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CHARACTERIZATIONS OF CENTRALIZERS AND DERIVATIONS ON SOME ALGEBRAS

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Abstract. A linear mapping ϕ on an algebra $\mathcal{A}$ is called a centralizable mapping at $G \in \mathcal{A}$ if $\phi(AB) = \phi(A)B = A\phi(B)$ for each A and B in $\mathcal{A}$ with $AB = G$, and ϕ is called a derivable mapping at $G \in \mathcal{A}$ if $\phi(AB) = \phi(A)B + A\phi(B)$ for each A and B in $\mathcal{A}$ with $AB = G$. A point G in $\mathcal{A}$ is called a full-centralizable point (resp. full-derivable point) if every centralizable (resp. derivable) mapping at G is a centralizer (resp. derivation). We prove that every point in a von Neumann algebra or a triangular algebra is a full-centralizable point. We also prove that a point in a von Neumann algebra is a full-derivable point if and only if its central carrier is the unit.

1. Introduction

Let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$, and ϕ be a linear mapping from $\mathcal{A}$ into itself. ϕ is called a *centralizer* if $\phi(AB) = \phi(A)B = A\phi(B)$ for each A and B in $\mathcal{A}$. Obviously, if $\mathcal{A}$ is an algebra with unit I, then ϕ is a centralizer if and only if $\phi(A) = \phi(I)A = A\phi(I)$ for every A in $\mathcal{A}$. ϕ is called a *derivation* if $\phi(AB) = \phi(A)B + A\phi(B)$ for each A and B in $\mathcal{A}$.

A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *centralizable mapping at* $G \in \mathcal{A}$ if $\phi(AB) = \phi(A)B = A\phi(B)$ for each A and B in $\mathcal{A}$ with $AB = G$, and ϕ is called a *derivable mapping at* $G \in \mathcal{A}$ if $\phi(AB) = \phi(A)B + A\phi(B)$ for each A and B in $\mathcal{A}$ with $AB = G$. An element G in $\mathcal{A}$ is called a *full-centralizable point* (*resp. full-derivable point*) if every centralizable (resp. derivable) mapping at G is a centralizer (resp. derivation).

In [3], Brešar proves that if $\mathcal{R}$ is a prime ring with a nontrival idempotent, then 0 is a full-centralizable point. In [18], X. Qi and J. Hou characterize centralizable and derivable mappings at 0 in triangular algebras. In [17], X. Qi proves that every nontrival idempotent in a prime ring is a full-centralizable point. In [19], W. Xu, R. An and J. Hou prove that every element in $B(\mathcal{H})$ is

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a full-centralizable point, where $\mathcal{H}$ is a Hilbert space. For more information on centralizable and derivable mappings, we refer to [2, 7, 11, 12, 14, 20].

For a von Neumann algebra $\mathcal{A}$, the *central carrier* $\mathcal{C}(A)$ of an element A in $\mathcal{A}$ is the projection $I-P$, where P is the union of all central projections P_α in $\mathcal{A}$ such that $P_\alpha A = 0$.

This paper is organized as follows. In Section 2, by using the techniques about central carriers, we show that every element in a von Neumann algebra is a full-centralizable point.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over the complex field $\mathbb{C}$, and $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule which is faithful both as a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. The algebra

$$Tri(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} : A \in \mathcal{A}, B \in \mathcal{B}, M \in \mathcal{M} \right\}$$

under the usual matrix addition and matrix multiplication is called a *triangular algebra*.

In Section 3, we show that if $\mathcal{A}$ and $\mathcal{B}$ are two unital Banach algebras, then every element in $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is a full-centralizable point.

In Section 4, we show that for every point G in a von Neumann algebra $\mathcal{A}$, if Δ is a derivable mapping at G, then $\Delta = D + \phi$, where $D : \mathcal{A} \to \mathcal{A}$ is a derivation and $\phi : \mathcal{A} \to \mathcal{A}$ is a centralizer. Moreover, we prove that G is a full-derivable point if and only if $\mathcal{C}(G) = I$.

2. Centralizers on von Neumann algebras

In this section, $\mathcal{A}$ denotes a unital algebra and $\phi : \mathcal{A} \to \mathcal{A}$ is a centralizable mapping at a given point $G \in \mathcal{A}$. The main result is the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then every element G in $\mathcal{A}$ is a full-centralizable point.*

Before proving Theorem 2.1, we need the following several lemmas.

Lemma 2.2. *Let $\mathcal{A}$ be a unital Banach algebra with the form $\mathcal{A} = \sum_{i\in\Lambda} \bigoplus \mathcal{A}_i$. Then $\phi(\mathcal{A}_i) \subseteq \mathcal{A}_i$. Moreover, suppose $G = \sum_{i\in\Lambda} G_i$, where $G_i \in \mathcal{A}_i$. If G_i is a full-centralizable point in $\mathcal{A}_i$ for every $i \in \Lambda$, then G is a full-centralizable point in $\mathcal{A}$.*

Proof. Let I_i be the unit in $\mathcal{A}_i$. Suppose that A_i is an invertible element in $\mathcal{A}_i$, and t is an arbitrary nonzero element in $\mathbb{C}$. It is easy to check that

$$(I - I_i + t^{-1}GA_i^{-1})((I - I_i)G + tA_i) = G.$$

So we have

$$(I - I_i + t^{-1}GA_i^{-1})\phi((I - I_i)G + tA_i) = \phi(G).$$

Considering the coefficient of t, since t is arbitrarily chosen, we have $(I - I_i)\phi(A_i) = 0$. It follows that $\phi(A_i) = I_i\phi(A_i) \in \mathcal{A}_i$ for all invertible elements. Since $\mathcal{A}_i$ is a Banach algebra, every element can be written into the sum of two invertible elements. So the above equation holds for all elements in $\mathcal{A}_i$. That is to say $\phi(\mathcal{A}_i) \subseteq \mathcal{A}_i$.

Let $\phi_i = \phi\mid_{\mathcal{A}_i}$. For every A in $\mathcal{A}$, we write $A = \sum_{i\in\Lambda} A_i$. Assume $AB = G$. Since $A_iB_i = G_i$ and $\phi(\mathcal{A}_i) \subseteq \mathcal{A}_i$, we have

$$\sum_{i\in\Lambda} \phi(G_i) = \sum_{i\in\Lambda} \phi(A_i) \sum_{i\in\Lambda} B_i = \sum_{i\in\Lambda} \phi(A_i)B_i.$$

It implies that $\phi_i(G_i) = \phi_i(A_i)B_i$. Similarly, we can obtain $\phi_i(G_i) = A_i\phi_i(B_i)$. By assumption, G_i is a full-centralizable point, so ϕ_i is a centralizer. Hence

$$\phi(A) = \sum_{i\in\Lambda} \phi_i(A_i) = \sum_{i\in\Lambda} \phi_i(I_i)A_i = \sum_{i\in\Lambda} \phi_i(I_i) \sum_{i\in\Lambda} A_i = \phi(I)A.$$

Similarly, we can prove $\phi(A) = A\phi(I)$. Hence G is a full-centralizable point. □

Lemma 2.3. *Let $\mathcal{A}$ be a C^*-algebra. If G^* is a full-centralizable point in $\mathcal{A}$, then G is a full-centralizable point in $\mathcal{A}$.*

Proof. Define a linear mapping $\widetilde{\phi} : \mathcal{A} \to \mathcal{A}$ by: $\widetilde{\phi}(A) = (\phi(A^*))^*$ for every A in $\mathcal{A}$. For each A and B in $\mathcal{A}$ with $AB = G^*$, we have $B^*A^* = G$. It follows that $\phi(G) = \phi(B^*)A^* = B^*\phi(A^*)$. By the definition of $\widetilde{\phi}$, we obtain $\widetilde{\phi}(G^*) = \widetilde{\phi}(A)B = A\widetilde{\phi}(B)$. Since G^* is a full-centralizable point in $\mathcal{A}$, we have that $\widetilde{\phi}$ is a centralizer. Thus ϕ is also a centralizer. Hence G is a full-centralizable point in $\mathcal{A}$. □

For a unital algebra $\mathcal{A}$ and a unital $\mathcal{A}$-bimodule $\mathcal{M}$, an element $A \in \mathcal{A}$ is called a *left separating point* (*resp. right separating point*) of $\mathcal{M}$ if $AM = 0$ implies $M = 0$ ($MA = 0$ implies $M = 0$) for every $M \in \mathcal{M}$.

Lemma 2.4. *Let $\mathcal{A}$ be a unital Banach algebra and G be a left and right separating point in $\mathcal{A}$. Then G is a full-centralizable point.*

Proof. For every invertible element X in $\mathcal{A}$, we have

$$\phi(I)G = \phi(G) = \phi(XX^{-1}G) = \phi(X)X^{-1}G.$$

Since G is a right separating point, we obtain $\phi(I) = \phi(X)X^{-1}$. It follows that $\phi(X) = \phi(I)X$ for each invertible element X and so for all elements in $\mathcal{A}$. Similarly, we have that $\phi(X) = X\phi(I)$. Hence G is a full-centralizable point. □

Lemma 2.5. *Let $\mathcal{A}$ be a von Neumann algebra. Then $G = 0$ is a full-centralizable point.*

Proof. For any projection P in $\mathcal{A}$, since $P(I - P) = (I - P)P = 0$, we have

$$\phi(P)(I - P) = P\phi(I - P) = \phi(I - P)P = (I - P)\phi(P) = 0.$$

It follows that $\phi(P) = \phi(I)P = P\phi(I)$. By [6, Proposition 2.4] and [4, Corollary 1.2], we know that ϕ is continuous. Since $\mathcal{A} = \overline{span\{P \in \mathcal{A} : P = P^* = P^2\}}$, it follows that $\phi(A) = \phi(I)A = A\phi(I)$ for every $A \in \mathcal{A}$. Hence G is a full-centralizable point. □

Lemma 2.6. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and P be the range projection of G. If $\mathcal{C}(P) = \mathcal{C}(I-P) = I$, then G is a full-centralizable point.*

Proof. Set $P_1 = P$, $P_2 = I - P$, and denote $P_i\mathcal{A}P_j$ by $\mathcal{A}_{ij}$, $i, j = 1, 2$. For every A in $\mathcal{A}$, denote P_iAP_j by A_{ij}.

Firstly, we claim that the condition $A\mathcal{A}_{ij} = 0$ implies $AP_i = 0$, and similarly, $\mathcal{A}_{ij}A = 0$ implies $P_jA = 0$. Indeed, since $\mathcal{C}(P_j) = I$, by [9, Proposition 5.5.2], the range of $\mathcal{A}P_j$ is dense in $\mathcal{H}$. So $AP_i\mathcal{A}P_j = 0$ implies $AP_i = 0$. On the other hand, if $\mathcal{A}_{ij}A = 0$, then $A^*\mathcal{A}_{ji} = 0$. Hence $A^*P_j = 0$ and $P_jA = 0$.

Besides, since $P_1 = P$ is the range projection of G, we have $P_1G = G$. Moreover, if $AG = 0$, then $AP_1 = 0$.

In the following, we assume that A_{ij} is an arbitrary element in $\mathcal{A}_{ij}$, $i, j = 1, 2$, and t is an arbitrary nonzero element in $\mathbb{C}$. Without loss of generality, we may assume that A_{11} is invertible in $\mathcal{A}_{11}$.

Claim 1 $\phi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

Since $(P_1 + tA_{12})G = G$, we have $\phi(G) = \phi(P_1 + tA_{12})G$. It implies that $\phi(A_{12})G = 0$. Hence $\phi(A_{12})P_1 = 0$.

By $(P_1 + tA_{12})G = G$, we also have $\phi(G) = (P_1 + tA_{12})\phi(G)$. It follows that $A_{12}\phi(G) = A_{12}\phi(P_1)G = 0$. So $A_{12}\phi(P_1)P_1 = 0$. Hence $P_2\phi(P_1)P_1 = 0$.

Since $(A_{11} + tA_{11}A_{12})(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = G$, we have

$$\phi(A_{11} + tA_{11}A_{12})(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = \phi(G). \tag{2.1}$$

Since t is arbitrarily chosen in (2.1), we obtain

$$\phi(A_{11})(A_{11}^{-1}G - A_{12}A_{22}) + \phi(A_{11}A_{12})A_{22} = \phi(G).$$

Since A_{12} is also arbitrarily chosen, we can obtain

$$\phi(A_{11})A_{12}A_{22} = \phi(A_{11}A_{12})A_{22}.$$

Taking $A_{22} = P_2$, since $\phi(A_{12})P_1 = 0$, we have

$$\phi(A_{11}A_{12}) = \phi(A_{11})A_{12}. \tag{2.2}$$

Taking $A_{11} = P_1$, since $P_2\phi(P_1)P_1 = 0$, we have

$$P_2\phi(A_{12}) = P_2\phi(P_1)A_{12} = 0. \tag{2.3}$$

So

$$\begin{aligned}\phi(A_{12}) &= \phi(A_{12})P_1 + P_1\phi(A_{12})P_2 + P_2\phi(A_{12})P_2 \\ &= P_1\phi(A_{12})P_2 \subseteq \mathcal{A}_{12}.\end{aligned}$$

Claim 2 $\phi(\mathcal{A}_{11}) \subseteq \mathcal{A}_{11}$.

Considering the coefficient of t^{-1} in (2.1), we have $\phi(A_{11})A_{22} = 0$. Thus $\phi(A_{11})P_2 = 0$. By (2.2), we obtain $P_2\phi(A_{11})A_{12} = P_2\phi(A_{11}A_{12}) = 0$. It follows that $P_2\phi(A_{11})P_1 = 0$. Therefore, $\phi(A_{11}) = P_1\phi(A_{11})P_1 \subseteq \mathcal{A}_{11}$.

Claim 3 $\phi(\mathcal{A}_{22}) \subseteq \mathcal{A}_{22}$.

By $(A_{11} + tA_{11}A_{12})(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = G$, we also have

$$(A_{11} + tA_{11}A_{12})\phi(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = \phi(G).$$

Through a similar discussion to equation (2.1), we can prove $P_1\phi(A_{22}) = 0$ and

$$\phi(A_{12}A_{22}) = A_{12}\phi(A_{22}). \tag{2.4}$$

Thus $A_{12}\phi(A_{22})P_1 = \phi(A_{12}A_{22})P_1 = 0$. It follows that $P_2\phi(A_{22})P_1 = 0$. Therefore, $\phi(A_{22}) = P_2\phi(A_{22})P_2 \subseteq \mathcal{A}_{22}$.

Claim 4 $\phi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Since $(A_{11} + tA_{11}A_{12})(A_{11}^{-1}G - A_{12}A_{21} + t^{-1}A_{21}) = G$, we have

$$(A_{11} + tA_{11}A_{12})\phi(A_{11}^{-1}G - A_{12}A_{21} + t^{-1}A_{21}) = \phi(G).$$

According to this equation, we can similarly obtain that $P_1\phi(A_{21}) = 0$ and

$$A_{12}\phi(A_{21}) = \phi(A_{12}A_{21}). \tag{2.5}$$

Hence $A_{12}\phi(A_{21})P_2 = \phi(A_{12}A_{21})P_2 = 0$. It follows that $P_2\phi(A_{21})P_2 = 0$. Therefore, $\phi(A_{21}) = P_2\phi(A_{21})P_1 \subseteq \mathcal{A}_{21}$.

Claim 5 $\phi(A_{ij}) = \phi(P_i)A_{ij} = A_{ij}\phi(P_j)$ for each $i, j \in \{1, 2\}$.

By taking $A_{11} = P_1$ in (2.2), we have $\phi(A_{12}) = \phi(P_1)A_{12}$. By taking $A_{22} = P_2$ in (2.4), we have $\phi(A_{12}) = A_{12}\phi(P_2)$.

By (2.2), we have $\phi(A_{11})A_{12} = \phi(A_{11}A_{12}) = \phi(P_1)A_{11}A_{12}$. It follows that $\phi(A_{11}) = \phi(P_1)A_{11}$. On the other hand, $\phi(A_{11})A_{12} = \phi(A_{11}A_{12}) = A_{11}A_{12}\phi(P_2) = A_{11}\phi(A_{12}) = A_{11}\phi(P_1)A_{12}$. It follows that $\phi(A_{11}) = A_{11}\phi(P_1)$.

By (2.4) and (2.5), through a similar discussion as above, we can obtain that $\phi(A_{22}) = A_{22}\phi(P_2) = \phi(P_2)A_{22}$ and $\phi(A_{21}) = A_{21}\phi(P_1) = \phi(P_2)A_{21}$.

Now we have proved that $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$ and $\phi(A_{ij}) = \phi(P_i)A_{ij} = A_{ij}\phi(P_j)$. It follows that

$$\begin{aligned}
\phi(A) &= \phi(A_{11} + A_{12} + A_{21} + A_{22}) \\
&= \phi(P_1)(A_{11} + A_{12} + A_{21} + A_{22}) + \phi(P_2)(A_{11} + A_{12} + A_{21} + A_{22}) \\
&= \phi(P_1 + P_2)(A_{11} + A_{12} + A_{21} + A_{22}) \\
&= \phi(I)A.
\end{aligned}$$

Similarly, we can prove that $\phi(A) = A\phi(I)$. Hence G is a full-centralizable point. □

Proof of Theorem 2.1. Suppose the range projection of G is P. Set $Q_1 = I - \mathcal{C}(I - P)$, $Q_2 = I - \mathcal{C}(P)$, and $Q_3 = I - Q_1 - Q_2$. Since $Q_1 \leq P$ and $Q_2 \leq I - P$, $\{Q_i\}_{i=1,2,3}$ are mutually orthogonal central projections. Therefore $\mathcal{A} = \sum_{i=1}^{3} \bigoplus \mathcal{A}_i = \sum_{i=1}^{3} \bigoplus (Q_i\mathcal{A})$. Obviously, $\mathcal{A}_i$ is also a von Neumann algebra acting on $Q_i\mathcal{H}$. For each element A in $\mathcal{A}$, we write $A = \sum_{i=1}^{3} A_i = \sum_{i=1}^{3} Q_iA$.

We divide our proof into two cases.

Case 1 $ker(G) = \{0\}$.

Since $Q_1 \le P$, we have $\overline{ranG_1} = \overline{ranQ_1G} = Q_1\mathcal{H}$. Since G is injective on $\mathcal{H}$, $G_1 = Q_1G$ is also injective on $Q_1\mathcal{H}$. Hence G_1 is a separating point (both right and left) in $\mathcal{A}_1$. By Lemma 2.4, G_1 is a full-centralizable point in $\mathcal{A}_1$.

Since $Q_2 \le I - P$, we have $G_2 = Q_2G = 0$. By Lemma 2.5, G_2 is a full-centralizable point in $\mathcal{A}_2$.

Note that $\overline{ranG_3} = \overline{ranQ_3G} = Q_3P = P_3$. Denote the central carrier of P_3 in $\mathcal{A}_3$ by $\mathcal{C}_{\mathcal{A}_3}(P_3)$. We have $Q_3 - \mathcal{C}_{\mathcal{A}_3}(P_3) \le Q_3 - P_3 = Q_3(I-P) \le I - P$. Obviously, $Q_3 - \mathcal{C}_{\mathcal{A}_3}(P_3)$ is a central projection orthogonal to Q_2, so $Q_3 - \mathcal{C}_{\mathcal{A}_3}(P_3) + I - \mathcal{C}(P) \le I - P$. That is $Q_3 - \mathcal{C}_{\mathcal{A}_3}(P_3) + P \le \mathcal{C}(P)$. It implies that $Q_3 - \mathcal{C}_{\mathcal{A}_3}(P_3) = 0$, i.e., $\mathcal{C}_{\mathcal{A}_3}(P_3) = Q_3$. Similarly, we can prove $\mathcal{C}_{\mathcal{A}_3}(Q_3 - P_3) = Q_3$. By Lemma 2.6, G_3 is a full-centralizable point in $\mathcal{A}_3$.

By Lemma 2.2, G is a full-centralizable point.

Case 2 $ker(G) \neq \{0\}$.

In this case, G_2 and G_3 are still full-centralizable points. Since $\overline{ranG_1} = Q_1H$, we have $ker(G_1^*) = \{0\}$. By Case 1, G_1^* is a full-centralizable point in $\mathcal{A}_1$. By Lemma 2.3, G_1 is also a full-centralizable point in $\mathcal{A}_1$.

By Lemma 2.2, G is a full-centralizable point. $\square$

3. Centralizers on triangular algebras

In this section, we characterize the full-centralizable points on triangular algebras. The following theorem is our main result.

Theorem 3.1. *Let $\mathcal{J} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ be a triangular algebra, where $\mathcal{A}$ and $\mathcal{B}$ are two unital Banach algebras. Then every G in $\mathcal{J}$ is a full-centralizable point.*

Proof. Let $\phi : \mathcal{J} \to \mathcal{J}$ be a centralizable mapping at G.

Since ϕ is linear, for every $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right]$ in $\mathcal{J}$, we write

$$\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(X) + g_{11}(Y) + h_{11}(Z) & f_{12}(X) + g_{12}(Y) + h_{12}(Z) \\ 0 & f_{22}(X) + g_{22}(Y) + h_{22}(Z) \end{bmatrix},$$

where $f_{11} : \mathcal{A} \to \mathcal{A}$, $f_{12} : \mathcal{A} \to \mathcal{M}$, $f_{22} : \mathcal{A} \to \mathcal{B}$, $g_{11} : \mathcal{M} \to \mathcal{A}$, $g_{12} : \mathcal{M} \to \mathcal{M}$, $g_{22} : \mathcal{M} \to \mathcal{B}$, $h_{11} : \mathcal{B} \to \mathcal{A}$, $h_{12} : \mathcal{B} \to \mathcal{M}$, $h_{22} : \mathcal{B} \to \mathcal{B}$, are all linear mappings.

In the following, we denote the units of $\mathcal{A}$ and $\mathcal{B}$ by I_1 and I_2, respectively. We write $G = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ and

(3.1)
$$\phi \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \begin{bmatrix} f_{11}(A) + g_{11}(M) + h_{11}(B) & f_{12}(A) + g_{12}(M) + h_{12}(B) \\ 0 & f_{22}(A) + g_{22}(M) + h_{22}(B) \end{bmatrix}.$$

We divide our proof into several steps.

Claim 1 $f_{12} = f_{22} = 0$.

Let $S = \left[\begin{smallmatrix} X & M \\ 0 & B \end{smallmatrix}\right]$ and $T = \begin{bmatrix} X^{-1}A & 0 \\ 0 & I_2 \end{bmatrix}$, where X is an invertible element in $\mathcal{A}$. Since $ST = G$, we have

$$\phi(G) = \phi(S)T$$
$$= \begin{bmatrix} f_{11}(X)+g_{11}(M)+h_{11}(B) & f_{12}(X)+g_{12}(M)+h_{12}(B) \\ 0 & f_{22}(X)+g_{22}(M)+h_{22}(B) \end{bmatrix} \begin{bmatrix} X^{-1}A & 0 \\ 0 & I_2 \end{bmatrix}$$
$$(3.2) \quad = \begin{bmatrix} * & f_{12}(X)+g_{12}(M)+h_{12}(B) \\ 0 & f_{22}(X)+g_{22}(M)+h_{22}(B) \end{bmatrix}.$$

By comparing (3.1) with (3.2), we obtain $f_{12}(X) = f_{12}(A)$ and $f_{22}(X) = f_{22}(A)$ for each invertible element X in $\mathcal{A}$. Noting that A is a fixed element, for any nonzero element λ in $\mathbb{C}$, we have $f_{12}(\lambda X) = f_{12}(A) = \lambda f_{12}(X) = \lambda f_{12}(A)$. It follows that $f_{12}(X) = 0$ for each invertible element X. Thus $f_{12}(X) = 0$ for all X in $\mathcal{A}$. Similarly, we can obtain $f_{22}(X) = 0$.

Claim 2 $h_{12} = h_{11} = 0$.

Let $S = \begin{bmatrix} I_1 & 0 \\ 0 & BZ^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} A & M \\ 0 & Z \end{bmatrix}$, where Z is an invertible element in $\mathcal{B}$. Since $ST = G$, we have

$$\phi(G) = S\phi(T)$$
$$= \begin{bmatrix} I_1 & 0 \\ 0 & BZ^{-1} \end{bmatrix} \begin{bmatrix} f_{11}(A)+g_{11}(M)+h_{11}(Z) & f_{12}(A)+g_{12}(M)+h_{12}(Z) \\ 0 & f_{22}(A)+g_{22}(M)+h_{22}(Z) \end{bmatrix}$$
$$(3.3) \quad = \begin{bmatrix} f_{11}(A)+g_{11}(M)+h_{11}(Z) & f_{12}(A)+g_{12}(M)+h_{12}(Z) \\ 0 & * \end{bmatrix}.$$

By comparing (3.1) with (3.3), we obtain $h_{12}(Z) = h_{12}(B)$ and $h_{11}(Z) = h_{11}(B)$ for each invertible element Z in $\mathcal{B}$. Similarly as the previous discussion, we can obtain $h_{12}(Z) = h_{11}(Z) = 0$ for all Z in $\mathcal{B}$.

Claim 3 $g_{22} = g_{11} = 0$.

For every Y in $\mathcal{M}$, we set $S = \begin{bmatrix} I_1 & M-Y \\ 0 & B \end{bmatrix}$, $T = \begin{bmatrix} A & Y \\ 0 & I_2 \end{bmatrix}$. Obviously, $ST = G$. Thus we have

$$\phi(G) = \phi(S)T$$
$$= \begin{bmatrix} * & * \\ 0 & f_{22}(I_1)+g_{22}(M-Y)+h_{22}(B) \end{bmatrix} \begin{bmatrix} A & Y \\ 0 & I_2 \end{bmatrix}$$
$$(3.4) \quad = \begin{bmatrix} * & * \\ 0 & f_{22}(I_1)+g_{22}(M-Y)+h_{22}(B) \end{bmatrix}.$$

By comparing (3.1) with (3.4), we obtain $f_{22}(I_1) + g_{22}(M - Y) + h_{22}(B) = f_{22}(A) + g_{22}(M) + h_{22}(B)$. Hence $g_{22}(Y) = f_{22}(I_1 - A)$. It means $g_{22}(Y) = 0$ immediately.

On the other hand,

$$\phi(G) = S\phi(T)$$
$$= \begin{bmatrix} I_1 & M-Y \\ 0 & B \end{bmatrix} \begin{bmatrix} f_{11}(A)+g_{11}(Y)+h_{11}(I_2) & * \\ 0 & * \end{bmatrix}$$
$$(3.5) \quad = \begin{bmatrix} f_{11}(A)+g_{11}(Y)+h_{11}(I_2) & * \\ 0 & * \end{bmatrix}.$$

By comparing (3.1) with (3.5), we obtain $g_{11}(Y) = g_{11}(M) + h_{11}(B - I_2)$. Hence $g_{11}(Y) = 0$.

According to the above three claims, we obtain that

$$\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(X) & g_{12}(Y) \\ 0 & h_{22}(Z) \end{bmatrix}$$

for every $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right]$ in $\mathcal{J}$.

Claim 4 $f_{11}(X) = f_{11}(I_1)X$ for all X in $\mathcal{A}$, and $g_{12}(Y) = f_{11}(I_1)Y$ for all Y in $\mathcal{M}$.

Let $S = \left[\begin{smallmatrix} X & M-XY \\ 0 & B \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} X^{-1}A & Y \\ 0 & I_2 \end{smallmatrix}\right]$, where X is an invertible element in $\mathcal{A}$, and Y is an arbitrary element in $\mathcal{M}$. Since $ST = G$, we have

$$\begin{aligned} \phi(G) &= \phi(S)T \\ &= \begin{bmatrix} f_{11}(X) & g_{12}(M - XY) \\ 0 & h_{22}(B) \end{bmatrix} \begin{bmatrix} X^{-1}A & Y \\ 0 & I_2 \end{bmatrix} \\ &= \begin{bmatrix} * & f_{11}(X)Y + g_{12}(M - XY) \\ 0 & * \end{bmatrix} \\ &= \begin{bmatrix} f_{11}(A) & g_{12}(M) \\ 0 & h_{22}(B) \end{bmatrix}. \end{aligned} \tag{3.6}$$

So we have $f_{11}(X)Y = g_{12}(XY)$. It follows that

$$g_{12}(Y) = f_{11}(I_1)Y \tag{3.7}$$

by taking $X = I_1$. Replacing Y in (3.7) with XY, we can obtain $g_{12}(XY) = f_{11}(I_1)XY = f_{11}(X)Y$ for each invertible element X in $\mathcal{A}$ and Y in $\mathcal{M}$. Since $\mathcal{M}$ is faithful, we have

$$f_{11}(X) = f_{11}(I_1)X \tag{3.8}$$

for all invertible elements X and so for all elements in $\mathcal{A}$.

Claim 5 $h_{22}(Z) = Zh_{22}(I_2)$ for all Z in $\mathcal{B}$, and $g_{12}(Y) = Yh_{22}(I_2)$ for all Y in $\mathcal{M}$.

Let $S = \left[\begin{smallmatrix} I_1 & Y \\ 0 & BZ^{-1} \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} A & M-YZ \\ 0 & Z \end{smallmatrix}\right]$, where Z is an invertible element in $\mathcal{B}$, and Y is an arbitrary element in $\mathcal{M}$. Since $ST = G$, we have

$$\begin{aligned} \phi(G) &= S\phi(T) \\ &= \begin{bmatrix} I_1 & Y \\ 0 & BZ^{-1} \end{bmatrix} \begin{bmatrix} f_{11}(A) & g_{12}(M - YZ) \\ 0 & h_{22}(Z) \end{bmatrix} \\ &= \begin{bmatrix} * & g_{12}(M - YZ) + Yh_{22}(Z) \\ 0 & * \end{bmatrix} \\ &= \begin{bmatrix} f_{11}(A) & g_{12}(M) \\ 0 & h_{22}(B) \end{bmatrix}. \end{aligned} \tag{3.9}$$

So we have $g_{12}(YZ) = Yh_{22}(Z)$. Through a similar discussion as the proof of Claim 4, we obtain $h_{22}(Z) = Zh_{22}(I_2)$ for all Z in $\mathcal{B}$ and $g_{12}(Y) = Yh_{22}(I_2)$ for all Y in $\mathcal{M}$.

Thus we have that

$$\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(I_1)X & f_{11}(I_1)Y \\ 0 & Zh_{22}(I_2) \end{bmatrix} = \begin{bmatrix} f_{11}(I_1)X & Yh_{22}(I_2) \\ 0 & Zh_{22}(I_2) \end{bmatrix}$$

for every $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right]$ in $\mathcal{J}$. So it is sufficient to show that $f_{11}(I_1)X = Xf_{11}(I_1)$ for all X in $\mathcal{A}$, and $h_{22}(I_2)Z = Zh_{22}(I_2)$ for all Z in $\mathcal{B}$. Since $f_{11}(I_1)Y = Yh_{22}(I_2)$ for all Y in $\mathcal{M}$, we have $f_{11}(I_1)XY = XYh_{22}(I_2) = Xf_{11}(I_1)Y$. It implies that $f_{11}(I_1)X = Xf_{11}(I_1)$. Similarly, $h_{22}(I_2)Z = Zh_{22}(I_2)$. Now we can obtain that $\phi(J) = \phi(I)J = J\phi(I)$ for all J in $\mathcal{J}$, where $I = \left[\begin{smallmatrix} I_1 & 0 \\ 0 & I_2 \end{smallmatrix}\right]$ is the unit of $\mathcal{J}$. Hence, G is a full-centralizable point. □

As applications of Theorem 3.1, we have the following corollaries.

Corollary 3.2. *Let $\mathcal{A}$ be a nest algebra on a Hilbert space $\mathcal{H}$. Then every element in $\mathcal{A}$ is a full-centralizable point.*

Proof. If $\mathcal{A} = B(\mathcal{H})$, then the result follows from Theorem 2.1. Otherwise, $\mathcal{A}$ is isomorphic to a triangular algebra. By Theorem 3.1, the result follows. □

Corollary 3.3. *Let $\mathcal{A}$ be a CDCSL* (*completely distributive commutative subspace lattice*) *algebra on a Hilbert space $\mathcal{H}$. Then every element in $\mathcal{A}$ is a full-centralizable point.*

Proof. It is known that $\mathcal{A} \cong \sum_{i\in\Lambda} \bigoplus \mathcal{A}_i$, where each $\mathcal{A}_i$ is either $B(\mathcal{H}_i)$ for some Hilbert space $\mathcal{H}_i$ or a triangular algebra $Tri(\mathcal{B}, \mathcal{M}, \mathcal{C})$ such that the conditions of Theorem 3.1 hold (see in [8] and [15]). By Lemma 2.2, the result follows. □

Remark. For the definition of a CDCSL algebra, we refer to [5].

4. Derivations on von Neumann algebras

In this section, we characterize the derivable mappings at a given point in a von Neumann algebra.

Lemma 4.1. *Let $\mathcal{A}$ be a von Neumann algebra. Suppose $\Delta : \mathcal{A} \to \mathcal{A}$ is a linear mapping such that $\Delta(A)B + A\Delta(B) = 0$ for each A and B in $\mathcal{A}$ with $AB = 0$. Then $\Delta = D + \phi$, where $D : \mathcal{A} \to \mathcal{A}$ is a derivation, and $\phi : \mathcal{A} \to \mathcal{A}$ is a centralizer. In particular, Δ is bounded.*

Proof. **Case 1.** $\mathcal{A}$ is an abelian von Neumann algebra. In this case, $\mathcal{A} \cong C(\mathcal{X})$ for some compact Hausdorff space $\mathcal{X}$. If $AB = 0$, then the supports of A and B are disjoint. So the equation $\Delta(A)B + A\Delta(B) = 0$ implies that $\Delta(A)B = A\Delta(B) = 0$. By Lemma 2.5, Δ is a centralizer.

Case 2. $\mathcal{A} \cong M_n(\mathcal{B})(n \geq 2)$, where $\mathcal{B}$ is also a von Neumann algebra. By [1, Theorem 2.3], Δ is a generalized derivation with $\Delta(I)$ in the center. That is to say, Δ is a sum of a derivation and a centralizer.

For general cases, we know $\mathcal{A} \cong \sum_{i=1}^{n} \bigoplus \mathcal{A}_i$, where each $\mathcal{A}_i$ coincides with either Case 1 or Case 2. We write $A = \sum_{i=1}^{n} A_i$ with $A_i \in \mathcal{A}_i$ and denote the restriction of Δ in $\mathcal{A}_i$ by Δ_i. It is not difficult to check that $\Delta(A_i) \in \mathcal{A}_i$. Moreover, setting $A_iB_i = 0$, we have $\Delta(A_i)B_i + A_i\Delta(B_i) = \Delta_i(A_i)B_i + A_i\Delta_i(B_i) = 0$. By Case 1 and Case 2, each Δ_i is a sum of a derivation and a centralizer. Hence, $\Delta = \sum_{i=1}^{n} \Delta_i$ is a sum of a derivation and a centralizer. $\square$

Remark. In [10], the authors prove that for a prime semisimple Banach algebra $\mathcal{A}$ with nontrival idempotents and a linear mapping Δ from $\mathcal{A}$ into itself, the condition $\Delta(A)B + A\Delta(B) = 0$ for each A and B in $\mathcal{A}$ with $AB = 0$ implies that Δ is bounded. By Lemma 4.1, we have that for a von Neumann algebra $\mathcal{A}$, the result holds still even if $\mathcal{A}$ is not prime.

Now we prove our main result in this section.

Theorem 4.2. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and G be a given point in $\mathcal{A}$. If $\Delta : \mathcal{A} \to \mathcal{A}$ is a linear mapping derivable at G, then $\Delta = D + \phi$, where D is a derivation, and ϕ is a centralizer. Moreover, G is a full-derivable point if and only if $\mathcal{C}(G) = I$.*

Proof. Suppose the range projection of G is P. We note that $\mathcal{C}(G) = \mathcal{C}(P)$.

Set $Q_1 = I - \mathcal{C}(I - P)$, $Q_2 = I - \mathcal{C}(P)$, and $Q_3 = I - Q_1 - Q_2$. Then we have $\mathcal{A} = \sum_{i=1}^{3} \bigoplus \mathcal{A}_i = \sum_{i=1}^{3} \bigoplus (Q_i\mathcal{A})$. For every A in $\mathcal{A}$, we write $A = \sum_{i=1}^{3} A_i = \sum_{i=1}^{3} Q_iA$.

For any central projection Q, setting $Q^{\perp} = I - Q$, we have

$$(Q^{\perp} + t^{-1}QGA^{-1})(Q^{\perp}G + tQA) = G,$$

where A is an arbitrary invertible element in $\mathcal{A}$, and t is an arbitrary nonzero element in $\mathbb{C}$. So we obtain

$$\Delta(G) = (Q^{\perp}+t^{-1}QGA^{-1})\Delta(Q^{\perp}G+tQA)+\Delta(Q^{\perp}+t^{-1}QGA^{-1})(Q^{\perp}G+tQA).$$

Considering the coefficient of t, we obtain $Q^{\perp}\Delta(QA) + \Delta(Q^{\perp})(QA) = 0$. Since the ranges of Q and $Q^{\perp}$ are disjoint, it follows that $Q^{\perp}\Delta(QA) = 0$ and so $\Delta(QA) \in Q\mathcal{A}$. Since Q_i are central projections, we have $\Delta(\mathcal{A}_i) \subseteq \mathcal{A}_i$.

Denote the restriction of Δ to $\mathcal{A}_i$ by Δ_i. Setting $A_iB_i = G_i$, it is not difficult to check that $\Delta_i(G_i) = \Delta(A_i)B_i + A_i\Delta(B_i)$.

Since $Q_1 \leq P$, we have $\overline{ranG_1} = \overline{ranQ_1G} = Q_1H$. So G_1 is a right separating point in $\mathcal{A}_1$. By [13, Corallary 2.5], Δ_1 is a Jordan derivation and so is a derivation on $\mathcal{A}_1$.

Since $Q_2 \leq I - P$, we have $G_2 = Q_2G = 0$. By Lemma 4.1, Δ_2 is a sum of a derivation and a centralizer on $\mathcal{A}_2$.

Note that $\overline{ranG_3} = \overline{ranQ_3G} = Q_3P = P_3$. As we proved before, $\mathcal{C}_{\mathcal{A}_3}(P_3) = \mathcal{C}_{\mathcal{A}_3}(Q_3 - P_3) = Q_3$. So by [16, Theorem 3.1], Δ_3 is a derivation on $\mathcal{A}_3$.

Hence, $\Delta = \sum_{i=1}^{3} \Delta_i$ is a sum of a derivation and a centralizer.

If $\mathcal{C}(G) = I$, then $Q_2 = 0$, $\mathcal{A} = \mathcal{A}_1 \bigoplus \mathcal{A}_3$ and $G = G_1 + G_3$ is a full-derivable point. If $\mathcal{C}(G) \neq I$, then $Q_2 \neq 0$. Define a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = A_2$ for all $A \in \mathcal{A}$. One can check that δ is not a derivation but derivable at G. Thus G is not a full-derivable point. □

As an application, we obtain the following corollary.

Corollary 4.3. *Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ is a factor if and only if every nonzero element G in $\mathcal{A}$ is a full-derivable point.*

Proof. If $\mathcal{A}$ is a factor, for each nonzero element G in $\mathcal{A}$, we know that $\mathcal{C}(G) = I$. By Theorem 4.2, G is a full-derivable point.

If $\mathcal{A}$ is not a factor, then there exists a nontrival central projection P. Define a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = (I - P)A$ for all $A \in \mathcal{A}$. One can check that δ is not a derivation but derivable at P. Thus P is not a full-derivable point. □

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