J. Korean Math. Soc. **54** (2017), No. 2, pp. 675–684
https://doi.org/10.4134/JKMS.j160234
pISSN: 0304-9914 / eISSN: 2234-3008

ON GRADED 2-ABSORBING PRIMARY AND GRADED WEAKLY 2-ABSORBING PRIMARY IDEALS

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Abstract. Let G be a group with identity e and let R be a G-graded ring. In this paper, we introduce and study graded 2-absorbing primary and graded weakly 2-absorbing primary ideals of a graded ring which are different from 2-absorbing primary and weakly 2-absorbing primary ideals. We give some properties and characterizations of these ideals and their homogeneous components.

1. Introduction and preliminaries

Throughout this paper, all rings are assumed to be commutative with identity elements. The concept of 2-absorbing primary ideal was introduced in [9] as a generalization of the notion of primary ideal. Let R be a ring. A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever a, b, $c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Weakly primary ideals are also generalizations of primary ideals. Recall from [6] that a proper ideal I of R is called a weakly primary ideal if whenever $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. The concept of weakly primary ideal was generalized to the concept of weakly 2-absorbing primary ideal in [10]. A proper ideal I of R is said to be a weakly 2-absorbing primary ideal of R if whenever $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Graded primary ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [16]. Graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative graded rings have been introduced and studied by Al-Zoubi, R. Abu-Dawwas and S. Ceken in [1]. We like to point out that the concept of weakly prime ideals was initiated by Anderson and Smith in [3] (Recall from [3] that a proper ideal I of R is called weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$) and Badawi in [7] introduced the concept of 2-absorbing ideals of commutative rings that is a generalization of the concept of prime ideals (recall from [7] that a proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$

Received April 7, 2016; Revised August 3, 2016.
2010 *Mathematics Subject Classification.* 13A02, 16W50, 13C11.
Key words and phrases. graded 2-absorbing primary ideal, graded weakly 2-absorbing primary ideal, graded weakly primary ideal, graded primary ideal.

and $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$). In this paper, we introduce and study graded 2-absorbing primary and graded weakly 2-absorbing primary ideals of graded rings.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [12] and [13] for these basic properties and more information on graded rings and modules. Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e., $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is called homogeneous component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Also, if $r \in R_g$ and r is a unit, then $r^{-1} \in R_{g-1}$. A G-graded ring $R = \oplus_{g\in G} R_g$ is called a crossed product if R_g contains a unit for every $g \in G$. Note that a G-crossed product $R = \oplus_{g\in G} R_g$ is a strongly graded ring (see [13, Remark 1.1.2.]). Let $R = \oplus_{g\in G} R_g$ be a G-graded ring and. An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g) := \oplus_{g\in G} I_g$. If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \oplus_{g\in G} \oplus (R/I)_g$ where $(R/I)_g = \{x+I : x \in R_g\}$. Let R_1 and R_2 be G-graded rings and $R = R_1 \times R_2$. Then R is a G-graded ring with $h(R) = h(R_1) \times h(R_2)$. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called graded ring of fractions. Indeed, $S^{-1}R = \oplus_{g\in G} \oplus (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (\deg s)^{-1}(\deg r)\}$. Let $R = \oplus_{g\in G} \oplus R_g$ be a G-graded ring. A right R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $M_g R_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called homogeneous. Note that M_g is an R_e-module for every $g \in G$. So, if $I = \oplus_{g\in G} I_g$ is a graded ideal of R, then I_g is an R_e-module for every $g \in G$. Let R be a G-graded ring. The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$. A proper graded ideal P of R is said to be a graded primary (resp. graded weakly primary) ideal if whenever $r, s \in h(R)$ with $rs \in P$ (resp. $0 \neq rs \in P$), then either $r \in P$ or $s \in Gr(P)$ (see[4, 16]). A proper graded ideal I of R is said to be a graded 2-absorbing (resp. graded weakly 2-absorbing) ideal of R if whenever $r, s, t \in h(R)$ with $rst \in I$ (resp. $0 \neq rst \in I$), then $rs \in I$ or $rt \in I$ or $st \in I$ (see [1]).

In this article, we define graded (weakly) 2-absorbing primary ideals of a graded ring.

A proper graded ideal I of a graded ring R is said to be a graded 2-absorbing primary (resp. graded weakly 2-absorbing primary) ideal of R if whenever $r, s, t \in h(R)$ with $rst \in I$ (resp. $0 \neq rst \in I$), then $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$.

These ideals are generalizations of (weakly) 2-absorbing primary ideals in a graded ring. But we show that the set of all graded 2-absorbing primary ideals and the set of all 2-absorbing primary graded ideals need not to be equal in a graded ring (see Example 2.2(i)). According to our definition, every graded primary ideal is a graded 2-absorbing primary ideal. But we show that not every graded 2-absorbing primary ideal is a graded primary ideal (see Example 2.2(i)). Also, every graded 2-absorbing ideal is a graded 2-absorbing primary ideal. But we show that not every graded 2-absorbing primary ideal is a graded 2-absorbing ideal (see Example 2.2(ii)). Various properties of graded (weakly) 2-absorbing ideals and their homogeneous components are considered. Note that every graded 2-absorbing primary ideal is clearly a graded weakly 2-absorbing primary ideal. However, the converse is not true. For example, 0 is always a graded weakly 2-absorbing primary ideal of R, but it is not always a graded 2-absorbing primary ideal.

2. Graded 2-absorbing primary ideals

Definition 2.1. Let $R = \oplus_{g \in G} R_g$ be a G-graded ring, $I = \oplus_{g \in G} I_g$ be a graded ideal of R and $g \in G$.

(i) We say that I is a g-2-absorbing primary ideal of R if $I_g \neq R_g$ and whenever $r, s, t \in R_g$ with $rst \in I$, then $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$.

(ii) We say that I is a graded 2-absorbing primary ideal of R if $I \neq R$ and whenever $r, s, t \in h(R)$ with $rst \in I$, then $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$.

Example 2.2. Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$.

(i) Let $I = 6R$. Then I is not 2-absorbing primary ideal of R by [9, Corollary 2.12]). Also, I is not a graded primary ideal of R. Because $2, 3 \in R_0 \subseteq h(R)$ and $2 \cdot 3 \in I$ but $2 \notin I$ and $3 \notin Gr(I)$. However an easy computation shows that I is a graded 2-absorbing primary ideal of R.

(ii) Let $J = 12R$. Then J is not a graded 2-absorbing ideal of R. Because $2, 3 \in R_0 \subseteq h(R)$ and $2 \cdot 2 \cdot 3 \in J$ but $2 \cdot 2 \notin J$ and $2 \cdot 3 \notin J$. However, it is clear that J is a graded 2-absorbing primary ideal of R.

The following result is an analogue of [9, Theorem 2.2].

Theorem 2.3. *Let R be a G-graded ring. If I is a graded 2-absorbing primary ideal of R. Then $Gr(I)$ is a graded 2-absorbing ideal of R.*

Proof. Let $r, s, t \in h(R)$ such that $rst \in Gr(I)$, $rt \notin Gr(I)$ and $st \notin Gr(I)$. Since $rst \in Gr(I)$, there exists a positive integer n such that $(rst)^n = r^n s^n t^n \in I$. Since I is a graded 2-absorbing primary ideal, $rt \notin Gr(I)$ and $st \notin Gr(I)$, we conclude that $r^n s^n = (rs)^n \in I$ and so $rs \in Gr(I)$. Thus $Gr(I)$ is a graded 2-absorbing primary ideal of R. □

The following result is an analogue of [9, Theorem 2.8].

Theorem 2.4. *Let R be a G-graded ring and I be a proper graded ideal of R. If $Gr(I)$ is a graded prime ideal of R, then I is a graded 2-absorbing primary ideal of R.*

Proof. Let $rst \in I$ and $rs \notin I$ for some $r, s, t \in h(R)$. Since $(rt)(st) = rst^2 \in I \subseteq Gr(I)$ and $Gr(I)$ is a graded prime ideal of R, we have $st \in Gr(I)$ or $rt \in Gr(I)$. Therefore I is a graded 2-absorbing primary ideal of R. □

In [11], the concept of 2-absorbing primary ideal of a ring was extended to the notion of 2-absorbing primary submodule of a module. Let R be a ring and M be an R-module. A proper submodule N of M is called 2-absorbing primary, if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$ or $ab \in (N :_R M)$.

Theorem 2.5. *Let R be a G-graded ring and $I = \oplus_{g \in G} I_g$ be a graded ideal of R. Then the following hold.*

(i) *If I is a graded 2-absorbing primary ideal of R, then I_g is a 2-absorbing primary submodule of the R_e-module R_g for every $g \in G$ with $I_g \neq R_g$.*
(ii) *If R is a crossed product and I_e is a 2-absorbing primary ideal of R_e, then I is a graded 2-absorbing primary ideal of R.*

Proof. (i) Let $g \in G$ and $I_g \neq R_g$. Assume that $a, b \in R_e$ and $m \in R_g$ with $abm \in I_g$. Since I is a graded 2-absorbing primary ideal of R, we have either $ab \in I$ or $am \in Gr(I)$ or $bm \in Gr(I)$. If $ab \in I$, then $ab \in (I_g :_{R_e} R_g)$. If $bm \in Gr(I)$ or $am \in Gr(I)$, then $bm \in Gr(I_g)$ or $am \in Gr(I_g)$, respectively. Therefore, I_g is a 2-absorbing primary R_e-submodule of R_g.

(ii) Clearly, $I \neq R$. First we show that if I_e is a 2-absorbing primary ideal of R_e, then I_g is a 2-absorbing primary submodule of the R_e-module R_g for every $g \in G$. Let $g \in G$. If $I_g = R_g$, then it can be easily seen that $I_e = R_e$, a contradiction. So $I_g \neq R_g$. Let $a, b \in R_e$, $m \in R_g$ such that $abm \in I_g$. Let u be a unit in $R_{g^{-1}}$. Then $ab(mu) \in I_e$. Since I_e is a 2-absorbing primary ideal of R_e, we have $ab \in I_e$ or $a(mu) \in Gr(I_e)$ or $b(mu) \in Gr(I_e)$. If $ab \in I_e$, then $ab \in (I_g :_{R_e} R_g)$. If $a(mu) \in Gr(I_e)$ or $b(mu) \in Gr(I_e)$, then $am \in Gr(I)$ or $bm \in Gr(I)$, respectively. Now, let $a, b, c \in h(R)$ with $abc \in I$. There exist $g, h, \lambda \in G$ such that $a \in R_g$, $b \in R_h$ and $c \in R_\lambda$. Also, $R_{g^{-1}}$ contains a unit, say a' and $R_{h^{-1}}$ contains a unit, say b'. Thus $(aa')(bb')c \in I_\lambda$. Since I_λ is a 2-absorbing primary submodule of the R_e-module R_λ, we have $(aa')c \in Gr(I_\lambda)$ or $(bb')c \in Gr(I_\lambda)$ or $(aa')(bb') \in (I_\lambda :_{R_e} R_\lambda)$. If $(aa')c \in Gr(I_\lambda)$ or $(bb')c \in Gr(I_\lambda)$, then $ac \in Gr(I)$ or $bc \in Gr(I)$, respectively.

If $(aa')(bb') \in (I_\lambda :_{R_e} R_\lambda)$, then $ab(a'b')R_\lambda \in I_\lambda$. Since R is strongly graded, $ab(a'b') \in I_e$, we conclude that t $ab \in I$. Therefore I is a graded 2-absorbing primary ideal of R. □

For G-graded rings R and R', a G-graded ring homomorphism $\varphi : R \to R'$ is a ring homomorphism such that $\varphi(R_g) \subseteq R'_g$ for every $g \in G$. The following result is an analogue of [9, Theorem 2.20].

Theorem 2.6. *Let R and R' be two G-graded rings and $\varphi : R \to R'$ be G-graded ring homomorphism. Then the following statements hold*:

(i) *If I' is a graded 2-absorbing primary ideal of R', then $\varphi^{-1}(I')$ is a graded 2-absorbing primary ideal of R.*

(ii) *If φ is a graded epimorphism and I is a graded 2-absorbing primary ideal of R containing $ker(\varphi)$, then $\varphi(I)$ is a graded 2-absorbing primary ideal of R'.*

Proof. (i) Suppose that I' is a graded 2-absorbing primary ideal of R' and let $r, s, t \in h(R)$ such that $rst \in \varphi^{-1}(I')$. Then $\varphi(rst) = \varphi(r)\varphi(s)\varphi(t) \in I'$. Since I' is a graded 2-absorbing primary ideal of R, $\varphi(r)\varphi(s) \in I'$ or $\varphi(s)\varphi(t) \in Gr(I')$ or $\varphi(r)\varphi(t) \in Gr(I')$, and hence $rs \in \varphi^{-1}(I')$ or $st \in \varphi^{-1}(Gr(I')) = Gr(\varphi^{-1}(I')$ or $rt \in \varphi^{-1}(Gr(I')) = Gr(\varphi^{-1}(I'))$. Therefore $\varphi^{-1}(I')$ is a graded 2-absorbing primary ideal of R.

(ii) Suppose that I is a graded 2-absorbing primary ideal of R containing $ker(\varphi)$ and let $r', s', t' \in h(R')$ such that $r's't' \in \varphi(I)$. Since φ is a graded epimorphism, there exist $r, s, t \in h(R)$ such that $\varphi(r) = r'$, $\varphi(s) = s'$, $\varphi(t) = t'$ and $\varphi(rst) = r's't' \in \varphi(I)$. Since $Kerf \subseteq I$, we have $rst \in I$. Since I is a graded 2-absorbing primary ideal of R, we have $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$. So $r's' \in \varphi(I)$ or $r't' \in \varphi(Gr(I)) \subseteq Gr(\varphi(I))$ or $s't' \in \varphi(Gr(I)) \subseteq Gr(\varphi(I))$. Therefore $\varphi(I)$ is a graded 2-absorbing primary ideal of R'. □

Let I be a proper graded ideal of G-graded ring R. Then $G\text{-}Z_I(R) = \{r \in h(R) \,|\, rs \in I$ for some $s \in h(R) - h(I)\}$.

The following result is an analogue of [9, Theorem 2.22].

Theorem 2.7. *Let R be a G-graded ring, $S \subseteq h(R)$ be a multiplicatively closed subset of R, and I be a proper graded ideal of R. Then the following hold*:

(i) *If I is a graded 2-absorbing primary ideal of R such that $I \cap S = \phi$, then $S^{-1}I$ is a graded 2-absorbing primary ideal of $S^{-1}R$.*

(ii) *If $S^{-1}I$ is a graded 2-absorbing primary ideal of $S^{-1}R$ and $S \cap G\text{-}Z_I(R) = \phi$, then I is a graded 2-absorbing primary ideal of R.*

Proof. (i) Suppose that $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{r_3}{s_3} \in S^{-1}I$ for some $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in h(S^{-1}R)$. Then there exists $v \in S$ such that $vr_1r_2r_3 \in I$. Since I is a graded 2-absorbing primary ideal, we conclude that either $vr_1r_2 \in I$ or $r_2r_3 \in G_r(I)$ or $vr_1r_3 \in G_r(I)$. If $vr_1r_2 \in I$, then $\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{vr_1r_2}{vs_1s_2} \in S^{-1}I$. If $r_2r_3 \in G_r(I)$, then $\frac{r_2}{s_2}\frac{r_3}{s_3} \in S^{-1}Gr(I) = Gr(S^{-1}I)$. If $vr_1r_3 \in G_r(I)$, then $\frac{r_1}{s_1}\frac{r_3}{s_3} = \frac{vr_1r_3}{vs_1s_3} \in Gr(S^{-1}I)$.

(ii) Suppose that $r_1r_2r_3 \in I$ for some $r_1, r_2, r_3 \in h(R)$. Then $\frac{r_1r_2r_3}{1} = \frac{r_1}{1}\frac{r_2}{1}\frac{r_3}{1} \in S^{-1}I$. Since $S^{-1}I$ is a graded 2-absorbing primary ideal of $S^{-1}R$, we conclude that either $\frac{r_1}{1}\frac{r_2}{1} \in S^{-1}I$ or $\frac{r_2}{1}\frac{r_3}{1} \in Gr(S^{-1}I)$ or $\frac{r_1}{1}\frac{r_3}{1} \in Gr(S^{-1}I)$. If $\frac{r_1}{1}\frac{r_2}{1} = \frac{r_1r_2}{1} \in S^{-1}I$, then $vr_1r_2 \in I$, for some $v \in S$. Since $v \in S$ and $S \cap G\text{-}Z_I(R) = \phi$, we have $r_1r_2 \in I$. If $\frac{r_2}{1}\frac{r_3}{1} = \frac{r_2r_3}{1} \in Gr(S^{-1}I) = S^{-1}Gr(I)$, then there exists $t \in S$ and $n \in \mathbb{Z}^+$ such that $(tr_2r_3)^n = t^n r_2^n r_3^n \in I$. Since $t \in S$, we have $t^n \notin G\text{-}Z_I(R)$. Thus $r_2^n r_3^n \in I$, and so $r_2r_3 \in Gr(I)$. With a same argument, we can show that If $\frac{r_1}{1}\frac{r_3}{1} \in Gr(S^{-1}I)$, then $r_1r_3 \in Gr(I)$. Therefore I is a graded 2-absorbing primary ideal of R. □

Lemma 2.8 ([16, Lemma 1.8])**.** *Let R be a G-graded ring and I be a graded primary ideal of R. Then $P = Gr(I)$ is a graded prime ideal of R, and we say that I is a graded P-primary ideal of R.*

The following result is an analogue of [9, Theorem 2.4(2)].

Lemma 2.9. *Let R be a G-graded ring. Suppose that I_1 is a graded P_1-primary ideal of R for some graded prime ideal P_1 of R and I_2 is a graded P_2-primary ideal of R for some graded prime ideal P_2 of R. Then $I_1 \cap I_2$ is a graded 2-absorbing primary ideal of R.*

Proof. Let $J = I_1 \cap I_2$. Then $Gr(J) = P_1 \cap P_2$. Suppose that $rst \in J$, $rt \notin Gr(J)$ and $st \notin Gr(J)$ for some $r, s, t \in h(R)$. Then $r, s, t \notin Gr(J) = P_1 \cap P_2$. Since $Gr(J) = P_1 \cap P_2$, we conclude that $Gr(J)$ is a graded 2-absorbing ideal of R. Since $Gr(J) = P_1 \cap P_2$ is a graded 2-absorbing ideal of R and $rt, st \notin Gr(J)$, we have $rs \in Gr(J)$. We show that $rs \in J$. Since $rs \in Gr(J) \subseteq P_1$, we may assume that $r \in P_1$. Since $r \notin Gr(J)$ and $rs \in Gr(J) \subseteq P_2$, we conclude that $r \notin P_2$ and $s \in P_2$. Since $s \in P_2$ and $s \notin Gr(J)$, we have $s \notin P_1$. If $r \in I_1$ and $s \in I_2$, then $rs \in J$ and we are done. Assume that $r \notin I_1$. Since I_1 is a graded P_1-primary ideal of R and $r \notin I_1$, we have $st \in P_1$. Since $s \in P_2$ and $st \in P_1$, we have $st \in Gr(J)$, which is a contradiction. So $r \in I_1$. Similarly, assume that $s \notin I_2$. Since I_2 is a graded P_2-primary ideal of R and $s \notin I_2$, we have $rt \in P_2$. Since $rt \in P_2$ and $r \in P_1$, we have $rt \in Gr(J)$, which is a contradiction. So $s \in I_2$. Thus $rs \in J$. □

The following result is an analogue of [9, Theorem 2.23].

Theorem 2.10. *Let R_1 and R_2 be two graded rings and let I_1 and I_2 be a proper graded ideals of R_1 and R_2, respectively. Then the following statement are equivalent.*

(i) *$I_1 \times I_2$ is a graded 2-absorbing primary ideal of $R_1 \times R_2$.*
(ii) *I_1 and I_2 are graded primary ideals of R_1 and R_2, respectively.*

Proof. (i) Assume that $I_1 \times I_2$ is a graded 2-absorbing primary ideal of $R_1 \times R_2$. Suppose that I_1 is not a graded primary ideal of R_1. Then there are $r, s \in h(R)$ such that $rs \in I_1$ but neither $r \in I_1$ nor $s \in Gr(I_1)$. Let $x = (r, 1)$, $y = (1, 0)$ and $z = (s, 1)$. Then $xyz = (rs, 0) \in I_1 \times I_2$ but neither $xy = (r, 0) \in I_1 \times I_2$

nor $xz = (rs, 1) \in Gr(I_1 \times I_2)$ nor $yz = (s, 0) \in Gr(I_1 \times I_2)$, which is a contradiction. Thus I_1 is a graded primary ideal of R_1. Similarly, we can show that I_2 is a graded primary ideal of R_2.

(ii) Assume that I_1 and I_2 are graded primary ideals of R_1 and R_2, respectively. Then $I = I_1 \times R_2$ and $J = R_1 \times I_2$ are graded primary ideals of R. Hence $I \cap J = I_1 \times I_2$ is a graded 2-absorbing primary ideal of R by Lemma 2.9. □

3. Graded weakly 2-absorbing primary ideals

Definition 3.1. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring, $I = \oplus_{g\in G} I_g$ be a graded ideal of R and $g \in G$.

(i) We say that I is a weakly g-2-absorbing primary ideal of R if $I_g \neq R_g$ and whenever $r, s, t \in R_g$ with $0 \neq rst \in I$, then $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$.

(ii) We say that I is a graded weakly 2-absorbing primary ideal of R if $I \neq R$ and whenever $r, s, t \in h(R)$ with $0 \neq rst \in I$, then $rs \in I$ or $rt \in Gr(I\ )$ or $st \in Gr(I)$.

The following result is an analogue of [10, Theorem 2.10].

Theorem 3.2. *Let $R = \oplus_{g\in G} R_g$ be a graded ring and $I = \oplus_{g\in G} \oplus I_g$ be a graded weakly 2-absorbing primary ideal of R. Then, for each $g \in G$, either I is a g-2-absorbing primary ideal of R or $I_g^3 = (0)$.*

Proof. It is enough to show that if $I_g^3 \neq (0)$ for $g \in G$, then I is a g-2-absorbing primary ideal of R. Let $abc \in I$ where $a, b, c \in R_g$. If $0 \neq abc$, then $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$ by the hypothesis. So we may assume that $abc = 0$. Suppose first that $abI_g \neq (0)$, then there exists $i \in I_g$ such that $abi \neq 0$. Hence $0 \neq ab(c+i) = abi \in I$. Since I is a graded weakly 2-absorbing primary ideal of R, we have $ab \in I$ or $a(c + i) \in Gr(I)$ or $b(c + i) \in Gr(I)$, and hence $ab \in I$ or $ac \in Gr(I)$ or $bc \in Gr(I)$. So we can assume that $abI_g = (0)$. Similarly, we can assume that $acI_g = (0)$ and $bcI_g = (0)$. If $aI_g^2 \neq (0)$, then there exist $u, v \in I_g$ such that $auv \neq 0$. Hence $0 \neq a(b + u)(c + v) = auv \in I$. Since I is a graded weakly 2-absorbing primary ideal of R, we have $a(b+u) \in I$ or $a(c+v) \in Gr(I)$ or $(b + u)(c + v) \in Gr(I)$ and hence $ab \in I$ or $ac \in Gr(I)$ or $bc \in Gr(I)$. So we can assume that $aI_g^2 = (0)$. Similarly, we can assume that $bI_g^2 = (0)$ and $cI_g^2 = (0)$. Since $I_g^3 \neq (0)$, there exist $i_1, i_2, i_3 \in I_g$ such that $i_1 i_2 i_3 \neq 0$. Hence $0 \neq (a+i_1)(b+i_2)(c+i_3) = i_1 i_2 i_3 \in I_g$. Since I is a graded weakly 2-absorbing primary ideal of R, we get that $(a + i_1)(b + i_2) \in I$ or $(b + i_2)(c + i_3) \in Gr(I)$ or $(a + i_1)(c + i_3) \in Gr(I)$ and hence $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Therefore, I is a g-2-absorbing primary ideal of R. □

The following result is an analogue of [10, Corollary 2.11]

Corollary 3.3. *Let $R = \oplus_{g\in G} R_g$ be a graded ring and $I = \oplus_{g\in G}\oplus I_g$ be a graded weakly 2-absorbing primary ideal of R such that I is not a g-2-absorbing primary ideal of R for every $g \in G$. Then $Gr(I) = Gr(0)$.*

Proof. Clearly, $Gr(0) \subseteq Gr(I)$. By Theorem 3.2, $I_g^3 = (0)$ for every $g \in G$. This implies that $Gr(I) \subseteq Gr(0)$. □

The following result is an analogue of [10, Theorem 2.22].

Theorem 3.4. *Let R_1 and R_2 be two graded rings, and let I_1 and I_2 be a non-zero proper graded ideals of R_1 and R_2, respectively. Then the following statements are equivalent.*

(i) *$I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of R.*
(ii) *I_1 and I_2 are graded primary ideal of $R_1$and R_2, respectively.*
(iii) *$I_1 \times I_2$ is a graded 2-absorbing primary ideal of R.*

Proof. (i)$\Longrightarrow$(ii). Suppose that $I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$. We show that I_2 is a graded primary ideal of R_2. Let $r, s \in h(R_2)$ with $rs \in I_2$ and let $0 \neq i \in h(I_1)$. Then $(0, 0) \neq (i, 1)(1, r)(1, s) = (i, rs) \in I_1 \times I_2$. Since $I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$ and $(1, r)(1, s) = (1, rs) \notin Gr(I_1 \times I_2)$, we conclude that either $(i, 1)(1, r) = (i, r) \in I_1 \times I_2$ or $(i, 1)(1, s) = (i, s) \in Gr(I_1 \times I_2)$, and so $r \in I_2$ or $s \in Gr(I_2)$. Thus I_2 is a graded primary ideal of R_2. Similarly, one can show that I_1 is a graded primary ideal of R_1.

(ii)$\Longrightarrow$(iii). The proof is clear by Theorem 2.10

(iii)$\Longrightarrow$(i). It is clear. □

Theorem 3.5. *Let R_1 and R_2 be two graded rings, and let I be a non-zero proper graded ideal of R_1. Then the following statements are equivalent.*

(i) *$I \times (0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$.*
(ii) *I is a graded weakly primary ideal of R_1 and (0) is a graded primary ideal of R_2.*

Proof. (i)$\Rightarrow$(ii). Suppose that $I \times (0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$. First we show I is a graded weakly primary ideal of R_1, let $r, s \in h(R)$ with $0 \neq rs \in I$. Hence $(0, 0) \neq (r, 1)(s, 1)(1, 0) = (rs, 0) \in I \times (0)$. Since $I \times (0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$ and $(r, 1)(s, 1) = (rs, 1) \notin Gr(I \times (0))$, we conclude that either $(r, 1)(1, 0) = (r, 0) \in I \times (0)$ or $(s, 1)(1, 0) = (s, 0) \in Gr(I \times (0))$ and hence either $r \in I$ or $s \in Gr(I)$. Thus I is a graded weakly primary ideal of R_1. Now we show that (0) is a graded primary ideal of R_2. Let $r, s \in h(R_2)$ with $rs \in (0)$, and let $0 \neq i \in h(I)$. Hence $(0, 0) \neq (i, rs) = (i, 1)(1, r)(1, s) \in I \times (0)$. Since $I \times (0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$ and $(1, r)(1, s) = (1, rs) \notin Gr(I \times (0))$, we conclude that $(i, 1)(1, r) = (i, r) \in I \times (0)$ or $(i, 1)(1, s) = (i, s) \in Gr(I \times (0))$ and hence either $r \in (0)$ or $s \in Gr(0)$. Thus (0) is a graded primary ideal of R_2.

(ii)⇒(i). Suppose that I is a graded weakly primary ideal of R_1 and (0) is a graded primary ideal of R_2. We show that $I\times(0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$. Suppose that $(0,0) \neq (r_1,r_2)(s_1,s_2)(t_1,t_2) \in I \times (0)$ for some $r_1, s_1, t_1 \in h(R_1)$ and for some $r_2, s_2, t_2 \in h(R_2)$. We conclude that $r_1s_1t_1 \neq 0$. Assume $(r_1,r_2)(s_1,s_2) \notin I \times (0)$ we consider three cases.

Case one: Suppose that $r_1s_1 \notin I$, but $r_2s_2 = 0$. Since I is a graded weakly primary ideal of R_1 and $0 \neq r_1s_1t_1 \in I$, we have $t_1 \in Gr(I)$. Since (0) is a graded primary ideal of R_2 and $r_2s_2 = 0$, we have $r_2 = 0$ or $s_2 \in Gr(0)$. Thus $(r_1,r_2)(t_1,t_2) = (r_1t_1, r_2t_2) \in Gr(I \times (0)) = Gr(I) \times Gr(0)$ or $(s_1,s_2)(t_1,t_2) = (s_1t_1, s_2t_2) \in Gr(I \times (0)) = Gr(I) \times Gr(0)$.

Case Two: Suppose that $r_1s_1 \in I$, but $r_2s_2 \neq 0$. Since $0 \neq r_1s_1 \in I$ and I is a graded weakly primary ideal of R_1, we have $r_1 \in I$ or $s_1 \in Gr(I)$. Since $r_2s_2 \neq 0$ and (0) is a graded primary ideal of R_2, we have $t_2 \in Gr(0)$. Thus (r_1,r_2) $(t_1,t_2) \in Gr(I \times (0))$ or $(s_1,s_2)(t_1,t_2) \in Gr(I \times (0))$. Hence $I \times (0)$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$.

Case Three: Suppose that $r_1s_1 \notin I$ and $r_2s_2 \neq 0$. Since I is a graded weakly primary ideal of R_1 and $0 \neq r_1s_1t_1 \in I$, we have $t_1 \in Gr(I)$. Since (0) is a graded primary ideal of R_2, $r_2s_2t_2 = 0$ and $r_2s_2 \neq 0$, we have $t_2 \in Gr(0)$. Then $(t_1,t_2) \in Gr(I)\times Gr(0) = Gr(I\times(0))$. Thus (r_1,r_2) $(t_1,t_2) \in Gr(I\times(0))$ or $(s_1,s_2)(t_1,t_2) \in Gr(I \times (0))$. □

The following result is an analogue of [10, Theorem 2.23].

Theorem 3.6. *Let R_1 and R_2 be two graded rings, and let I_1 be a non-zero proper graded ideal of R_1, and I_2 be a proper graded ideal of R_2. Then the following statements are equivalent.*

(i) *$I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$ that is not a graded 2-absorbing primary ideal.*
(ii) *$I_2 = (0)$ is a graded primary ideal of R_2 and I_1 is a graded weakly primary ideal of R_1 that is not a graded primary ideal.*

Proof. (i)⇒(ii). Suppose that $I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$ that is not a graded 2-absorbing primary ideal. Theorem 3.4 implies that $I_2 = (0)$. By Theorem 3.5, $I_2 = (0)$ is a graded primary ideal of R_2 and I_1 is a graded weakly primary ideal of R_1. Now suppose that I_1 is a graded primary ideal of R_1. Then $I_1 \times I_2$ is a graded 2-absorbing primary ideal by Theorem 2.10 which contradicts the assumption. Thus I_1 is not a graded primary ideal of R_1.

(i)⇒(ii). Suppose that I_1 is a graded weakly primary ideal of R_1 that is not a graded primary ideal and $I_2 = (0)$ is a graded primary ideal of R_2. By Theorem 3.5, $I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $R_1 \times R_2$. Since I_1 is not a graded primary ideal of R_1, $I_1 \times (0)$ is not a graded 2-absorbing primary ideal of R by Theorem 2.10. □

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