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A MEASURE ZERO STABILITY OF A FUNCTIONAL EQUATION ASSOCIATED WITH INNER PRODUCT SPACE

Jaeyoung Chung and John Michael Rassias

Abstract. Let X, Y be real normed vector spaces. We exhibit all the solutions $f : X \to Y$ of the functional equation $f(rx + sy) + rsf(x - y) = rf(x) + sf(y)$ for all $x, y \in X$, where r, s are nonzero real numbers satisfying $r + s = 1$. In particular, if Y is a Banach space, we investigate the Hyers-Ulam stability problem of the equation. We also investigate the Hyers-Ulam stability problem on a restricted domain of the following form $\Omega \cap \{(x, y) \in X^2 : \|x\| + \|y\| \geq d\}$, where Ω is a rotation of $H \times H \subset X^2$ and H^c is of the first category. As a consequence, we obtain a measure zero Hyers-Ulam stability of the above equation when $f : \mathbb{R} \to Y$.

1. Introduction

Throughout this paper we denote by X, Y a real normed space and a Banach space, respectively. A mapping $f : X \to Y$ is called *additive mapping* if f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. The Hyers-Ulam stability problem for functional equations was originated by S. M. Ulam in 1940 (see [23]). One of the first assertions to be obtained is the following result, essentially due to D. H. Hyers [10] that gives an answer to a famous stability question of S. M. Ulam [23].

Theorem 1.1. *Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$. Then there exists a unique additive mapping $a : X \to Y$ such that

$$\|f(x) - a(x)\| \leq \epsilon$$

holds for all $x \in X$.

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In 1950 T. Aoki [3] provided a generalization of Theorem 1.1 for additive mappings by allowing the Cauchy difference to be unbounded, that is, controlled by the sum of two powered norms. In 1978 Th. M. Rassias [20] generalized the Hyers theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). This stability concept is also applied to the case of other functional equations. Among the pioneering results, F. Skof [22] solved the Hyers-Ulam stability problem for additive mappings on a restricted domain. Also S. M. Jung [12, 14] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains and J. M. Rassias [18] investigated the Hyers-Ulam stability of mixed type mappings on restricted domains as well. For more analogous results on functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions we refer the reader to pertinent papers [4, 6, 7, 8, 9, 13, 21, 22].

A mapping $f : X \to Y$ is called *quadratic mapping* if f satisfies the following quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$. It is well known that a normed vector space X with norm $\|\cdot\|$ is an inner product space if and only if $f(x) = \|x\|^2$ is a quadratic mapping (see P. Jordan and J. von Neumann [11]). In [16], A. Najati and S. M. Jung gave another characterization of inner product space.

Theorem 1.2. *A normed vector space X with norm $\|\cdot\|$ is an inner product space if and only if there exists a pair (r, s) of nonzero real numbers r, s with $r + s = 1$, such that $f(x) = \|x\|^2$ satisfies the functional equation*

$$f(rx + sy) + rsf(x - y) = rf(x) + sf(y) \tag{1.1}$$

for all $x, y \in X$.

As we see in the following Section 2, the above equation (1.1) is not exactly equivalent to a quadratic functional equation and therefore we can call the equation (1.1) *quasi-quadratic functional equation.* For the proof of the above result the authors investigated the general solutions of the equation (1.1) under the assumption that f is an even function and r is a rational number. As the main result, they also proved the stability of the equation (1.1) on a restricted domain.

Theorem 1.3. *Let r, s be nonzero real numbers with $r + s = 1$ and $d > 0$ and $\delta \geq 0$. Assume that an even mapping $f : X \to Y$ satisfies the following (r, s)-quasi-quadratic functional inequality*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta \tag{1.2}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{19(2 + |r| + |s|)}{2|rs|}\delta \tag{1.3}$$

for all $x \in X$.

In this paper we first exhibit all the solutions of the functional equation (1.1) without the assumptions that f is even and r is rational. Secondly, we prove the Hyers-Ulam stability of the equation (1.1) defined in the whole domain. Thirdly, we investigate the stability of the equation (1.1) on restricted domains satisfying the condition (C) (see Section 4 for the condition). Finally, by finding subsets satisfying the condition (C) we prove the stability of equation (1.1) on the restricted domain $\Omega \cap \{(x, y) \in X^2 : \|x\| + \|y\| \geq d\}$, where Ω is a rotation of $H \times H \subset X^2$ and H^c is of the first category, which generalizes and refines the result of A. Najati and S. M. Jung [16] (we deal with more restricted domain, without the said assumption of evenness of f and obtain a much better constant than that one in (1.3)). As a consequence, we also obtain the stability of the equation (1.1) when $f : \mathbb{R} \to Y$ satisfies the above inequality (1.2) on a set $\Omega \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of Lebesgue measure zero.

2. General solutions

In this section, we establish the general solutions of the (r, s)-quasi-quadratic functional equation (1.1).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation* (1.1) *if and only if*

$$f(x) = B(x, x) + a(x) \tag{2.1}$$

holds for all $x \in X$, where $B : X \times X \to Y$ is a symmetric bi-additive mapping and $a : X \to Y$ is an additive mapping, which satisfy

$$B(rx, y) = rB(x, y), \quad a(rx) = r^2 a(x) \tag{2.2}$$

for all $x, y \in X$. In particular, if r is an algebraic number or $t \to f(tx)$ is a continuous function of $t \in \mathbb{R}$ for all $x \in X$, then $a = 0$.

Proof. Let D be the difference

$$D(x, y) = f(rx + sy) + rsf(x - y) - rf(x) - sf(y)$$

for all $x, y \in X$. Then we obtain the four difference relations:

$$\begin{aligned}
D(rx + 2y, rx + y) &= f(rx + (r + 1)y) + rsf(y) - rf(rx + 2y) - sf(rx + y),\\
D(x + 2y, y) &= f(rx + (r + 1)y) + rsf(x + y) - rf(x + 2y) - sf(y),\\
D(x + 2y, 2y) &= f(rx + 2y) + rsf(x) - rf(x + 2y) - sf(2y),\\
D(x + y, y) &= f(rx + y) + rsf(x) - rf(x + y) - sf(y)
\end{aligned}$$

for all $x, y \in X$. Thus, we can write

$$\begin{aligned}
0 &= D(rx + 2y, rx + y) - D(x + 2y, y) + rD(x + 2y, 2y) + sD(x + y, y)\\
&= rs\big(f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y)\big)
\end{aligned}$$

for all $x, y \in X$, and we obtain the special functional equation

$$f(x) + f(x + 2y) = 2f(x + y) - 2f(y) + f(2y) \tag{2.3}$$

for all $x, y \in X$. Let f^e and f^o be the even part and the odd part of f respectively. Then from (2.3) we have the two pertinent functional equations:

$$f^e(x) + f^e(x + 2y) = 2f^e(x + y) - 2f^e(y) + f^e(2y), \tag{2.4}$$

$$f^o(x) + f^o(x + 2y) = 2f^o(x + y) - 2f^o(y) + f^o(2y) \tag{2.5}$$

for all $x, y \in X$.

First, we consider the even part f^e. Putting $x = y = 0$ and $x = -y$ in (2.4) we have

$$f^e(0) = 0, \quad f^e(2y) = 4f^e(y) \tag{2.6}$$

for all $y \in X$. Replacing x by $x - y$ in (2.4) and using (2.6) we get

$$f^e(x - y) + f^e(x + y) = 2f^e(y) + 2f^e(y) \tag{2.7}$$

for all $x, y \in X$. Thus, $f^e := q$ is a quadratic mapping. Secondly, we consider the odd part f^o. Putting $x = y = 0$ and then replacing x by $-y$ in (2.5) we have $f^o(0) = 0$ and $f^o(2y) = 2f^o(y)$ for all $y \in X$. Thus, we have

$$\begin{aligned} f^o(x) + f^o(x + 2y) &= 2f^o(x + y) - 2f^o(y) + f^o(2y) \\ &= 2f^o(x + y) = f^o(2x + 2y) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. Replacing x by u and y by $\frac{1}{2}(v - u)$ in (2.8) we have

$$f^o(u) + f^o(v) = f^o(u + v)$$

for all $u, v \in X$ and $f^o := a$ is an additive mapping. It is well known that a function $q : X \to Y$ between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric bi-additive function $B : X \times X \to Y$ such that $q(x) = B(x, x)$ for all $x \in X$(see [2, p. 166] or [15]). Thus, all general solutions of the equation (1.1) is of the form (2.1).

Now, we find the sufficient conditions. From (1.1) we have the two functional relations

$$q(rx + sy) + rsq(x - y) = rq(x) + sq(y), \tag{2.9}$$

$$a(rx + sy) + rsa(x - y) = ra(x) + sa(y) \tag{2.10}$$

for all $x, y \in X$. From (2.9) we have

$$\begin{aligned} rB(x, x) + sB(y, y) &= B(rx + sy, rx + sy) + rsB(x - y, x - y) \\ &= B(rx, rx) + 2B(rx, sy) + B(sy, sy) \\ &\quad + rsB(x, x) - 2rsB(x, y) + rsB(y, y) \end{aligned} \tag{2.11}$$

for all $x, y \in X$. Putting $y = 0$ and $x = 0$ separately in (2.11) we have

$$\begin{aligned} B(rx, rx) &= (r - rs)B(x, x) = r^2B(x, x), \\ B(sy, sy) &= (s - rs)B(y, y) = s^2B(y, y) \end{aligned} \tag{2.12}$$

for all $x, y \in X$, respectively. From (2.11) and (2.12) we have

$$rB(x, x) + sB(y, y) = r^2B(x, x) + 2B(rx, sy) + s^2B(y, y) \tag{2.13}$$

$$+ rsB(x,x) - 2rsB(x,y) + rsB(y,y)$$
$$= rB(x,x) + 2B(rx,sy) - 2rsB(x,y) + sB(y,y)$$

for all $x, y \in X$. Thus, from (2.13) we have

$$B(rx, sy) = rsB(x,y) \tag{2.14}$$

for all $x, y \in X$. Using (2.12) we have

$$\begin{aligned} r^2B(x+y, x+y) &= B(rx+ry, rx+ry) \\ &= B(rx, rx) + 2B(rx, ry) + B(ry, ry) \\ &= r^2B(x,x) + 2B(rx, ry) + r^2B(y,y) \end{aligned} \tag{2.15}$$

for all $x, y \in X$, and on the other hand we also have

$$r^2B(x+y, x+y) = r^2B(x,x) + 2r^2B(x,y) + r^2B(y,y) \tag{2.16}$$

for all $x, y \in X$. Equating (2.15) and (2.16) we have

$$B(rx, ry) = r^2B(x,y) \tag{2.17}$$

for all $x, y \in X$. From (2.14) and (2.17) we have

$$\begin{aligned} B(rx, y) &= B(rx, (r+s)y) = B(rx, ry) + B(rx, sy) \\ &= r^2B(x,y) + rsB(x,y) = rB(x,y) \end{aligned} \tag{2.18}$$

for all $x, y \in X$.

Conversely, if B satisfies (2.18) we have

$$B(sx, y) = B(x - rx, y) = B(x,y) - rB(x,y) = sB(x,y) \tag{2.19}$$

for all $x, y \in X$. Thus, $B(x,x)$ satisfies the equation (2.11) which implies that $q(x) = B(x,x)$ satisfies the functional equation (1.1).

Now, putting $y = 0$ in (2.10) we have

$$a(rx) = (r - rs)a(x) + sa(0) = r^2a(x) \tag{2.20}$$

for all $x \in X$.

Conversely, if a satisfies (2.20), then we have

$$\begin{aligned} a(rx+sy) + rsa(x-y) &= a(rx) + a(y - ry) + rsa(x) - rsa(y) \\ &= r^2a(x) + a(y) - r^2a(y) + rsa(x) - rsa(y) \\ &= r(r+s)a(x) + (1 - r^2 - rs)a(y) \\ &= ra(x) + sa(y) \end{aligned} \tag{2.21}$$

for all $x, y \in X$. Thus, if B and a satisfy (2.2), then $f(x) = B(x,x) + a(x)$ is a solution of the functional equation of (1.1).

Finally we show that if r is an algebraic number or $t \to f(tx)$ is continuous function of $t \in \mathbb{R}$ for all $x \in X$, then $a = 0$. In fact, if $r = -1$, then by (2.20) we have $a = 0$. Assume that $r \neq -1$ is an algebraic number and $a \neq 0$. By iteration we have

$$a(r^k x) = r^{2k} a(x) \tag{2.22}$$

for all $x \in X$ and all $k = 1, 2, 3 \ldots$.

Let $p(x)$ be an irreducible polynomial such that $p(r) = 0$. Then using (2.22) we have

$$0 = a(0) = a(p(r)x) = p(r^2)a(x) \tag{2.23}$$

for all $x \in X$. From (2.23) we have $p(r^2) = 0$. Now, using (2.22) again we have

$$0 = a(0) = a(p(r^2)x) = p(r^4)a(x) \tag{2.24}$$

for all $x \in X$ and hence $p(r^4) = 0$. Continuing this inductive process we obtain $p(r^{2^k}) = 0$ for all $k = 1, 2, 3, \ldots$. This implies that $p(x) = 0$ has infinitely many roots $r, r^2, r^4, \ldots$. Thus, we conclude that if r is an algebraic number then $a = 0$. Finally, we assume that $t \to f(tx)$ is continuous. Then $t \to a(tx)$ is also continuous. It remains that r is a transcendental number. Choose a sequence t_n, $n = 1, 2, \ldots$, of rational numbers such that $t_n \to r$ as $n \to \infty$. Then we have

$$a(rx) = \lim_{n\to\infty} a(r_n x) = \lim_{n\to\infty} r_n a(x) = ra(x)$$

for all $x \in X$. Thus, we have $ra(x) = r^2 a(x)$ for all $x \in X$ and hence $a = 0$. Now, the proof is complete. □

Remark 2.2. It is known that there exists a nonzero additive function $a : \mathbb{R} \to \mathbb{R}$ satisfying $a(rx) = r^2 a(x)$ provided that r is a transcendental number (see [1, p. 70] and references therein).

3. Classical stability

In this section we prove the Hyers-Ulam stability of the functional equation (1.1). We first prove the following Lemma 3.1 for a better bound.

Lemma 3.1. *Suppose that $f : X \to Y$ satisfies the mixed additive-quadratic functional inequality*

$$\|f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y)\| \leq \epsilon \tag{3.1}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - q(x) - A(x)\| \leq 2\epsilon \tag{3.2}$$

for all $x \in X$.

Proof. Let f^e and f^o be the even part and the odd part of f, respectively. Then we have

$$\|f^e(x) + f^e(x + 2y) - 2f^e(x + y) + 2f^e(y) - f^e(2y)\| \leq \epsilon, \tag{3.3}$$

$$\|f^o(x) + f^o(x + 2y) - 2f^o(x + y) + 2f^o(y) - f^o(2y)\| \leq \epsilon \tag{3.4}$$

for all $x, y \in X$. First, we consider the even part f^e. Putting $x = y = 0$ in (3.3) we have $\|f^e(0)\| \le \epsilon$. Replacing x by $-y$ in (3.3), dividing the result by 4 and using the triangle inequality we have

$$\left\| f^e(y) - \frac{1}{4} f^e(2y) \right\| \le \frac{3}{4}\epsilon. \tag{3.5}$$

By the well-known induction method employed by D. H. Hyers in [10] we can show that

$$q(x) := \lim_{m\to\infty} 4^{-m} f^e(2^m x) \tag{3.6}$$

exists and satisfies

$$\|f^e(x) - q(x)\| \le \epsilon \tag{3.7}$$

for all $x \in X$, and the functional equation

$$q(x) + q(x + 2y) - 2q(x + y) + 2q(y) - q(2y) = 0 \tag{3.8}$$

for all $x, y \in X$. Following the proof of Theorem 2.1, equation (3.8) implies that q is a quadratic mapping. Now, we consider the odd part f^o. Replacing x by $-y$ in (3.4) and dividing the result by 2, we have

$$\left\| f^o(y) - \frac{1}{2} f^o(2y) \right\| \le \frac{\epsilon}{2}. \tag{3.9}$$

Again, by the well-known induction method as in [10], we can show that

$$A(x) = \lim_{m\to\infty} 2^{-m} f^o(2^m x) \tag{3.10}$$

exists and satisfies

$$\|f^o(x) - A(x)\| \le \epsilon \tag{3.11}$$

for all $x \in X$, and the functional equation

$$A(x) + A(x + 2y) - 2A(x + y) + 2A(y) - A(2y) = 0 \tag{3.12}$$

for all $x, y \in X$. Thus, following the proof of Theorem 2.1, equation (3.12) implies that A is an additive mapping. From (3.7) and (3.11), using the triangle inequality we get (3.2). Now, the proof is complete. □

Theorem 3.2. *Let r, s be nonzero real numbers with $r + s = 1$ and $\delta \ge 0$. Suppose that $f : X \to Y$ satisfies the (r, s)-quasi-quadratic functional inequality*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \le \delta \tag{3.13}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $q : X \to Y$ satisfying $B(rx, y) = rB(x, y)$ and a unique additive mapping $a : X \to Y$ satisfying $a(rx) = r^2 a(x)$ such that

$$\|f(x) - B(x, x) - a(x)\| \le K(r)\delta \tag{3.14}$$

for all $x \in X$, where

$$K(r) = \begin{cases} \min\left\{\frac{1}{|r(|r|-1)|}, \ \frac{2(2+|r|+|1-r|)}{|r(1-r)|}\right\}, & \text{if } r \neq -1, \\ \frac{3}{2}, & \text{if } r = -1. \end{cases} \tag{3.15}$$

Proof. We use two approaches in order to get a better bound. First, we use Lemma 3.1. Let D be the difference

$$D(x, y) = f(rx + sy) + rsf(x - y) - rf(x) - sf(y)$$

for all $x, y \in X$. Then we obtain

$$\begin{aligned} &D(rx + 2y, rx + y) - D(x + 2y, y) + rD(x + 2y, 2y) + sD(x + y, y) \\ &\quad = rs\big(f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y)\big) \end{aligned} \tag{3.16}$$

for all $x, y \in X$. From (3.13) and (3.16) we have

$$\|f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y)\| \leq \frac{2 + |r| + |s|}{|rs|}\delta \tag{3.17}$$

for all $x, y \in X$. By Lemma 3.1, there exist a unique quadratic mapping $q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - q(x) - A(x)\| \leq \frac{2(2 + |r| + |s|)}{|rs|}\delta \tag{3.18}$$

for all $x \in X$.

Secondly, we use a direct approach. Putting $x = y = 0$ in (3.13) we have $|f(0)| \leq \frac{\delta}{|rs|}$. Putting $y = 0$ in (3.13) and using the triangle inequality we have

$$\|f(rx) - r^2 f(x)\| \leq \delta + |sf(0)| \leq \left(\frac{|r| + 1}{|r|}\right)\delta \tag{3.19}$$

for all $x, y \in X$. If $|r| > 1$, dividing (3.19) by r^2 we have

$$\left\|f(x) - \frac{1}{r^2}f(rx)\right\| \leq \left(\frac{|r| + 1}{|r|^3}\right)\delta \tag{3.20}$$

for all $x \in X$.

Using the well-known induction method as in [10] we can show that

$$g(x) = \lim_{m\to\infty} \frac{1}{r^{2m}} f(r^m x) \tag{3.21}$$

exists and satisfies

$$\|f(x) - g(x)\| \leq \left(\frac{|r| + 1}{|r|^3}\right)\delta \times \left(\frac{r^2}{r^2 - 1}\right) = \frac{\delta}{|r|(|r| - 1)} \tag{3.22}$$

for all $x \in X$ and the functional equation (1.1).

If $|r| < 1$, replacing x by $\frac{x}{r}$ in (3.19) we have

$$\left\|f(x) - r^2 f\left(\frac{x}{r}\right)\right\| \leq \left(\frac{|r| + 1}{|r|}\right)\delta \tag{3.23}$$

for all $x, y \in X$.

Using the well-known induction method as in [10] we can show that

$$g(x) = \lim_{m\to\infty} r^{2m} f(r^{-m}x) \tag{3.24}$$

exists and satisfies

$$\|f(x) - g(x)\| \le \left(\frac{|r|+1}{|r|}\right)\delta \times \left(\frac{1}{1-|r|^2}\right) = \frac{\delta}{|r|(1-|r|)} \tag{3.25}$$

for all $x \in X$ and the functional equation (1.1). Finally, if $r = -1$, then $s = 2$ and we get the inequality

$$\|f(-x+2y) - 2f(x-y) + f(x) - 2f(y)\| \le \delta \tag{3.26}$$

for all $x, y \in X$. Putting $y = 0$ in (3.26) we have

$$\|f(-x) - f(x) - 2f(0)\| \le \delta. \tag{3.27}$$

Putting $x = 0$ in (3.26) and using (3.27) and the triangle inequality we have

$$\|f(2y) - 4f(y)\| \le \frac{9}{2}\delta. \tag{3.28}$$

Using the well-known induction method as in [10] we can show that

$$g(x) = \lim_{m\to\infty} 4^{-m} f(2^m x)$$

exists and satisfies

$$\|f(x) - g(x)\| \le \frac{3}{2}\delta \tag{3.29}$$

for all $x \in X$ and the functional equation

$$g(-x+2y) - 2g(x-y) + g(x) - 2g(y) = 0$$

for all $x, y \in X$. From (3.22), (3.25) and (3.29) we get

$$\|f(x) - g(x)\| \le C(r)\delta \tag{3.30}$$

for all $x \in X$, where

$$C(r) = \begin{cases} \frac{1}{|r(|r|-1)|}\,, & \text{if } r \ne -1, \\ \frac{3}{2}\,, & \text{if } r = -1. \end{cases}$$

By Theorem 2.1 we have

$$g(x) = B(x,x) + a(x) \tag{3.31}$$

for all $x \in X$, where $B : X \times X \to Y$ is a symmetric bilinear mapping satisfying $B(rx, y) = rB(x, y)$ and $a : X \to Y$ is an additive mapping satisfying $a(rx) = r^2 a(x)$. Now, from (3.18), (3.30) and (3.31) we have

$$\|B(x,x) + a(x) - q(x) - A(x)\| \le \left(C(r) + \frac{2(2+|r|+|s|)}{|rs|}\right)\delta \tag{3.32}$$

for all $x \in X$. Replacing x by kx, $k \in \mathbb{N}$ in (3.32) and using the triangle inequality

$$k^2\|B(x,x) - q(x)\| - k\|a(x) - A(x)\| \leq \left(C(r) + \frac{2(2+|r|+|s|)}{|rs|}\right)\delta$$

for all $x \in X$ and $k \in \mathbb{N}$, which implies that $B(x,x) = q(x)$ and $a(x) = A(x)$ for all $x \in X$. From (3.18) and (3.30) we get (3.14). Now, the proof is complete. □

4. Stability on restricted domains

In this section we consider the generalized Hyers-Ulam stability of the quasi-quadratic functional equation on restricted domains $\Omega \subset X \times X$ satisfying the following condition (C):

(C) Let $(\gamma_j, \lambda_j) \in \mathbb{R}^2$, $j = 1, 2, \ldots, r$, with $\gamma_j^2 + \lambda_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$, be given.
For any $p_j, q_j \in X$, $j = 1, 2, \ldots, r$, there exists $t \in X$ such that

$$\{(p_j + \gamma_j t,\ q_j + \lambda_j t) : j = 1, 2, \ldots, r\} \subset \Omega.$$

As a main result we prove the following Theorem 4.1.

Theorem 4.1. *Let r, s be nonzero real numbers with $r + s = 1$ and $\delta \geq 0$. Suppose that $f : X \to Y$ satisfies the following (r, s)-quasi-quadratic functional inequality*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta \tag{4.1}$$

for all $(x, y) \in \Omega$. Then there exist a unique quadratic mapping $q : X \to Y$ and a unique additive mapping $a : X \to Y$ such that

$$\|f(x) - q(x) - a(x)\| \leq \frac{6(2+|r|+|s|)}{|rs|}\delta \tag{4.2}$$

for all $x \in X$.

Proof. Let D be the difference

$$D(x, y) = f(rx + sy) + rsf(x - y) - rf(x) - sf(y)$$

for all $x, y \in X$. Then we obtain

$$\begin{aligned} &D(rx + 2y,\ rx + y) - D(x + 2y, y) + rD(x + 2y, 2y) + sD(x + y, y) \\ &\quad = rs\big(f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y)\big) \end{aligned} \tag{4.3}$$

for all $x, y \in X$. Now, let F be the difference

$$F(x, y) = f(x) + f(x + 2y) - 2f(x + y) + 2f(y) - f(2y) \tag{4.4}$$

for all $x, y \in X$. Then we obtain the four relations:

$$F(x - t, y) = f(x - t) + f(x - t + 2y) - 2f(x - t + y) + 2f(y) - f(2y),$$
$$F(x + t, y) = f(x + t) + f(x + t + 2y) - 2f(x + t + y) + 2f(y) - f(2y),$$

$$F(x-t,t) = f(x-t) + f(x+t) - 2f(x) + 2f(t) - f(2t),$$
$$F(x-t+2y,t) = f(x-t+2y) + f(x+t+2y) - 2f(x+2y) + 2f(t) - f(2t),$$
$$2F(x-t+y,t) = 2f(x-t+y) + 2f(x+t+y) - 4f(x+y) + 4f(t) - 2f(2t)$$

for all $x, y, t \in X$. Thus, the following equality

$$\begin{aligned}(4.5)\qquad & F(x-t,y)+F(x+t,y)-F(x-t,t)-F(x-t+2y,t)+2F(x-t+y,t) \\ & = 2F(x,y)\end{aligned}$$

holds for all $x, y, t \in X$. It follows from (4.3), (4.4) and (4.5) that we can write

$$\begin{aligned}(4.6)\qquad 2F(x,y) &= F(x-t,y) + F(x+t,y) - F(x-t,t) - F(x-t+2y,t) \\ &\quad + 2F(x-t+y,t) \\ &= \frac{1}{rs}\sum_{j=1}^{20} c_j D(p_j + \alpha_j t, q_j + \beta_j t)\end{aligned}$$

for some $c_j, \alpha_j, \beta_j \in \mathbb{R}$ with $\sum |c_j| = 6(2 + |r| + |s|)$ and $(\alpha_j(r), \beta_j(r)) \neq (0, 0)$ for all $j = 1, 2, \ldots, 20$, and $p_j, q_j, j = 1, 2, \ldots, 20$, are linear combinations of x and y. Since Ω satisfies the above condition (C), for any $x, y \in X$, there exists $t \in X$ such that

$$\{(p_j + \alpha_j(r)t,\ q_j + \beta_j(r)t) : j = 1, 2, \ldots, 20\} \subset \Omega,$$

and hence from (4.1) and (4.6) we obtain

$$\begin{aligned}\|F(x,y)\| &\le \frac{1}{2|rs|}\sum_{j=1}^{20} |c_j| \|D(p_j + \alpha_j t, q_j + \beta_j t)\| \\ &\le \frac{6(2+|r|+|s|)}{2|rs|}\delta = \frac{3(2+|r|+|s|)}{|rs|}\delta\end{aligned}$$

for all $x, y \in X$. By Lemma 3.1 we get (4.2). Now, the proof is complete. $\square$

It is easy to see that $\Omega := \{(x, y) \in X \times X : \|x\| + \|y\| \ge d\}$ satisfies the condition (C). As a direct consequence of Theorem 4.1 we obtain a refined result of [11, Theorem 3.4].

Corollary 4.2. *Let $d > 0$. Suppose that $f : X \to Y$ satisfies the (r, s)-quasi-quadratic functional inequality (4.1) for all $x, y \in X$ with $\|x\| + \|y\| \ge d$. Then there exist a unique quadratic mapping $q : X \to Y$ and a unique additive mapping $a : X \to Y$ such that*

$$(4.7)\qquad \|f(x) - q(x) - a(x)\| \le \frac{6(2+|r|+|s|)}{|rs|}\delta$$

for all $x \in X$.

Remark 4.3. In particular, if f is even, then replacing x by $-x$ in (4.7) we get

$$(4.8)\qquad \|f(x) - q(x) + a(x)\| \le \frac{6(2+|r|+|s|)}{|rs|}\delta$$

for all $x \in X$. Using the triangle inequality with (4.7) and (4.8) we get $\|a(x)\| \le \frac{12(2+|r|+|s|)}{|rs|}\delta$ for all $x \in X$ and hence $a = 0$.

5. Stability on a set of Lebesgue measure zero

Throughout this section we assume that X is complete. By constructing a subset $\Omega \subset X \times X$ satisfying the condition (C) we prove the Hyers-Ulam stability of the functional equation (1.1) satisfied in a set of Lebesgue measure zero when $X = \mathbb{R}$.

Definition 5.1. A subset K of a topological space E is said to be of the first category if K is a countable union of nowhere dense subsets of E, and otherwise it is said to be of the second category.

Theorem 5.2 (Baire category theorem)**.** *Every nonempty open subset of a compact Hausdorff space or a complete metric space is of the second category.*

Lemma 5.3. *Let H be a subset of X such that $H^c := X \setminus H$ is of the first category. Then, for any countable subsets $U \subset X, \Gamma \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $t \in X$ with $\|t\| \ge M$ such that*

$$U + \Gamma t = \{u + \gamma t : u \in U, \gamma \in \Gamma\} \subset H. \tag{5.1}$$

Proof. Let $H^c_{u,\gamma} = \gamma^{-1}(H^c - u)$, $u \in U, \gamma \in \Gamma$. Then, since H^c is of the first category, $H^c_{u,\gamma}$ are also of the first category for all $u \in U$, $\gamma \in \Gamma$. Since each $H^c_{u,\gamma}$ consists of a countable union of nowhere dense subsets of X, by the Baire category theorem, the countable union of all $\{H^c_{u,\gamma} : u \in U, \gamma \in \Gamma\}$ cannot cover $X_0 := \{t \in X : \|t\| \ge M\}$, i.e.,

$$X_0 \not\subset \bigcup_{(u,\gamma)\in U\times\Gamma} H^c_{u,\gamma}.$$

Choose a $t \in X_0$ such that $t \notin H^c_{u,\gamma}$ for all $u \in U$, $\gamma \in \Gamma$. Then we have $u + \gamma t \in H$ for all $u \in U$, $\gamma \in \Gamma$. Now, the proof is complete. $\square$

From now on we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Lemma 5.4. *Let $P = \{(p_j + \gamma_j t, q_j + \lambda_j t) : j = 1, 2, \ldots, r\}$, where $p_j, q_j, t \in X, \gamma_j, \lambda_j \in \mathbb{R}$ with $\gamma_j^2 + \lambda_j^2 \ne 0$ for all $j = 1, 2, \ldots, r$. Then there exists $\theta_0 \in [0, 2\pi)$ such that $e^{-i\theta_0}P := \{(p'_j + \gamma'_j t, q'_j + \lambda'_j t) : j = 1, 2, \ldots, r\}$ satisfies $\gamma'_j\lambda'_j \ne 0$ for all $j = 1, 2, \ldots, r$.*

Proof. The coefficients γ'_j and λ'_j are given by

$$\gamma'_j = \gamma_j \cos\theta + \lambda_j \sin\theta, \quad \lambda'_j = \lambda_j \cos\theta - \gamma_j \sin\theta$$

for all $j = 1, 2, \ldots, r$. Now, the following equation

$$\Pi_{j=1}^r(\gamma_j \cos\theta + \lambda_j \sin\theta)(\lambda_j \cos\theta - \gamma_j \sin\theta) = 0$$

has only a finite number of zeros $\theta \in [0, 2\pi)$. Thus, we can choose a $\theta_0 \in [0, 2\pi)$ such that $\Pi_{j=1}^r \gamma'_j\lambda'_j \ne 0$. Now, the proof is complete. $\square$

Theorem 5.5. *Let H be a subset of X such that H^c is of the first Baire category. Then there exists a $\theta_0 \in [0, 2\pi)$ such that $\Omega_{\theta_0,d} := (e^{i\theta_0}H^2) \cap \{(x,y) \in X^2 : \|x\| + \|y\| \geq d\}$ satisfies* (C) *for all $d > 0$.*

Proof. Let θ_0 be in Lemma 5.4. It suffices to show that let $(\gamma_j, \lambda_j) \in \mathbb{R}^2$, $j = 1, 2, \ldots, r$, be given with $\gamma_j^2 + \lambda_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$, then for any $p_j, q_j \in X$, $j = 1, 2, \ldots, r$, there exists $t \in X$ such that

$$e^{-i\theta_0}P \subset H^2, \; P \subset \{(x,y) : \|x\| + \|y\| \geq d\}, \tag{5.2}$$

where $P = \{(p_j + \gamma_j t, q_j + \lambda_j t) : j = 1, 2, \ldots, r\}$. Let $e^{-i\theta_0}P = \{(p'_j + \gamma'_j t, q'_j + \lambda'_j t) : j = 1, 2, \ldots, r\}$. Then by Lemma 5.4, we have $\gamma'_j\lambda'_j \neq 0$ for all $j = 1, 2, \ldots, r$. Let $U = \{p'_j, q'_j : j = 1, 2, \ldots, r\}$, $\Gamma = \{\gamma'_j,\, \lambda'_j : j = 1, 2, \ldots, r\}$. Then we have

$$\{u, v : (u,v) \in e^{-i\theta_0}P\} \subset U + \Gamma t. \tag{5.3}$$

Now, by Lemma 5.3, there exists $t \in X$ with $\|t\| \geq \max_{1\leq j\leq r}(|\gamma_j| + |\lambda_j|)^{-1}(|p_j| + |q_j| + d)$ such that

$$U + \Gamma t \subset H. \tag{5.4}$$

From (5.3) and (5.4) we have

$$e^{-i\theta_0}P \subset H^2.$$

By the choice of t, we have $P \subset \{(x,y) : \|x\| + \|y\| \geq d\}$. This completes the proof. □

Remark 5.6. The set $\mathbb{R}$ of real numbers can be partitioned as follows:

$$\mathbb{R} = H \cup (\mathbb{R} \setminus H),$$

where H is of Lebesgue measure zero and $\mathbb{R}\setminus H$ is of the first category [17, Theorem 1.6]. Thus, in view of Theorem 5.5 we can find a subset $\Omega_d \subset \{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of Lebesgue measure zero satisfying (C). Now, as a consequence of Theorem 4.1 we obtain the following.

Corollary 5.7. *Let r, s be nonzero real numbers with $r + s = 1$ and $\delta \geq 0$, $d > 0$. Then there exists a subset $\Omega_d \subset \{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of Lebesgue measure zero such that if $f : \mathbb{R} \to Y$ satisfies the functional inequality*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta$$

for all $(x,y) \in \Omega_d$, then f satisfies

$$\|f(x) - q(x) - a(x)\| \leq \frac{6(2 + |r| + |s|)}{|rs|}\delta$$

for all $x \in \mathbb{R}$, where $q : \mathbb{R} \to Y$ is a quadratic mapping and $a : \mathbb{R} \to Y$ is an additive mapping.

Using the method as in [8, 14, 18, 19] we obtain the following asymptotic behavior of the (r,s)-quasi-quadratic functional equation.

Corollary 5.8. *Let r, s be nonzero real numbers with $r+s=1$. Then there exists a subset $\Omega \subset \mathbb{R}^2$ of Lebesgue measure zero such that if $f:\mathbb{R}\to Y$ satisfies the condition*

$$\|f(rx+sy)+rsf(x-y)-rf(x)-sf(y)\|\to 0$$

as $|x|+|y|\to\infty$ only for $(x,y)\in\Omega$, then f is of the form

$$f(x)=q(x)+a(x)$$

for all $x\in\mathbb{R}$, where $q:\mathbb{R}\to Y$ is a quadratic mapping and $a:\mathbb{R}\to Y$ is an additive mapping.

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Jaeyoung Chung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail address: jychung@kunsan.ac.kr

John Michael Rassias
Pedagogical Department E. E., Section of Mathematics and Informatics
National and Capodistrian University of Athens
4, Agamemnonos Str.
Aghia Paraskevi 15342 Attikis, Athens, Greece
E-mail address: jrassias@primedu.uoa.gr; jrass@otenet.gr