}(x_i x_j x_k) = \mu_i \mu_j \mu_k + \mu_i S_{jk} + \mu_j S_{ik} + \mu_k S_{ij},\\
&\mathbf{E}(x_i x_j x_k x_l) = \mu_i \mu_j \mu_k \mu_l + S_{ij} S_{kl} + S_{ik} S_{lj} + S_{il} S_{jk}\\
&\qquad + \mu_i \mu_j S_{kl} + \mu_i \mu_k S_{jl} + \mu_i \mu_l S_{jk}\\
&\qquad + \mu_j \mu_k S_{il} + \mu_j \mu_l S_{ik} + \mu_k \mu_l S_{ij},\\
&\text{where}\\
&\mu_h = \mathbf{E}(x_h), \quad S_{pq} = \mathbf{E}\left[(x_p - \mu_q)(x_q - \mu_q)\right],
\end{aligned} \tag{19}$$

and standard matrix manipulations.

*Proof of Theorem 5.1.* We are now is a position to derive the first MSE (15). For simplicity we omit time index, i.e., $x_t \to x$, $\hat{x}_t \to \hat{x}$, $P_t \to P$, .... Then using (3) and (7), the estimation error can be written as

$$\begin{aligned}
&\varepsilon = z - \hat{z}^{opt} = x^T\Omega x + d^T x - \mathrm{tr}\left[\Omega\left(P + \hat{x}\hat{x}^T\right)\right] - d^T\hat{x} \\
&\quad = x^T\Omega x - \hat{x}^T\Omega\hat{x} - \mathrm{tr}\,(\Omega P) + d^T e = (e+\hat{x})^T\,\Omega\,(e+\hat{x}) - \hat{x}^T\Omega\hat{x} \\
&\quad -\mathrm{tr}\,(\Omega P) + d^T e = e^T\Omega e + 2e^T\Omega\hat{x} + d^T e - \mathrm{tr}\,(\Omega P),
\end{aligned}$$

where

$$e = x - \hat{x}, \quad \mathrm{tr} = \left(\Omega\hat{x}\hat{x}^T\right) = \hat{x}^T\Omega\hat{x}, \quad \hat{x}^T\Omega e = e^T\Omega\hat{x}.$$

Next, using the unbiased and orthogonality properties of the Kalman estimate (13) we obtain the optimal MSE

$$\begin{aligned}
P_z^{opt} &= \mathbf{E}\left(\varepsilon^2\right) = \mathbf{E}\left(e^T\Omega ee^T\Omega e\right) + 4\mathbf{E}\left(e^T\Omega\hat{x}\hat{x}^T\Omega e\right) + d^T P d \\
&+ \mathrm{tr}^2\,(\Omega P) + 4\mathbf{E}\left(e^T\Omega ee^T\Omega\hat{x}\right) + 2\mathbf{E}\left(e^T\Omega ee^T\right)d \\
&- 2\mathrm{tr}^2\,(\Omega P) + 4\mathbf{E}\left(e^T\Omega\hat{x}e^T\right)d.
\end{aligned} \tag{20}$$

Using Lemma 5.1 we can calculate high-order moments in (20). We have

$$\begin{aligned}
&\text{(a) } \mathbf{E}\left(e^T\Omega ee^T\Omega e\right) = 2\mathrm{tr}\,(\Omega P\Omega P) + \mathrm{tr}^2\,(\Omega P), \\
&\quad U = e,\ V = \Omega e. \\
&\text{(b) } \mathbf{E}\left(e^T\Omega\hat{x}\hat{x}^T\Omega e\right) = \mathrm{tr}\,(P\Omega P_{\hat{x}\hat{x}}\Omega) + \mu^T\Omega P\Omega\mu \\
&\quad = \mathrm{tr}\,(P\Omega CP) - \mathrm{tr}\,(\Omega P\Omega P) + \mu^T\Omega P\Omega\mu, \\
&\quad U = e,\ V = \Omega\hat{x}. \\
&\text{(c) } \mathbf{E}\left(e^T\Omega ee^T\Omega\hat{x}\right) = \mathbf{E}\left(e^T\Omega e\hat{x}^T\Omega e\right) = 0, \\
&\quad U = e,\ V = \Omega e,\ W = \Omega\hat{x}. \\
&\text{(d) } \mathbf{E}\left(e^T\Omega ee^T\right) = 0,\ U = e,\ V = \Omega e,\ W = e. \\
&\text{(e) } \mathbf{E}\left(e^T\Omega\hat{x}e^T\right) = \mu^T\Omega P,\ U = e,\ V = \Omega\hat{x},\ W = e.
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\mu = \mathbf{E}(x) = \mathbf{E}\,(\hat{x}),\ C = \mathrm{Cov}(x,x),\ P = \mathrm{Cov}(e,e),\ \mathbf{E}(e) = 0, \\
&\mathbf{E}\,(\Omega\hat{x}) = \Omega\mu, P_{\hat{x}\hat{x}} = \mathrm{Cov}\,(\hat{x},\hat{x}) = C - P,\ \mathrm{Cov} = (e,\Omega e) = P\Omega, \\
&\mathrm{Cov} = (\Omega e,\Omega e) = \Omega P\Omega.
\end{aligned}$$

Substituting (21) to (20), and after some manipulations, we get the optimal MSE (15).

The unknown mean $\mu = \mathbf{E}\,(x_t)$ and covariance $\mathrm{Cov}\,(x_t, x_t)$ of random process (1) satisfy the Lyapunov equations (17).
This completes the derivation (15).

In the case of the suboptimal estimate $\hat{z}_t^{sub}$, the derivation of the MSE (16) is similar.

Thus, (15) and (16) completely define the true MSEs of the optimal and suboptimal estimates $\hat{z}_t^{opt}$ and $\hat{z}_t^{sub}$ for the NIQF, respectively.

Corollary 5.1. *Comparison of the MSEs* $P_{z,t}^{opt}$ *and* $P_{z,t}^{sub}$ *shows that the difference between them is equal*

$$P_{z,t}^{sub} - P_{z,t}^{opt} = tr^2\left(\Omega_t, P_t\right),$$

*where* $P_t$ *is error covariance determined by the KF equations (5).*

Corollary 5.2. *In particular case with* $\Omega_t = 1$ *and* $d_t = 0$*, the NIQF, optimal and suboptimal estimates, and MSEs take the form*

$$\begin{aligned}
&z_t = \|x_t\|^2 = x_t^T x_t,\ \hat{z}_t^{opt} = \|\hat{x}_t\|^2 + tr\left(P_t\right),\ \hat{z}_t^{sub} = \|\hat{x}_t\|^2,\\
&P_{z,t}^{opt} = 4tr\left(P_t C_t\right) - 2tr\left(P_t^2\right) + 4\mu_t^T P_t \mu_t,\\
&P_{z,t}^{sub} = 4tr\left(P_t C_t\right) - 2tr\left(P_t^2\right) + 4\mu_t^T P_t \mu_t + tr^2\left(P_t\right).
\end{aligned}$$

Next we derive the actual MSEs for the integral function (4),

$$P_{u,t}^{opt} = \mathbf{E}\left(\delta_t^2\right),\ P_{u,t}^{sub} = \mathbf{E}\left(\tilde{\delta}_t^2\right),\ \delta_t = u - \hat{u}_t^{opt},\ \tilde{\delta}_t = u - \hat{u}_t^{sub}. \tag{22}$$

Theorem 5.2. *The actual mean square error* $P_{u,t}^{opt}$ *for the IQF is described by the differential equation*

$$\dot{P}_{u,t}^{opt} = 2\mathbf{E}\left(\delta_t e_t^T \Omega_t e_t\right) + 4\mathbf{E}\left(\delta_t e_t^T \Omega_t \hat{x}_t\right) + 2\mathbf{E}\left(\delta_t d_t^T e_t\right),\ P_{u,0}^{opt} = 0. \tag{23}$$

*Here*

$$\begin{aligned}
&\mathbf{E}\left(\delta_t d_t^T e_t\right) = \sum_{i=1}^{n} d_{i,t} m_{i,t},\ \mathbf{E}\left(\delta_t e_t^T \Omega_t e_t\right) = \sum_{i,j=1}^{n} \Omega_{ij,t}\alpha_{ij,t},\\
&\mathbf{E}\left(\delta_t e_t^T \Omega_t \hat{x}_t\right) = \sum_{i,j=1}^{n} \Omega_{ij,t}\beta_{ij,t},\ m_{i,t} = \mathbf{E}\left(\delta_t e_{i,t}\right),\\
&\alpha_{ij,t} = \mathbf{E}\left(\delta_t e_{i,t} e_{j,t}\right), \beta_{ij,t} = \mathbf{E}\left(\delta_t e_{i,t}\hat{x}_{j,t}\right),\\
&P_t, F_t, A_t, \Omega_t, C_t \in \Re^{n\times n},\ K_t \in \Re^{n\times m},\\
&H_t \in \Re^{m\times n},\ d_t, \mu_t \in \Re^n,
\end{aligned} \tag{24}$$

*and the moments* $m_{i,t}$, $\alpha_{i,t}$, $\beta_{i,t}$ $(i,j=1,\ldots,n)$ *are determined by*

$$
\begin{aligned}
\dot{m}_{i,t} &= 2\sum_{k,h=1}^{n}\Omega_{kh,t}\mu_{h,t}P_{ik,t}+\sum_{k=1}^{n}d_{k,t}P_{ik,t}\\
&\quad+\sum_{k=1}^{n}A_{ik,t}m_{i,t},\ \ m_{i,0}=0,\\
\dot{\alpha}_{ij,t} &= \sum_{k,h=1}^{n}\Omega_{kh,t}\left(P_{ij,t}P_{kh,t}+P_{ik,t}P_{jh,t}+P_{ih,t}P_{jk,t}\right)\\
&\quad-tr\left(\Omega_t P_t\right)P_{ij,t}+\sum_{k=1}^{n}\left(A_{ik,t}\alpha_{jk,t}+A_{kj,t}\alpha_{ik,t}\right),\\
&\quad\alpha_{ij,0}=0, A_t=F_t-K_tH_t,\\
\dot{\beta}_{ij,t} &= 2\sum_{k,h=1}^{n}\Omega_{kh,t}P_{ik,t}\left(C_{jh,t}-P_{jh,t}+\mu_{j,t}\mu_{h,t}\right)\\
&\quad+\sum_{k=1}^{n}d_{k,t}\mu_{j,t}P_{ik,t}+\sum_{k=1}^{n}\left(A_{ik,t}\beta_{kj,t}+F_{jk,t}\beta_{ik,t}\right)\\
&\quad+\sum_{l=1}^{m}\sum_{h=1}^{n}K_{jl}H_{lh}\alpha_{ih,t},\ \ \beta_{ij,0}=0.
\end{aligned}
\tag{25}
$$

*Proof.* For simplicity we omit time index. Then using (1), (2), (5), and (10), (11), the Kalman estimate $\hat{x}$, and estimation errors $e=x-\hat{x}$ and $\delta=u-\hat{u}$ are determined by the equations

$$
\begin{aligned}
d\hat{x} &= (F\hat{x}+KHe)\,dt+Kdw,\\
de &= Aedt-Kdw,\quad A=F-KH,\\
d\delta &= \left[e^T\Omega e+2e^T\Omega\hat{x}+d^Te-\mathrm{tr}(\Omega P)\right]dt,
\end{aligned}
\tag{26}
$$

respectively. Using the Ito formula of a function $\delta_t^2$ by virtue on the third equation of (26) we get the equation (23) for the MSE $P_{u,t}^{opt}=\mathbf{E}\left(\delta_t^2\right)$. The expectations (24) represent linear functions of the elements (moments) $m_{i,t}$, $\alpha_{ij,t}$, $\beta_{ij,t}$. Using the Ito formula and equation (26) we obtain the differential equations for the moments

$$
\begin{aligned}
\dot{m}_i &= \tfrac{d}{dt}\mathbf{E}\left(\delta e_i\right) = \sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_k e_h\right) + \sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_k \hat{x}_h\right) \\
&\quad + \sum_{k=1}^{n} d_k \mathbf{E}\left(e_i e_k\right) - \mathbf{E}\left(e_i\right)\operatorname{tr}\left(\Omega P\right) + \sum_{k=1}^{n} A_{ik}\mathbf{E}\left(\delta e_k\right), \\
\dot{\alpha}_{ij} &= \tfrac{d}{dt}\mathbf{E}\left(\delta e_i e_j\right) = \sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_j e_k e_h\right) \\
&\quad + 2\sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_j e_k \hat{x}_h\right) \\
&\quad + \sum_{k=1}^{n} d_k \mathbf{E}\left(e_i e_j e_k\right) - \operatorname{tr}\left(\Omega P\right)\mathbf{E}\left(e_i e_j\right) \\
&\quad + \sum_{k=1}^{n} \left[A_{ik}\mathbf{E}\left(\delta e_j e_k\right) + A_{kj}\mathbf{E}\left(\delta e_i e_k\right)\right], \\
\dot{\beta}_{ij} &= \tfrac{d}{dt}\mathbf{E}\left(\delta e_i \hat{x}_j\right) = \sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_k e_h \hat{x}_j\right) \\
&\quad + 2\sum_{k,h=1}^{n} \Omega_{kh}\mathbf{E}\left(e_i e_k \hat{x}_j \hat{x}_h\right) \\
&\quad + \sum_{k=1}^{n} d_k \mathbf{E}\left(e_i e_k \hat{x}_j\right) - \operatorname{tr}\left(\Omega P\right)\mathbf{E}\left(e_i \hat{x}_j\right) \\
&\quad + \sum_{k=1}^{n} \left(A_{ik}\beta_{kj} + F_{jk}\beta_{ik}\right) + \sum_{l=1}^{m}\sum_{h=1}^{n} K_{jl} H_{lh} \alpha_{ih}.
\end{aligned}
\tag{27}
$$

Then the third- and fourth-order expectations in the right-hand sides of (27) are calculated by using the formulas (19) and orthogonality properties (13). After some manipulations, we get the equations (25). □

This completes the derivation (23)-(25).

In the case of the suboptimal estimate $\hat{u}_t^{sub}$, the derivation of the MSE $P_{u,t}^{sub} = \mathbf{E}\left(\tilde{\delta}_t^2\right)$ is similar. We have

THEOREM 5.3. *The actual mean square error $P_{u,t}^{sub}$ for the IQF is described by the differential equation*

$$
\begin{aligned}
\dot{P}_{u,t}^{sub} &= 2\mathbf{E}\left(\tilde{\delta}_t e_t^T \Omega_t e_t\right) + 4\mathbf{E}\left(\tilde{\delta}_t e_t^T \Omega_t \hat{x}_t\right) + 2\mathbf{E}\left(\tilde{\delta}_t d_t^T e_t\right), \\
P_{u,0}^{sub} &= 0.
\end{aligned}
\tag{28}
$$

*Here*

$$
\begin{aligned}
&\mathbf{E}\left(\tilde{\delta}_t d_t^T e_t\right) = \sum_{i=1}^{n} d_{i,t}\tilde{m}_{i,t},\ \mathbf{E}\left(\tilde{\delta}_t e_t^T \Omega_t e_t\right) = \sum_{i,j=1}^{n} \Omega_{ij,t}\tilde{\alpha}_{ij,t},\\
&\mathbf{E}\left(\tilde{\delta}_t e_t^T \Omega_t \hat{x}_t\right) = \sum_{i,j=1}^{n} \Omega_{ij,t}\tilde{\beta}_{ij,t},\\
&\tilde{m}_{i,t} = \mathbf{E}\left(\tilde{\delta}_t e_{i,t}\right), \tilde{\alpha}_{ij,t} = \mathbf{E}\left(\tilde{\delta}_t e_{i,t} e_{j,t}\right),\ \tilde{\beta}_{ij,t} = \mathbf{E}\left(\tilde{\delta}_t e_{i,t}\hat{x}_{j,t}\right),
\end{aligned} \tag{29}
$$

*and the moments* $\tilde{m}_{i,t}$*,* $\tilde{\alpha}_{ij,t}$*,* $\tilde{\beta}_{ij,t}$ $(i,j=1,\ldots,n)$ *are determined by*

$$
\begin{aligned}
\dot{\tilde{m}}_{i,t} &= 2\sum_{k,h=1}^{n} \Omega_{kh,t}\mu_{h,t}P_{ik,t} + \sum_{k=1}^{n} d_{k,t}P_{ik,t} + \sum_{k=1}^{n} A_{ik,t}\tilde{m}_{i,t},\\
\tilde{m}_{i,0} &= 0,\\
\dot{\tilde{\alpha}}_{ij,t} &= \sum_{k,h=1}^{n} \Omega_{kh,t}\left(P_{ij,t}P_{kh,t} + P_{ik,t}P_{jh,t} + P_{ih,t}P_{jk,t}\right)\\
&\quad + \sum_{k=1}^{n}\left(A_{ij,t}\tilde{\alpha}_{jk,t} + A_{kj,t}\tilde{\alpha}_{ik,t}\right),\ \tilde{\alpha}_{ij,0} = 0,\\
\dot{\tilde{\beta}}_{ij,t} &= 2\sum_{k,h=1}^{n} \Omega_{kh,t}P_{ik,t}\left(C_{jh,t} - P_{jh,t} + \mu_{j,t}\mu_{h,t}\right) + \sum_{k=1}^{n} d_{k,t}\mu_{j,t}P_{ik,t}\\
&\quad + \sum_{k=1}^{n}\left(A_{ij,t}\tilde{\beta}_{kj,t} + F_{jk,t}\tilde{\beta}_{ik,t}\right) + \sum_{l=1}^{m}\sum_{h=1}^{n} K_{jl}H_{lh}\tilde{\alpha}_{ij,t},\ \tilde{\beta}_{ij,0} = 0.
\end{aligned} \tag{30}
$$

In next Section we consider practical example of using the NIQF and IQF.

## 6. Application of NIQF and IQF. Estimation of Power of Signal

If $x_t$ is a scalar random signal measured in additive white noise then the signal and observation equations (1) and (2) are

$$
\begin{aligned}
dx_t &= ax_t dt + dv_t,\ a < 0,\ x_0 \sim \mathcal{N}\left(\bar{x}_0, \sigma_0^2\right),\\
dy_t &= x_t dt + dw_t,\ t \geq 0,
\end{aligned} \tag{31}
$$

where $v_t$ and $w_t$ are independent scalar Wiener processes (noises) with intensities $q$ and $r$, respectively, $a = const$.

The KF equation (5) gives the following

$$
\begin{aligned}
d\hat{x}_t &= a\hat{x}_t dt + K_t\left(dy_t - \hat{x}_t dt\right),\ \hat{x}_0 = \bar{x}_0,\ K_t = P_t/r,\\
dP_t &= \left(2aP_t - P_t^2/r + q\right)dt,\ P_0 = \sigma_0^2,\ P_t = \mathbf{E}\left[\left(x_t - \hat{x}_t\right)^2\right].
\end{aligned} \tag{32}
$$

Analytical solution of the Riccati equation takes the form

$$P_t = k_2 + \frac{k_1+k_2}{\left[\left(\sigma_0^2+k_1\right)/\left(\sigma_0^2-k_2\right)\right]e^{2bt}-1}, \quad k_1 = r(b-a),\ k_2 = r(b+a), b = \sqrt{a^2+q/r}. \tag{33}$$

**6.1. Example of NIQF – Estimation of a current power of signal**

Further, we consider a specific NIQF which represents a current power of a signal, i.e.,

$$z_t = x_t^2. \tag{34}$$

Using (7) and (12) we obtain the best optimal and suboptimal estimates of a power of a signal,

$$\hat{z}_t^{opt} = \hat{x}_t^2 + P_t,\ \hat{z}_t^{sub} = \hat{x}_t^2, \tag{35}$$

where $\hat{x}_t$ and $P_t$ are determined by (32) and (33), respectively.

Let compare estimation accuracy of the optimal and suboptimal estimates (35).

Using Theorem 5.1 we obtain precise formulas for the actual MSEs of these estimates,

$$P_{z,t}^{opt} = \mathbf{E}\left[\left(x_t^2 - \hat{x}_t^2 - P_t\right)^2\right] = 4P_tC_t - 2P_t^2 + 4\mu_t^2P_t, \quad P_{z,t}^{sub} = \mathbf{E}\left[\left(x_t^2 - \hat{x}_t^2\right)^2\right] = 4P_tC_t - P_t^2 + 4\mu_t^2P_t, \tag{36}$$

where the mean $\mu_t$ and covariance $C_t$ of the signal $x_t$ are determined by Lyapunov equations

$$\dot{\mu}_t = a\mu_t,\ \mu_0 = \bar{x}_0,\ \dot{C}_t = 2aC_t + q,\ C_0 = \sigma_0^2, \tag{37}$$

with solutions

$$\mu_t = \bar{x}_0 e^{at},\ C_t = \left(\sigma_0^2 + q/2a\right)e^{2at} - q/2a. \tag{38}$$

Thus, the analytical solutions (33) and (37) with formulas (36) completely establish the actual MSEs for the optimal and suboptimal estimates (35).

According to Corollary 5.1 the difference between the MSEs is equal $P_{z,t}^{sub} - P_{z,t}^{opt} = P_t^2$. Figure 1 shows the numerical values of the MSEs for the values $a = -1$, $q = 0.5$, $\bar{x}_0 = 0$, $\sigma_0^2 = 4$, and $r = 0.1$.

From Figure 1 we observe that the relative error $\Delta_t(\%) = \left|\left(P_{z,t}^{sub} - P_{z,t}^{opt}\right)/P_{z,t}^{opt}\right|100\%$ varies from 3% to 6% within the time zone $t \in [0.1;\ 1.1]$, and then it increases. In steady-state zone $t > 4$ the relative error is reached the value $\Delta_\infty = 20.4\%$ and at the same time zone the absolute

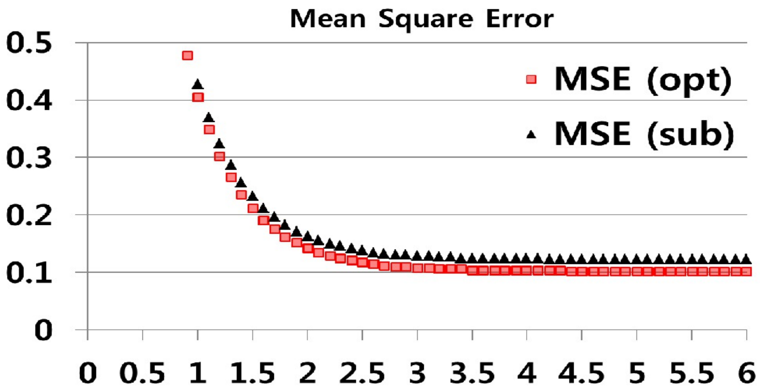


FIGURE 1. Optimal and suboptimal MSEs for power of signal

values of the MSEs are equal $P_{z,\infty}^{opt} = 0.1029$ and $P_{z,\infty}^{sub} = 0.1239$. Thus the numerical results show that the suboptimal estimate $\hat{z}_t^{sub} = \hat{x}_t^2$ may be seriously worse than the optimal one $\hat{z}_t^{opt} = \hat{x}_t^2 + P_t$.

**6.2. Example of IQF – Estimation of an accumulated power of signal**

Here we consider an accumulated power of a signal, then an IQF is represented as

$$u_t = \int_0^t x_s^2 ds. \tag{39}$$

Using (11) and (12) the best optimal and suboptimal estimates of an accumulated power satisfy the differential equations

$$\dot{\hat{u}}_t^{opt} = \hat{x}_t^2 + P_t, \ \ \dot{\hat{u}}_t^{sub} = \hat{x}_t^2, \ \ \hat{u}_0^{opt} = \hat{u}_0^{sub} = 0. \tag{40}$$

Using Theorems 5.2 and 5.3 we obtain the differential equations for the actual MSEs of these estimates, $P_{u,t}^{opt} = \mathbf{E}\left[\left(u_t - \hat{u}_t^{opt}\right)^2\right]$ and $P_{u,t}^{sub} = \mathbf{E}\left[\left(u_t - \hat{u}_t^{sub}\right)^2\right]$, respectively,

$$\begin{aligned} &\dot{P}_{u,t}^{opt} = 2\alpha_{11,t} + 4\beta_{11,t}, \ \ P_{u,0}^{opt} = 0, \\ &\dot{\alpha}_{11,t} = 2P_t^2 + 2\left(a - K_t\right)\alpha_{11,t}, \ \ \alpha_{11,0} = 0, \ \ K_t = P_t/r \\ &\dot{\beta}_{11,t} = 2P_t\left(C_t - P_t + \mu_t^2\right) + \left(2a - K_t\right)\beta_{11,t} + K_t\alpha_{11,t}, \ \ \beta_{11,0} = 0, \end{aligned} \tag{41}$$

and

$$
\begin{aligned}
&\dot{P}^{sub}_{u,t} = 2\tilde{\alpha}_{11,t} + 4\tilde{\beta}_{11,t},\ P^{sub}_{u,0} = 0,\\
(42)\quad &\dot{\tilde{\alpha}}_{11,t} = 3P_t^2 + 2\left(a - K_t\right)\tilde{\alpha}_{11,t},\ \tilde{\alpha}_{11,0} = 0,\\
&\dot{\tilde{\beta}}_{11,t} = 2P_t\left(C_t - P_t + \mu_t^2\right) + \left(2a - K_t\right)\tilde{\beta}_{11,t} + K_t\tilde{\alpha}_{11,t},\ \tilde{\beta}_{11,0} = 0,
\end{aligned}
$$

where $P_t$, $K_t$, $C_t$ and $\mu_t$ are determined by (32), (33) and (37), respectively. Thus, the equations (41) and (42) completely establish the actual MSEs for the optimal and suboptimal estimates (40).

## 7. Conclusion

In many application problems, a quadratic function of signal brings useful information of the signal for control. In order to estimate an arbitrary NIQF and IQF, an optimal and suboptimal algorithms are proposed. The estimates are a comprehensively investigated, including derivation of compact matrix forms for an optimal and suboptimal estimates and their MSEs. In a view of importance of a quadratic functions for practice, the obtained algorithms are illustrated on example of estimation of a power of random signal which shows that the optimal estimate yields a reasonably good estimation accuracy.

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